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LOWER BOUNDS FOR THE NORMS OF PROJECTIONS WITH SMALL KERNELS

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Popov has recently introduced a class of subspaces of $L_p(\mu)$ (μ nonatomic) which generalise the finite codimensional ones, and proved that for $p \neq 2$ any projection onto such a subspace has a norm strictly greater than one. In this paper we give the quantitative version of Popov's result computing the best possible lower bound for the norms of the considered projections.

1. INTRODUCTION

In a recent paper [**3**], Popov has shown the existence of a non-trivial lower bound for the norms of projections from $L_p(\mu)$ onto subspaces with "small" codimension. These subspaces, which include the finite codimensional ones, are called rich. In Section 2 of this paper we compute the best possible lower bound, denoted by $\wedge_p$, for the norm of the projections onto rich subspaces of $L_p(\mu)$. Essentially, this is done by proving that the property used to define rich subspaces is in fact equivalent to a stronger one. This number $\wedge_p$ is equal to the norm of the minimal projection onto hyperplanes in $L_p[0, 1]$ (the value being independent of the chosen hyperplane). In Section 3 we show that for any fixed n there exist subspaces of codimension n in $L_p[0, 1]$ which admit a projection whose norm is exactly $\wedge_p$; the same is also true for some infinite codimensional rich subspaces.

Our notations are standard. If A is a set, χ_A denotes its characteristic function.

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Let (T, σ, μ) be a measure space with a nonnegative, finite, non-atomic measure μ and $1 \leqslant p \leqslant \infty$, Popov has given the following

DEFINITION: (Popov [**3**]). A subspace $\vee$ of $L_p[0, 1]$ is called rich if it has the following property:
for any $A \in \sigma$ and $\varepsilon > 0$ there is in σ a partition $\{A_1, A_2\}$ of A with $\mu(A_1) = 1/2\,\mu(A)$ and $v_\varepsilon \in \vee$ such that $\|v_\varepsilon - y\| < \varepsilon$ for $y = \chi_{A_1} - \chi_{A_2}$.

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We now define numbers $\wedge_p$:

$$\wedge_p = \begin{cases} 2, & \text{for } 1 = p \\ \max\{\varphi_p(t), t \in [0, 1]\}, & 1 < p < \infty \end{cases}$$

where $\varphi_p(t) = [t^{1/p-1} + (1-t)^{1/p-1}]^{p-1/p}[t^{p-1} + (1-t)^{p-1}]^{1/p}$. Note that if $1/p + 1/q = 1$ then $\varphi_p = \varphi_q$, $\wedge_p = \wedge_q$; also $\varphi_p(t) = \varphi_p(1-t)$, $\varphi_2(t) = 1$, $\varphi_p(t) \geqslant 1$. If $p \neq 2$, $\varphi_p(t) = 1$ only if $t \in \{0, 1/2, 1\}$, so that if $p \neq 2$ then $\wedge_p > t$. ▯

We shall prove the following:

THEOREM 1. *Assume $1 \leqslant p < \infty$. If $\vee$ is a proper rich subspace of $L_p(\mu)$ and $L\colon L_p(\mu) \to \vee$ is a projection, then $\|P\| \geqslant \wedge_p$.*

Before proving this theorem, let us note that under its assumptions every subspace $\vee$ of $L_p(\mu)$ of finite codimension is rich.

This fact (in a somewhat different form) is stated by Popov in [**3**], where for its proof he refers to [4], for completeness we include here statement and proof of the above mentioned result.

THEOREM 2. (see Popov, [4] and [**3**]). *Any finite codimensional subspace $\vee$ of $L_p(\mu)$ is rich, here $1 \leqslant p < \infty$ and μ is nonatomic.*

PROOF: The proof is an almost immediate consequence of the following theorem of Blackwell ([**1**], Theorem 2):

If (T, σ, μ) is as above and $\{f_i\}$ are measurable functions on T with $\int_T f_i d\mu < \infty$, $i = 1, \dots, n$, then there is a sigma algebra $\sigma_1 \subset \sigma$ such that μ is nonatomic on σ_1 and, for every $D \in \sigma_1$, $\int_D f_i d\mu = \mu(D) \int_T f_i d\mu$.

