ON MAXIMAL REGULAR IDEALS AND IDENTITIES IN THE TENSOR PRODUCT OF COMMUTATIVE BANACH ALGEBRAS

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1. Introduction. Let A_1 and A_2 be commutative Banach algebras and $A_1 \odot A_2$ their algebraic tensor product over the complex numbers C. There is always at least one norm, namely the greatest cross-norm γ (**2**), on $A_1 \odot A_2$ that renders it a normed algebra. We shall write $A_1 \otimes_\alpha A_2$ for the α-completion of $A_1 \odot A_2$ when α is an algebra norm on $A_1 \odot A_2$. Gelbaum (**2**; **3**), Tomiyama (**9**), and Gil de Lamadrid (**4**) have shown that for certain algebra norms α on $A_1 \odot A_2$, every complex homomorphism on $A_1 \odot A_2$ is α-continuous. In § 3 of this paper, we present a condition on an algebra norm α which is equivalent to the α-continuity of every complex homomorphism on $A_1 \odot A_2$. Also, in § 3, we give an example of an algebra norm on a particular tensor product that is not one of the types discussed by the above-mentioned authors but does satisfy our condition. In § 4 we characterize those pairs (A_1, A_2) for which the radical of $A_1 \odot A_2$ is the intersection of the kernels of the complex homomorphisms on $A_1 \odot A_2$. We also characterize those pairs (A_1, A_2) for which every maximal regular ideal in $A_1 \odot A_2$ has co-dimension 1. Section 5 is devoted to a study of identities in $A_1 \otimes_\alpha A_2$ versus identities in A_1 and A_2.

2. Preliminaries. If A is a commutative complex algebra, then $H(A)$ denotes the collection of all complex homomorphisms from A onto the complex numbers, $R(A)$ denotes the radical of A and we set $K(A) = \bigcap_{h \in H(A)} h^{-1}(0)$. As usual, if A is a commutative Banach algebra, then the set $H(A)$ endowed with the Gelfand topology is denoted by Φ_A (**7**).

If A_i, $i = 1, 2$, are complex algebras, then it is known that the elements in $H(A_1 \odot A_2)$ can be identified in a natural way with the set $H(A_1) \times H(A_2)$. More precisely, if $h_i \in H(A_i)$ and $h_1 \otimes h_2$ is defined on $A_1 \odot A_2$ by setting

$$h_1 \otimes h_2\left(\sum_{j=1}^{m} a_j \otimes b_j \right) = \sum_{j=1}^{m} h_1(a_j) h_2(b_j),$$

then $h_1 \otimes h_2 \in H(A_1 \odot A_2)$. Conversely, if $h \in H(A_1 \odot A_2)$ and h_1 is defined on A_1 by setting $h_1(a_1) = h(a_1 a_0 \otimes b_0)/h(a_0 \otimes b_0)$, where $h(a_0 \otimes b_0) \neq 0$, then $h_1 \in H(A_1)$. If h_2 is defined similarly, then $h = h_1 \otimes h_2$;

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see (**9**) for details. The natural identification of $H(A_1 \odot A_2)$ with $H(A_1) \times H(A_2)$ is given by $h_1 \otimes h_2 \to (h_1, h_2)$.

LEMMA 1. *$K(A_1 \odot A_2) = K(A_1) \odot A_2 + A_1 \odot K(A_2)$ and*

$$A_1 \odot A_2/K(A_1 \odot A_2)$$

is isomorphic with $A_1/K(A_1) \odot A_2/K(A_2)$, the isomorphism being $\sum a_i \otimes b_i + K(A_1 \odot A_2) \to \sum (a_i + K(A_1)) \otimes (b_i + K(A_2))$.

Proof. It is easy to verify that $K(A_1) \odot A_2 + A_1 \odot K(A_2)$ is contained in $K(A_1 \odot A_2)$.

