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AN INEQUALITY FOR ELEMENTARY SYMMETRIC FUNCTIONS

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Let E_r denote the rth elementary symmetric function on $\alpha_1, \alpha_2, \ldots, \alpha_m$ which is defined by

(1) $$E_r = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le m} \prod_{j=1}^{r} \alpha_{i_j}$$

$E_0 = 1$ and $E_r = 0$ $(r > m)$.

We define the rth symmetric mean by

(2) $$p_r = \binom{m}{r}^{-1} E_r$$

where $\binom{m}{r}$ denote the binomial coefficient. If $\alpha_1, \alpha_2, \ldots, a_m$ are positive reals then we have two well-known inequalities

(3) $$E_r^{1/r} \ge E_{r+1}^{1/(r+1)}$$

and

(4) $$p_r^{1/r} \ge p_{r+1}^{1/(r+1)}.$$

In this paper we consider a generalization of these inequalities. The inequality (4) is known as Newton's inequality which contains the arithmetic and geometric mean inequality.

We define

$$p_r(q) = \frac{E_r}{\begin{bmatrix} m \\ r \end{bmatrix}}$$

where $\begin{bmatrix} m \\ r \end{bmatrix}$ denote the q-binomial coefficient defined by

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1})\ldots(1-q^{m-k+1})}{(1-q)(1-q^2)\ldots(1-q^k)}$$

$$\begin{bmatrix} m \\ o \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} m \\ k \end{bmatrix} = 0 \quad (k < 0).$$

We note that when q tends to 1,

$$p_r(1) = p_r$$

and when $q=0$

$$p_r(0) = E_r.$$

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In [1] it is proved that if $\alpha_1, \alpha_2, \ldots, \alpha_m$ are real numbers then

$$p_r^2 \geq p_{r-1}p_{r+1}. \tag{5}$$

THEOREM. *If* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *are positive reals then*

$$\{p_r(q)\}^{1/r} \geq \{p_{r+1}(q)\}^{1/(r+1)}$$

where $0 \leq q \leq 1$.

Proof. We first prove that

$$\frac{\begin{bmatrix} m \\ r-1 \end{bmatrix}\begin{bmatrix} m \\ r+1 \end{bmatrix}}{\begin{bmatrix} m \\ r \end{bmatrix}^2} \frac{\binom{m}{r}^2}{\binom{m}{r-1}\binom{m}{r+1}} \geq 1 \quad (1 \leq r \leq m). \tag{6}$$

Indeed (6) is equivalent to

$$\frac{(m-r+1)(r+1)}{(m-r)r} \frac{(1-q^{m-r})(1-q^r)}{(1-q^{m-r+1})(1-q^{r+1})} \geq 1 \quad (1 \leq r \leq m). \tag{7}$$

Since $(1-q^\alpha)/(1-q) \geq \alpha$ if $0 \leq \alpha \leq 1$, we have

$$\frac{1-q^{m-r}}{1-q^{m-r+1}} = \frac{1-(q^{m-r+1})^{(m-r)/(m-r+1)}}{1-(q^{m-r+1})} \geq \frac{m-r}{m-r+1}$$

and similarly

$$\frac{1-q^r}{1-q^{r+1}} \geq \frac{r}{r+1}.$$

Hence (7) is proved.

Now from (5) and (6) we have

$$\frac{\begin{bmatrix} m \\ r-1 \end{bmatrix}\begin{bmatrix} m \\ r+1 \end{bmatrix}}{\begin{bmatrix} m \\ r \end{bmatrix}^2} \frac{\binom{m}{r}^2}{\binom{m}{r-1}\binom{m}{r+1}} \frac{p_2^r}{p_{r-1}p_{r+1}} \geq 1$$

or

$$\frac{\binom{m}{r}p_r^2}{\begin{bmatrix} m \\ r \end{bmatrix}^2} \geq \frac{\binom{m}{r-1}\binom{m}{r+1}p_{r-1}p_{r+1}}{\begin{bmatrix} m \\ r-1 \end{bmatrix}\begin{bmatrix} m \\ r+1 \end{bmatrix}}.$$

Hence

$$\{p_r(q)\}^2 \geq p_{r-1}(q)p_{r+1}(q). \tag{8}$$

It must be noted that (8) is true for all real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$. Now (8) leads to the theorem by the same argument that [1, p. 11] uses for the case $q=1$.

This proof shows that although the statement of the theorem implies that it contains the Newton or McLaurin) inequality, it really is less sharp than the

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Newton's inequality. In fact to see this we only need to look at the Arithmetic mean–Geometric mean case. We have as a consequence of the theorem

$$\frac{\alpha_1+\alpha_2+\cdots+\alpha_m}{1+q+q^2+\cdots+q^{m-1}} \geq \sqrt[m]{\alpha_1\alpha_2\ldots\alpha_m}$$

The left-hand side is greater than or equal to $(\alpha_1+\alpha_2+\cdots+\alpha_m)/m$ and hence the theorem has not given us anything new.

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