Let $A \in \sigma$ and $c \in [0, 1]$ be given; we can select a finite set $\{f_1, \dots, f_n\}$ of elements of Lpm such that $\vee = \{x \in L_p(\mu)\colon \int_T x f_i d\mu = 0, i = 1, \dots, n\}$.

We apply Blackwell's theorem to (A, σ, μ): select an $A_1 \in \sigma_1$ with $\mu(A_1) = c\mu(A)$ and $\int_{A_1} f_i d\mu = c\mu(A) \int_A f_i d\mu$. If $A_1 = A \setminus A_1$ and $y = (1-c)\chi_{A_1} - c\chi_{A_2}$ we have

$$\int_T f_i y = \int_{A_1} f_i y + \int_{A_2} f_i y = 0,$$

that is $y \in \vee$. Taking $c = 1/2$, we see that $\vee$ is rich. ▯

We remark that, in the above proof, what is required for $c = 1/2$ was proved for any $c \in [0, 1]$. This apparently stronger property is actually true for any rich subspace: this fact will be crucial in the proof of Theorem 1.

LEMMA 1. *Under the assumption of Theorem 1, if $\vee$ is a rich subspace of $L_p(\mu)$ then:*

given any $A \in \sigma$, $c \in [0,1]$ and $\varepsilon > 0$, there is in σ a partition $\{C, D\}$ of A with $\mu(C) = c\mu(A)$ and $v_\varepsilon \in \mathrm{V}$ such that $\|v_\varepsilon - y\| < \varepsilon$ for $y = (1-c)\chi_c - c\chi_D$.

PROOF: We first show that given $A \in \sigma$ and $\varepsilon > 0$ there is in σ a sequence of partitions $\{A_n, B_n\}$ of A with $A_i \cap A_j = \emptyset$ for $i \neq j$, $\mu(A_n) = 2^{-n}\mu(A)$ and elements $v_n \in \mathrm{V}$ such that $\|v_n - y_n\| < \varepsilon$ for $y_n = (1-2^{-n})\chi_{A_n} - 2^{-n}\chi_{B_n}$.

Let A and $\varepsilon > 0$ be given and select $\varepsilon_i > 0$ with $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon$; since V is rich we can find in σ a partition $\{A_1, B_1\}$ of A with $\mu(A_1) = 1/2\,\mu(A)$ and $v_1 \in \mathrm{V}$ such that $\|v_1 - y_1\| < \varepsilon_1$ for $y_1 = (1/2)\chi_{A_1} - (1/2)\chi_{B_1}$. Assume now that for $i = 1, \ldots, n-1$ we have selected in σ partitions $\{A_i, B_i\}$ of A with $\mu(A_i) == 2^{-i}\mu(A)$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $v_i \in \mathrm{V}$ such that $\|v_i - y_i\| < \varepsilon_i < \varepsilon$ for $y_i = \left(1-2^{-1}\right)\chi_{A_i} - 2^{-i}\chi_{B_i}$. We now construct A_n, B_n and $v_n \in \mathrm{V}$ with the required properties. Set $F = [0,1] \setminus \bigcup_{i=1}^{n-1} A_i$. We have $\mu(F) = 2^{-(n-1)}\mu(A)$, since V is rich there is in σ a partition $\{A_n F_n\}$ of F with $\mu(A_n) = 1/2\,\mu(F) = 2^{-n}\mu(A)$ and a $w_n \in \mathrm{V}$ with $\|w_n - z_n\| < \varepsilon_n$ for $z_n = \chi_{A_n} - \chi_{F_n}$. We define $B_n = A \setminus A_n$ and $v_n = -1/2(v_1 + v_2 + \ldots + v_{n-1} - w_n)$. Since $v_n \in V$ it remains to prove that $\|v_n - y_n\| < \varepsilon$, where $y_n = (1-2^{-n})\chi_{A_n} - 2^{-n}\chi_{B_n}$. This is true since, as it is easily seen, we have $y_n = -1/2(y_1 + y_2 + \ldots + y_{n-1} - z_n)$.