Suppose that $\tau = \sum_{i=1}^{n} a_i \otimes b_i \in K(A_1 \odot A_2)$. τ can be expressed in the form $\sum_{j=1}^{m} a_j' \otimes b_j' + \tau'$, where $\tau' \in A_1 \odot K(A_2)$ and no non-trivial linear combination of the elements $b_1', \ldots, b_m'$ is in $K(A_2)$. This follows from the fact that there exists a subset $\{b_1', \ldots, b_m'\}$ of $\{b_1, \ldots, b_n\}$ which, modulo $K(A_2)$, is a basis for the linear span of the b_i, $i = 1, \ldots, n$. Since $\tau' \in A_1 \odot K(A_2) \subseteq K(A_1 \odot A_2)$, $\sum_{j=1}^{m} a_j' \otimes b_j' \in K(A_1 \odot A_2)$, and hence

$$0 = h_1 \otimes h_2\left(\sum_{j=1}^{m} a_j' \otimes b_j'\right) = \sum_{j=1}^{m} h_1(a_j')h_2(b_j') = h_2\left(\sum_{j=1}^{m} h_1(a_j')b_j'\right)$$

for all $h_1 \in H(A_1)$ and $h_2 \in H(A_2)$. This means that $\sum_{j=1}^{m} h_1(a_j')b_j' \in K(A_2)$ for all $h_1 \in H(A_1)$. Hence, $a_j' \in K(A_1)$ since $h_1(a_j') = 0$ for all $h_1 \in H(A_1)$ and $j = 1, \ldots, m$. Thus, $\sum_{j=1}^{m} a_j' \otimes b_j' \in K(A_1) \odot A_2$ and it follows that $K(A_1 \odot A_2) = K(A_1) \odot A_2 + A_1 \odot K(A_2)$.

The last assertion of the lemma is well known; see, for example, (**5**).

COROLLARY 1. *If A_1 and A_2 are complex algebras for which $K(A_i) = (0)$, $i = 1, 2$, then $A_1 \odot A_2$ is semisimple.*

The corollary, of course, follows from the lemma and the fact that $R(A_1 \odot A_2) \subseteq K(A_1 \odot A_2)$, the latter ideal being equal to (0) in the situation of the corollary.

The inclusion $R(A_1 \odot A_2) \subseteq K(A_1 \odot A_2)$ suggests the following question: When is $R(A_1 \odot A_2) = K(A_1 \odot A_2)$? We shall completely answer this question in § 4 for the case where A_1 and A_2 are commutative Banach algebras.

3. Spectral tensor norms. Throughout the remainder of this paper, A_i will always denote a commutative Banach algebra with norm $||\cdot||_i$ and spectral radius ν_i, $i = 1, 2$; α will always denote an algebra norm on $A_1 \odot A_2$. We set $\nu_\alpha(\tau) = \lim_{n\to+\infty} (\alpha(\tau^n))^{1/n}$ for $\tau \in A_1 \odot A_2$. The space $\Phi_{A_1 \otimes_\alpha A_2}$ can be identified with the set of α-continuous complex homomorphisms on $A_1 \odot A_2$, and hence can be viewed in a natural way as a subset of $\Phi_{A_1} \times \Phi_{A_2}$. If $\Phi_{A_1 \otimes_\alpha A_2}$ exhausts $\Phi_{A_1} \times \Phi_{A_2}$, we say that $\Phi_{A_1 \otimes_\alpha A_2}$ is *full.* In this section, we show that $\Phi_{A_1 \otimes_\alpha A_2}$ is full if and only if $\nu_\alpha(a \otimes b) = \nu_1(a)\nu_2(b)$ holds for all simple tensors $a \otimes b \in A_1 \odot A_2$. Smith (**8**) has presented necessary and sufficient conditions for $\Phi_{A_1 \otimes_\alpha A_2}$ to be full when α is an algebra cross-norm.

Our results show that these conditions are always satisfied. A norm which satisfies (1): $\nu_\alpha(a \otimes b) = \nu_1(a)\nu_2(b)$ for all $a \otimes b \in A_1 \odot A_2$ will be called a spectral tensor norm.

The norms that have been studied by Gelbaum (**3**), Tomiyama (**9**) and Gil de Lamadrid (**4**) are all spectral tensor norms. For each of these norms, there is a positive number k such that α satisfies (2): $k\|a\|_1\|b\|_2 \leqq \alpha(a \otimes b)$ for all simple tensors $a \otimes b \in A_1 \odot A_2$. Now, if (2) holds, then

$$\nu_1(a)\nu_2(b) = \lim_{n\to+\infty} k^{1/n}\|a^n\|_1^{1/n}\|b^n\|_2^{1/n} \leqq \nu_\alpha(a \otimes b), \quad a \otimes b \in A_1 \odot A_2.$$

Since $\nu_\alpha(a \otimes b) \leqq \nu_1(a)\nu_2(b)$ is always true, it follows that (2) implies (1). Spectral tensor norms, however, need not satisfy (2). We offer below an example of a spectral tensor norm which does not satisfy $\alpha(a \otimes b) \geqq k\|a\|_1\|b\|_2$ for any $k > 0$.

Example. Let $A_1 = A_2 = C^1[0, 1]$, the algebra of continuously differentiable complex-valued functions on $[0, 1]$, with $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $f \in C^1[0, 1]$. Then $A_1 \odot A_2$ is isomorphic to the set of all functions on the unit square S of the form

$$\sum_{i=1}^{n} f_i(x)g_i(y), \qquad f_i, g_i \in C^1[0, 1].$$

Hence, $A_1 \odot A_2$ can be viewed as a subalgebra of $A = \{f \in C(S)\colon \partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous on $S\}$. For $f \in A$, we set

$$\alpha(f) = \|f\|_\infty + \left\|\frac{\partial f}{\partial x}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty.$$

It is easy to verify that A is a Banach algebra under the norm α. It follows from a theorem of Butzer[1] (**1**) that $A_1 \odot A_2$ is α-dense in A. Now, set $f_n(x) = x^n$, $n = 1, 2, \ldots$. Since

$$\lim_{n\to+\infty} \frac{\alpha(f_n \otimes f_n)}{\|f_n\|^2} = \lim_{n\to+\infty} \frac{1 + n + n}{(1 + n)^2} = 0,$$

there exists no $k > 0$ such that $\alpha(f \otimes g) \geqq k\|f\|\,\|g\|$ for all $f \otimes g \in A_1 \odot A_2$.

We commented above that $\Phi_{A_1 \otimes_\alpha A_2}$ can be viewed as a subset of $H(A_1 \odot A_2)$. The following proposition describes the topological aspects of the embedding.

PROPOSITION 1. *$\Phi_{A_1 \otimes_\alpha A_2}$ is a closed subset of $\Phi_{A_1} \times \Phi_{A_2}$, and the Gelfand topology on $\Phi_{A_1 \otimes_\alpha A_2}$ is the relativization of the product topology on $\Phi_{A_1} \times \Phi_{A_2}$.*

The space $\Phi_{A_1 \otimes_\alpha A_2}$ is closed in $\Phi_{A_1} \times \Phi_{A_2}$ since

$$\operatorname{lub}\{|h_1 \otimes h_2(\tau)|\colon h_1 \otimes h_2 \in \operatorname{cl}(\Phi_{A_1 \otimes_\alpha A_2})\} = \operatorname{lub}\{|h_1 \otimes h_2(\tau)|\colon h_1 \otimes h_2 \in \Phi_{A_1 \otimes_\alpha A_2}\} \leqq \alpha(\tau),$$

[1]We would like to thank Professor G. G. Lorentz for suggesting this reference.

for all $\tau \in A_1 \odot A_2$. Since the Gelfand topology on $\Phi_{A_1 \otimes_\alpha A_2}$ is identical with the weak topology induced by $(A_1 \odot A_2)^\wedge$ on $\Phi_{A_1 \otimes_\alpha A_2}$, the last assertion follows from the fact that the product topology on $\Phi_{A_1} \times \Phi_{A_2}$ is the weak topology induced by $(A_1 \odot A_2)^\wedge$ (see **9**, p. 150) and the relative product topology is the weak topology induced by $(A_1 \odot A_2)^\wedge$ on $\Phi_{A_1 \otimes_\alpha A_2}$.

THEOREM 1. *Let A_1 and A_2 be commutative Banach algebras. Then $\Phi_{A_1 \otimes_\alpha A_2}$ is full if and only if α is a spectral tensor norm.*

Proof. Suppose that $\Phi_{A_1 \otimes_\alpha A_2}$ is full. Then

$$\begin{aligned}\nu_\alpha(a \otimes b) &= \text{lub}\{|h_1 \otimes h_2(a \otimes b)|: h_i \in \Phi_{A_i}, i = 1, 2\} \\ &= \text{lub}\{|h_1(a)|: h_1 \in \Phi_{A_1}\} \text{ lub}\{|h_2(b)|: h_2 \in \Phi_{A_2}\} \\ &= \nu_1(a)\nu_2(b).\end{aligned}$$

Thus, α is a spectral tensor norm on $A_1 \odot A_2$.