Let now $A \in \sigma$, $c \in [0,1]$ and $\varepsilon > 0$ be fixed. According to its binary representation we write $c = \sum_k w^{-n_k}$ with $n_1 < n_2 < \ldots < n_k < \ldots$, and we choose A_{n_k} and $v_{n_k} \in \mathrm{V}$ as above $\left(\left\|v_{n_k} - y_{n_k}\right\| < \varepsilon_{n_k}\right)$. Setting $C = \bigcup_k A_{n_k}$, so that $\mu(C) = c\mu(A)$, we define $\overline{w} = \sum_k v_{n_k}$ and $y = \sum_k y_{n_k}$: it is immediate to see that $y = (1-c)\chi_C - c\chi_D$ (where $D = A \setminus C$) and that $\|\overline{w} - y\| < \varepsilon$. If V is closed, $\overline{w} \in \mathrm{V}$ (the same is true if the sum is finite); in any case, we can approximate $\overline{w}$ with a finite sum $w = \sum_{k=1}^{N} v_{n_k} \in \mathrm{V}$ so that $\|w - y\| < \varepsilon$. □

We remark that, if the property defining a rich subspace V holds for $\varepsilon = 0$ (as in the finite codimensional case), the same is true for the extended property when V is closed.

LEMMA 2. *Assume that $x, y \in \mathbb{R}$, $\lambda \in [0,1]$ and set $a = a(x, y, \lambda) = \lambda x + (1-\lambda)y$. Then the extremum problem*

$$\max\{\lambda\,|x-a|^p + (1-\lambda)\,|y-a|^p : \lambda\,|x|^p + (1-\lambda)\,|y|^p = 1\}$$

has the value $\wedge_p^p$. Moreover the max can be attained with $x > 0$ and $y < 0$.

PROOF: For $p = 1$ the proof is straightforward. For $p > 1$ just observe that any optimal triple x, y, λ must satisfy the orthogonality condition

$$\lambda\,|x|^{p-1}\operatorname{sgn} x + (1-\lambda)\,|y|^{p-1}\operatorname{sgn} y = 0.$$

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$\square$

We note that the number $\wedge_p$ was shown in [2] to be the value of the minimal projection onto a hyperplane of $L_p[0,1]$ (the value being independent of the chosen hyperplane); see Section 3 for further discussion.

With the help of the two above Lemmas we have an easy

PROOF OF THEOREM 1: Assume that $P\colon L_p(\mu) \to \vee$ is a projection and $\varepsilon > 0$; we want to approximate with a simple function a chosen element $u \in L_p(\mu)$ such that $\|u\| = 1$ and $Pu = 0$. In fact, we can find disjoint sets $A_i \in \sigma$, $i = 1, 2, \dots, m$, and numbers c_i such that $\|x - u\| < k\varepsilon$ for $x = \sum_{i=1}^{m} c_i \chi_{A_i}$ (here k is a constant). Let α, $-\beta$, λ with $\alpha, \beta > 0$ and $\lambda \in [0,1]$ be optimal elements for the extremum problem of Lemma 2; we apply to the set A_i Lemma 1 with $c = \lambda$: there exist in σ a partition $\{A_{i_1}, A_{i_2}\}$ of A_i with $\mu(A_{i_1}) = \lambda\mu(A_i)$ and $v_i \in \vee$ such that $\|v_i - y_i\|\, Mk\varepsilon$ for $y_i = (1-\lambda)\chi_{A_{i_1}} - \lambda\chi_{A_{i_2}}$. We define $z_i = (\alpha+\beta)y_i$ and $w_i = (\alpha+\beta)v$; then $z_i = (\alpha - z)\chi_{A_{i_1}} - (\beta + a)\chi_{A_{i_2}}$ (recall that $a = \lambda\alpha - (1-\lambda)\beta$). We now write $x = \sum_{i=1}^{m} d_i a \chi_{A_i} (d_i = c_i/a)$ and set $z = \sum_{i=1}^{m} d_i z_i$, $w = \sum_{i=1}^{m} d_i w_i$; note that $w \in \vee$. We have:

$$\|z + x\|^p = \left\| \sum_{i=1}^{m} d_i \left(\alpha\chi_{A_{i_1}} - \beta\chi_{A_{i_2}} \right) \right\|^p i$$

$$= \sum_{i=1}^{m} |d_i|^p \mu(A_i)(\lambda\alpha^p + (1-\lambda)\beta^p) = \sum_{i=1}^{m} |d_i|^p \mu(A_i)$$

$$\|z\|^p = \sum_{i=1}^{m} |d_i|^p (\lambda(\alpha - a)^p + (1-\lambda)(\beta + a)^p)\mu(A_i) = \wedge_p^p \sum_{i=1}^{m} |d_i|^p \mu(A_i).$$

We also have

$$\|P\| \geqslant \frac{\|P(w+u)\|}{\|w+u\|} = \frac{\|w\|}{\|w+u\|}$$

since w is approximated by z and u by x; choosing k small, we have

$$\|P\| - \varepsilon \geqslant \frac{\|z\|}{\|z+x\|} \geqslant \wedge_p.$$

$\square$

We shall show in the next section that $\wedge_p$ is the best possible lower bound.

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THEOREM 3. *Let $p \geqslant 1$ and I be any subset of $\mathbb{N}$. There exists a subspace $\vee$ with $\operatorname{codim} \vee = \operatorname{card} I$ and a projection $P\colon L_p[0,1] \to \vee$ such that $\|P\| \wedge_p$ (as a consequence of Theorem 1, P is a minimal projection onto $\vee$, if $p > 1$ P is unique).*

PROOF: We apply a remark used by Rolewicz in [6]. Let $\{A_i\}_{i\in I}$ be a partition of $[0, 1]$ in nondegenerated subintervals, let $\varphi_i \in L_q[0, 1]$ be such that the support of φ_i is contained in A_i; define $\vee = \{x \in L_p[0, 1]\colon \int_0^1 \varphi_i(x) = 0, i \in I\}$. If I is infinite, $\vee$ is an example of a rich subspace of infinite codimension ($\vee$ is rich since $\mu(A_i) < \varepsilon$ for all but a finite number of indices). Define

$$X_i = \{x \in L_p[0, 1]\colon \operatorname{supp} x \subset A_i\};$$

then $L_p[0, 1]$ is the direct sum of the X_i. If $x = \sum\limits_{i\in I} x_i$, then $\|x\|^p = \sum\limits_{i\in I} \|x_i\|^p$; moreover, X_i is isometric to $L_p[A_i]$. Let P_i be a minimal projection from X_i onto its hyperplane $\vee \cap X_i$. If $x = \sum\limits_{i\in I} x_i$, define $Px = \sum\limits_{i\in I} P_i x_i$; P os a projection from $L_p(\mu)$ onto $\vee$ and $\|Px\|^p = \sum\limits_{i\in I} \|P_i x_i\|^p \leqslant \sum\limits_{i\in I} \|P_i\|^p \|x_i\|^p$. In [5] it was proved that all the $\|P_i\|$ are equal and their common value was shown in [2] to be $\wedge_p$. We thus have $\|Px\| \leqslant \wedge_p \big(\sum_{i\in I} \|x_i\|^p\big)^{1/p} = \wedge_p \|x\|$. The proof is complete since by Theorem 1 we have $\|P\| \geqslant \wedge_p$. ∎

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