Suppose that α is a spectral tensor norm on $A_1 \odot A_2$. By the above proposition, we know that $\Phi_{A_1 \otimes_\alpha A_2}$ is a closed subset of $\Phi_{A_1} \times \Phi_{A_2}$. We first show that if ∂_{A_i} denotes the Šilov boundary of A_i, $i = 1, 2$, then $\partial_{A_1} \times \partial_{A_2} \subseteq \Phi_{A_1 \otimes_\alpha A_2}$. To this end, suppose that $h_1 \otimes h_2 \in \partial_{A_1} \times \partial_{A_2} \backslash \Phi_{A_1 \otimes_\alpha A_2}$. By a characterization of the Šilov boundary, there exist open neighbourhoods V_1 and V_2 of h_1 and h_2, respectively, such that $V_1 \times V_2 \cap \Phi_{A_1 \otimes_\alpha A_2} = \emptyset$, and elements $a \in A_1$ and $b \in A_2$ such that $|\hat{a}(h_1')| < \nu_1(a)$ for $h_1' \notin V_1$ and $|\hat{b}(h_2')| < \nu_2(b)$ for $h_2' \notin V_2$. On the other hand, there exists $h_1^0 \otimes h_2^0 \in \Phi_{A_1 \otimes_\alpha A_2}$ such that $\nu_\alpha(a \otimes b) = |\hat{a}(h_1^0)|\,|\hat{b}(h_2^0)|$. Since $h_1^0 \otimes h_2^0 \notin V_1 \times V_2$, then $h_i^0 \notin V_i$ for either $i = 1$ or 2. Hence, $\nu_\alpha(a \otimes b) < \nu_1(a)\nu_2(b)$, which contradicts the hypothesis that α is a spectral tensor norm. Thus, $\partial_{A_1} \times \partial_{A_2} \subseteq \Phi_{A_1 \otimes_\alpha A_2}$. If $h_1 \otimes h_2 \in \Phi_{A_1} \times \Phi_{A_2}$ and $\tau \in A_1 \odot A_2$, then $|\hat{\tau}(h_1 \otimes h_2)| \leqq \text{lub}\{|\hat{\tau}(h_1' \otimes h_2')|: h_1' \otimes h_2' \in \partial_{A_1} \times \partial_{A_2}\}$; see (**3**, Theorem 2). Since the right-hand side is equal to or less than $\nu_\alpha(\tau)$, we have that every complex homomorphism on $A_1 \odot A_2$ is α-continuous.

COROLLARY 2. *If A_1 and A_2 are semisimple and regular, then any algebra norm on $A_1 \odot A_2$ is a spectral tensor norm.*

Proof. The argument is a modification of one that appears in (**7**, p. 175). Suppose that α is an algebra norm on $A_1 \odot A_2$ that is not a spectral tensor norm. Then there is an element $h_1 \otimes h_2 \in \Phi_{A_1} \times \Phi_{A_2} \backslash \Phi_{A_1 \otimes_\alpha A_2}$. We can choose open neighbourhoods U_i and V_i of h_i such that $U_1 \times U_2$ is disjoint from $\Phi_{A_1 \otimes_\alpha A_2}$, $\bar{V}_i$ is compact and $\bar{V}_i \subseteq U_i$, for $i = 1, 2$. There are elements $a_i \in A_i$ and $b_i \in A_i$ such that $\hat{a}_i(h_i) = 1$ and $\hat{a}_i$ is identically 0 off V_i, $\hat{b}_i$ is identically 1 on V_i, and $\hat{b}_i$ is identically 0 off U_i. Now the simple tensors $u = a_1 \otimes a_2$ and $v = b_1 \otimes b_2$ have the property that $uv - u \in K(A_1 \odot A_2)$. However, $K(A_1 \odot A_2) = (0)$ since A_1 and A_2 are semisimple. Hence $uv = u$. Since $\hat{v}$ is identically 0 on $\Phi_{A_1 \otimes_\alpha A_2}$, $v \in R(A_1 \otimes_\alpha A_2)$. Thus, v has a quasi-inverse $w \in A_1 \otimes_\alpha A_2$, from which it follows that $0 = v \circ w = (u \circ v) \circ w = u \circ (v \circ w) = u$. This is impossible since $u \neq 0$.

The above proof yields the following stronger result. Suppose that A is a subalgebra of $C_0(\Omega)$, Ω a locally compact Hausdorff space, and that each complex homomorphism of A onto C is given by point evaluation at some point of Ω. Suppose, further, that for each closed set $K \subset \Omega$ and for each point $w \in \Omega \backslash K$ there exists an $f \in A$ which vanishes on K and is identically 1 in a neighbourhood of w. Then every algebra norm on A majorizes the supremum norm over Ω. This result is not new for the case where A is also a Banach algebra under some norm; see (**7**, p. 176).

4. On the radical and maximal regular ideals of infinite co-dimension in $A_1 \odot A_2$. In (**2**), Gelbaum assumed that A_1, A_2, and α were such that $A_1 \odot A_2$ had no α-dense maximal regular ideals; that is, $A_1 \odot A_2$ was a Q-algebra with respect to α. On the basis of this assumption, he showed that $\Phi_{A_1 \otimes_\alpha A_2}$ was full. (If α is taken to be the greatest cross-norm, then this assumption can be dropped, as examination of his proof shows.) It is natural to ask: under what conditions is $A_1 \odot A_2$ a Q-algebra under α? Clearly, α must be a spectral tensor norm since $h^{-1}(0)$ is α-dense if h is α-discontinuous. Furthermore, every maximal regular ideal must have co-dimension 1; that is, it must be the kernel of a complex homomorphism. In Theorem 3, we characterize those pairs (A_1, A_2) for which every maximal regular ideal in $A_1 \odot A_2$ has co-dimension 1. In the investigation leading to Theorem 3, we obtained a characterization (Theorem 2) of those pairs (A_1, A_2) for which $R(A_1 \odot A_2) = K(A_1 \odot A_2)$.

LEMMA 2. *Let A be a commutative Banach algebra and $r \in A$ with $\|r\| \leqq \frac{1}{2}$. If $\hat{r}$ has infinite range or $r \in R(A)$ and r is not nilpotent, then $\sum_{n=1}^{\infty} \mu_n r^n = 0$, where $\{\mu_n\}$ is a bounded sequence of complex numbers, implies that $\mu_n = 0$ for $n = 1, 2, \ldots$.*

Proof. Suppose that $\hat{r}$ has infinite range. To show that r satisfies the property of the lemma, suppose that $\{\mu_n\}$ is a bounded sequence of complex numbers and that $\sum_{n=1}^{\infty} \mu_n r^n = 0$. Consider the power series $f(z) = \sum_{n=1}^{\infty} \mu_n z^n$. Since $\{\mu_n\}$ is a bounded sequence, this power series converges absolutely for $|z| < 1$. By assumption, $f(\hat{r}(h)) = 0$ for all $h \in \Phi_A$ so that f has infinitely many zeros of moduli less than or equal to $\frac{1}{2}$. Thus, $f(z)$ is identically zero and $\mu_n = 0$, $n = 1, 2, \ldots$.

Next suppose that $r \in R(A)$ is not nilpotent. For this part, we can assume that A has an identity e since the adjunction of an identity does not change the radical. Let $\{\mu_n\}$ be a bounded sequence of complex numbers such that $\sum_{n=1}^{\infty} \mu_n r^n = 0$ and let n_0 be the smallest integer such that $\mu_{n_0} \neq 0$. Then

$$\sum_{n=n_0}^{\infty} \mu_n r^n = r^{n_0}\left(\mu_{n_0} e + \sum_{n=n_0+1}^{\infty} \mu_n r^{n-n_0}\right) = 0.$$

Since $\sum_{n=n_0+1}^{\infty} \mu_n r^{n-n_0} \in R(A)$ and $\mu_{n_0} \neq 0$, the right-hand factor is invertible in A, and hence $r^{n_0} = 0$, a contradiction. This completes the proof of the lemma.

It is shown in (**6**) that if A is semisimple and infinite-dimensional, then A has an element with infinite spectrum. Hence, if Φ_A is infinite, then there exists $a \in A$ such that $\hat{a}$ has infinite range.

LEMMA 3. *Suppose that* $r \otimes s \in A_1 \odot A_2$, *with* $||r||_1 \leqq \frac{1}{2}$, $||s||_2 \leqq \frac{1}{2}$, *satisfies one of the following conditions:*

(i) $r \in R(A_1)$, *not nilpotent, and* $s \in R(A_2)$, *not nilpotent,*

(ii) $r \in R(A_1)$, *not nilpotent, and* $\hat{s}$ *has infinite range; or* $\hat{r}$ *has infinite range and* $s \in R(A_2)$, *not nilpotent,*

(iii) $\hat{r}$ *and* $\hat{s}$ *both have infinite range,*

then $r \otimes s$ *is the relative identity for a maximal regular ideal which has infinite co-dimension.*

Proof. We first show that if any of the above conditions obtain, then $r \otimes s$ is quasi-singular in $A_1 \odot A_2$. Suppose that $r \otimes s$ is quasi-regular in $A_1 \odot A_2$. Since $\gamma(r \otimes s) = ||r||_1||s||_2 \leqq \frac{1}{4}$, the quasi-inverse $(r \otimes s)^0$ of $r \otimes s$ in $A_1 \otimes_\gamma A_2$ is given by $-\sum_{n=1}^{\infty} (r \otimes s)^n = -\sum_{n=1}^{\infty} r^n \otimes s^n$. On the other hand, $(r \otimes s)^0 = \sum_{i=1}^{N} a_i \otimes b_i \in A_1 \odot A_2$. Let f belong to the dual A_1^* of A_1 and define

$$T_f\left(\sum_{n=1}^{\infty} a_n' \otimes b_n'\right) = \sum_{n=1}^{\infty} f(a_n')b_n',$$

a continuous linear mapping of $A_1 \otimes_\gamma A_2$ into A_2; see (**9**). Thus,

$$T_f\left(\sum_{i=1}^{N} a_i \otimes b_i\right) = \sum_{i=1}^{N} f(a_i)b_i,$$

so that for all $f \in A_1^*$, $\sum_{n=1}^{\infty} f(r^n)s^n$ lies in the finite-dimensional subspace of A_2 spanned by $b_1, \ldots, b_N$. By Lemma 2, we have that the r^n's are linearly independent. Hence, there exist $f_1, \ldots, f_{N+1} \in A_1^*$ such that $f_i(r^j) = \delta_{ij}$, $1 \leqq i, j \leqq N+1$. Now, there are complex numbers $\lambda_1, \ldots, \lambda_{N+1}$, not all zero, such that

$$0 = \sum_{i=1}^{N+1} \lambda_i\left(\sum_{n=1}^{\infty} f_i(r^n)s^n\right) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{N+1} \lambda_i f_i(r^n)\right)s^n.$$

Since

$$\left|\sum_{i=1}^{N+1} \lambda_i f_i(r^n)\right| \leqq \sum_{i=1}^{N+1} |\lambda_i|\, ||f_i|| \quad \text{for all } n \geqq 1,$$

Lemma 2 implies that $\sum_{i=1}^{N+1} \lambda_i f_i(r^n) = 0$, $n \geqq 1$. In particular, if $1 \leqq n \leqq N+1$, we have that $\lambda_n = \lambda_n f_n(r^n) = 0$, a contradiction. Thus, $r \otimes s$ must be quasi-singular in $A_1 \odot A_2$ and the ideal $I = \{(r \otimes s)\tau - \tau : \tau \in A_1 \odot A_2\}$ is a proper ideal with relative identity $r \otimes s$. Hence, I is contained in a maximal regular ideal, say M. Now, M is not the kernel of any complex homomorphism $h_1 \otimes h_2 \in \Phi_{A_1} \times \Phi_{A_2}$. For this would mean that $1 = h_1 \otimes h_2(r \otimes s) = h_1(r)h_2(s) \leqq ||r||_1||s||_2 \leqq \frac{1}{4}$.

If either (i) or (ii) holds in the above lemma, then it is obvious that $r \otimes s \in K(A_1 \odot A_2)$.

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It is convenient to introduce a name for a commutative Banach algebra which has a radical which is a nil ideal and has only a finite number of distinct complex homomorphisms. We shall simply refer to such an algebra as a *fini* Banach algebra.

THEOREM 2. *$R(A_1 \odot A_2) = K(A_1 \odot A_2)$ if and only if one of the following conditions hold:*

(i) *$R(A_1)$ and $R(A_2)$ are nil ideals;*

(ii) *A_1 is a fini Banach algebra;*

(iii) *A_2 is a fini Banach algebra.*

Proof. Suppose first that $R(A_1)$ and $R(A_2)$ are nil ideals. Then $K(A_1 \odot A_2)$ is a nil ideal, and hence $K(A_1 \odot A_2) \subseteq R(A_1 \odot A_2)$. Since $R(A_1 \odot A_2) \subseteq K(A_1 \odot A_2)$, we have equality. Suppose next that A_1 is a fini Banach algebra. From Lemma 1, we know that $K(A_1 \odot A_2) = R(A_1) \odot A_2 + A_1 \odot R(A_2)$. Since the sum of nilpotent elements is again nilpotent, $R(A_1) \odot A_2$ consists entirely of nilpotent elements, and hence $R(A_1) \odot A_2$ is contained in $R(A_1 \odot A_2)$. In order to show that $A_1 \odot R(A_2)$ is contained in $R(A_1 \odot A_2)$, let $\Phi_{A_1} = \{h_1, \ldots, h_k\}$ and $\{e_1, \ldots, e_k\}$ be the set of orthogonal idempotents in A_1 such that $\hat{e}_i(h_j) = \delta_{ij}$, $1 \leqq i, j \leqq k$ (**7**). Then we have that $A_1 = e_1A_1 \oplus \ldots \oplus e_kA_1 \oplus (1 - e_1 - \ldots - e_k)A_1$, where the last ideal is contained in $R(A_1)$. If $a \otimes s \in A_1 \odot R(A_2)$, then $a \otimes s = (e_1a \otimes s) + \ldots + (e_ka \otimes s) + ((1 - e_1 - \ldots - e_k)a \otimes s)$. Observe that the last term is in $R(A_1 \odot A_2)$. It suffices to show that $e_1a \otimes s \in R(A_1 \odot A_2)$. By a standard characterization of the radical of an algebra, all we need to show is that

$$\tau = \left(\sum_{j=1}^{n} a_j \otimes b_j\right)(e_1a \otimes s) + \xi(e_1a \otimes s) = \sum_{j=1}^{n} e_1a_ja \otimes b_js + (\xi e_1a \otimes s)$$

is quasi-regular for all $a_j \in A_1$, $b_j \in A_2$, $j = 1, \ldots, n$, and all complex numbers ξ. Since $e_1A_1 = Ce_1 \oplus R(e_1A_1)$, we can write $e_1a_ja = \xi_je_1 + r_j$, $\xi e_1a = \xi_0e_1 + r_0$. Now $\tau = e_1 \otimes s' + \tau'$, where τ' is nilpotent. Thus, τ is the sum of a quasi-regular element and a nilpotent element. Hence, τ is quasi-regular, by direct calculation of the quasi-inverse, and $e_1a \otimes s \in R(A_1 \odot A_2)$. Similarly, if A_2 is a fini Banach algebra, then $R(A_1 \odot A_2) = K(A_1 \odot A_2)$.

To establish the converse, it suffices to consider the case where $R(A_1)$ is not a nil ideal and A_2 is not a fini Banach algebra. Then either (i) or (ii) in Lemma 3 is satisfied. Hence, there exists a maximal regular ideal with relative identity u and $u \in K(A_1 \odot A_2)$. Since $u \notin R(A_1 \odot A_2)$, $R(A_1 \odot A_2)$ is a proper subset of $K(A_1 \odot A_2)$. This completes the proof of the theorem.

In (**5**), Jacobson proved that if A_1 is finite-dimensional over a field ϕ and A_2 is a radical algebra over ϕ, then $A_1 \odot A_2$ (over ϕ) is a radical algebra. For commutative Banach algebras, this also follows from the above theorem. Moreover, it follows that $A_1 \odot A_2$ is a radical algebra if and only if A_1 or A_2 is a radical algebra and one of the three conditions of the theorem holds.

THEOREM 3. *Every maximal regular ideal in $A_1 \odot A_2$ has co-dimension one if and only if A_1 or A_2 is a fini Banach algebra.*

Proof. If either A_1 or A_2 is a fini Banach algebra, then $R(A_1 \odot A_2) = K(A_1 \odot A_2)$ by Theorem 2. Let M be a maximal regular ideal in $A_1 \odot A_2$. Since $M \supset K(A_1 \odot A_2)$, $M + K(A_1 \odot A_2)$ is a maximal regular ideal in $A_1 \odot A_2/K(A_1 \odot A_2)$. By Lemma 1, $A_1 \odot A_2/K(A_1 \odot A_2) \cong A_1/R(A_1) \odot A_2/R(A_2)$. Now, if A_1 is a fini Banach algebra, then $A_1/R(A_1) \cong C^k$, where k is the dimension of $A_1/R(A_1)$. Hence,

$$A_1/R(A_1) \odot A_2/R(A_2) \cong \sum_{i=1}^{k} \oplus A_2/R(A_2).$$

Since the latter algebra is a Banach algebra, every maximal regular ideal has co-dimension one. Hence, both $M + K(A_1 \odot A_2)$ and M have co-dimension one.

To prove the converse, suppose that A_1 and A_2 are both not fini Banach algebras. This implies that one of the three statements in Lemma 3 is satisfied, and hence there exists a maximal regular ideal M with relative identity $r \otimes s$, where $||r||_1 \leqq \frac{1}{2}$ and $||s||_2 \leqq \frac{1}{2}$. Therefore, $|h_1 \otimes h_2(r \otimes s)| \leqq \frac{1}{4}$ for all $h_1 \otimes h_2 \in \Phi_{A_1} \times \Phi_{A_2}$. If $M = (h_1 \otimes h_2)^{-1}(0)$ for some $h_1 \otimes h_2 \in \Phi_{A_1} \times \Phi_{A_2}$, then $h_1 \otimes h_2(r \otimes s) = 1$, which is impossible. Hence, M has infinite co-dimension.

COROLLARY 3. *$A_1 \odot A_2$ is a Q-algebra with respect to α if and only if α is a spectral tensor norm and A_1 or A_2 is a fini Banach algebra.*

5. The identity in $A_1 \otimes_\alpha A_2$. If both A_1 and A_2 have identities, then of course $A_1 \odot A_2$ will also have an identity; hence, for any algebra norm α, $A_1 \otimes_\alpha A_2$ will also have an identity. Gelbaum (**3**) has shown that when A_1 and A_2 are semisimple, then $A_1 \otimes_\gamma A_2$ has an identity if and only if both A_1 and A_2 have identities. It follows from the theorem below that a similar result is valid for $A_1 \otimes_\alpha A_2$, where α is any spectral tensor norm, even without the semisimplicity assumption.

As usual, we view $\Phi_{A_1 \otimes_\alpha A_2}$ as a closed subset of $\Phi_{A_1} \times \Phi_{A_2}$ and denote by π_i the natural mapping of $\Phi_{A_1 \otimes_\alpha A_2}$ into Φ_{A_i}.

THEOREM 4. *Let α be an algebra norm on $A_1 \odot A_2$. If $A_1 \otimes_\alpha A_2$ has an identity and if the mappings π_i are onto, then A_1 and A_2 have identities.*

Proof. If $A_1 \otimes_\alpha A_2$ has an identity u, then $\Phi_{A_1 \otimes_\alpha A_2}$ is compact. Since π_i is continuous and onto, Φ_{A_i} is compact for $i = 1, 2$. Hence, there exist idempotents $e_i \in A_i$ such that $\hat{e}_i$ is identically 1 on Φ_{A_i} for $i = 1, 2$ (**7**, p. 168). The element $u_1 = e_1 \otimes e_2$ is an idempotent in $A_1 \odot A_2$ and $\hat{u}_1$ is identically 1 on $\Phi_{A_1 \otimes_\alpha A_2}$. Thus, u_1 has an inverse in $A_1 \otimes_\alpha A_2$, and since $u_1(u - u_1) = 0$, it follows that $u = u_1$.

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To show that $e_1a = a$ for all $a \in A_1$, we note that $(e_1a - a) \otimes e_2 = ((e_1a - a) \otimes e_2)(e_1 \otimes e_2) = (e_1a - e_1a) \otimes e_2 = 0 \otimes e_2 = 0$. Since $e_2 \neq 0$, $e_1a - a = 0$ for all $a \in A_1$. Thus, e_1 is an identity in A_1. Similarly, we conclude that e_2 is an identity in A_2.

COROLLARY 4. *If α is a spectral tensor norm, then $A_1 \otimes_\alpha A_2$ has an identity if and only if A_1 and A_2 have identities.*

As easily constructed examples show, if the mappings π_i are not onto, then $A_1 \otimes_\alpha A_2$ may have an identity without either A_1 or A_2 possessing identities.

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