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REPRESENTATIONS OF MINIMALLY ALMOST PERIODIC GROUPS

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Abstract

For any group G, we introduce the subset $S(G)$ of elements g which are conjugate to $g^{2^k}, g^{3^k}, g^{4^k}, \ldots$ for some positive integer k. We show that, for any bounded representation π of G and any g in $S(G)$, either $\pi(g) = 1$ or the spectrum of $\pi(g)$ is the full unit circle in $\mathbb{C}$. As a corollary, $S(G)$ is in the kernel of any homomorphism from G to the unitary group of a post-liminal C^*-algebra with finite composition series.

Next, for a topological group G, we consider the subset of elements approximately conjugate to 1, and we prove that it is contained in the kernel of any uniformly continuous bounded representation of G, and of any strongly continuous unitary representation in a finite von Neumann algebra.

We apply these results to prove triviality for a number of representations of isotropic simple algebraic groups defined over various fields.

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0. Introduction

Let G be a topological group; the intersection $n(G)$ of the kernels of the finite-dimensional continuous unitary representations of G is the *von Neumann kernel* of G; this closed normal subgroup of G can be completely characterized when G is locally compact and connected (see [13], [14]). G is said to be *minimally almost periodic* (m.a.p.) if $G = n(G)$, i.e. if G has no non-trivial finite-dimensional continuous unitary representations. In their paper [11], von Neumann and Wigner obtained the following useful sufficient condition for an element g in G to belong to $n(G)$: assume that there exists a function f from the set $\mathbb{N}_0$ of

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positive integers to itself, such that, for any n in $\mathbb{N}_0$, n divides $f(n)$ and g is conjugate to $g^{f(n)}$ inside G; then g belongs to $n(G)$, when G is endowed with the discrete topology.

In Section 1 of this paper, we consider the subset $S(G)$ of elements in G for which the function f above can be taken of the form $f(n) = n^k$, for some k in $\mathbb{N}_0$. The interest of this case comes from the example of $\mathrm{SL}_2(k)$, where k is a field of characteristic 0; for any a in k and any λ in $k^\times$, the multiplicative group of k, we have:

$$(*) \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^2 a \\ 0 & 1 \end{pmatrix}.$$

In particular, taking for λ a positive integer n, we see that $g = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ is conjugate to g^{n^2}, for any n. We will extend the von Neumann and Wigner result by showing that, for any g in $S(G)$ and any bounded representation π of G on a (complex) Banach space, the following alternative holds: either $\pi(g) = 1$ or the spectrum of $\pi(g)$ is the unit circle S^1 in $\mathbb{C}$. As a corollary, we will see that isotropic simple algebraic groups over k (of characteristic 0) are m.a.p. in a very strong sense: roughly speaking, we will show that any homomorphism from such a group to the unitary group of a post-liminal C^*-algebra with finite composition series, is trivial (see Corollary 2 and Theorem 2 for the precise statement).

In Section 2, we consider, for a topological group G, two classes of continuous representations which are "close to" finite-dimensional ones, namely uniformly continuous bounded representations on a Banach space, and strongly continuous unitary representations in a finite von Neumann algebra (we refer to [12] for terminology). To explain what we have in mind, let us mention the following result, proved by Singer [16] for unitary representations, and extended by Gurarie [3] to more general representations: a connected Lie group G admits a faithful uniformly continuous bounded representation if and only if G is the direct product of a (finite-dimensional) real vector space and a compact Lie group. This in turn is equivalent to the existence of a faithful strongly continuous unitary representation of G in a finite von Neumann algebra (a result due to Kadison and Singer [8, p. 64]). Moreover, it was shown by Kallman [6] that the disintegration of any uniformly continuous unitary representation of a locally compact connected group involves a measure which is compactly supported in the set of finite-dimensional irreducible unitary representations. Finally, there are results saying that, roughly speaking, irreducible uniformly continuous representations are finite-dimensional (see [3], [4], [16]). We shall give a unified proof of the following result: if k is a non-discrete locally compact field (of any characteristic), and if G_k is the group of k-rational points of some isotropic simple algebraic group defined over k, endowed with its natural locally compact topology, then any strongly continuous unitary representation of G_k in a finite von Neumann algebra and any uniformly continuous bounded representation of G_k on a Banach space factorize through a compact abelian group. (Moreover, the second assertion is even true without the boundedness assumption if $k \neq \mathbb{R}, \mathbb{C}$.) The results are related to the existence in G_k of a "large" (we quote Tits [17, p.314]) normal subgroup G_k^0 which is m.a.p. Our proof was motivated by the proof of von Neumann and Segal [10] of the fact that strongly continuous unitary representations of a simple non-compact Lie group in a finite von Neumann algebra are trivial; our idea is to consider elements g of a topological group G which are *approximately conjugate to* 1, i.e. such that the closure of the conjugacy class of g contains the identity 1 of G. For example, in $\mathrm{SL}_2(k)$, the element $g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ has this property, as we see from formula (∗) by letting λ tend to 0 in k. The proof will proceed by introducing a class $\mathscr{C}$ of topological groups such that, if H belongs to $\mathscr{C}$, and if G is any topological group, the set of elements in G which are approximately conjugate to 1 is in the kernel of any continuous homomorphism $G \to H$.

1. The set $S(G)$

We recall that, for any group G, we have defined $S(G)$ to be the set of g's in G such that, for some k in $\mathbb{N}_0$, g is conjugate to g^{n^k} for any n in $\mathbb{N}_0$. For any element x in a Banach algebra A, we denote by $\operatorname{sp} x$ (respectively $r(x)$) the spectrum (respectively spectral radius) of x.

PROPOSITION 1. *Let A be a unital Banach algebra, and let x be an element in $S(A^{-1})$. Then either $\operatorname{sp} x = \{1\}$ or $\operatorname{sp} x = S^1$.*

PROOF. First, we claim that $\operatorname{sp} x$ is contained in S^1. Indeed, since x belongs to $S(A^{-1})$, there exists a k in $\mathbb{N}_0$ such that

$$\operatorname{sp} x = \operatorname{sp}(x^{n^k}) = (\operatorname{sp} x)^{n^k} \quad \text{for any } n \text{ in } \mathbb{N}_0.$$

So, if some z in $\operatorname{sp} x$ is such that $|z| \neq 1$, then $\operatorname{sp} x$ contains the sequence $(z^{n^k})_{n \in \mathbb{N}_0}$, which contradicts the compactness of $\operatorname{sp} x$ in $\mathbb{C}^\times$. Now, we identify S^1 with $\mathbb{R}/\mathbb{Z}$ by $s \to \exp(2\pi i s)$. The result then follows from the next lemma.

LEMMA 1. *Let X be a non-empty closed subset of $\mathbb{R}/\mathbb{Z}$, such that $n^k X = X$ for any n in $\mathbb{N}_0$. Then either $X = \{0\}$ or $X = \mathbb{R}/\mathbb{Z}$.*

PROOF. We use several steps.

Step 1. If X is finite, then $X = \{0\}$. Indeed, let σ be the permutation of X defined by $\sigma(x) = 2^k x$ ($x \in X$). Let m be the cardinal of X; iterating $m!$ times the preceding relation (although the order of σ in the symmetric group Sym m would suffice), we get

$$x = (2^k)^{m!}x \quad \text{i.e.} \quad 0 = (2^{k.m!} - 1)x.$$

So, for some positive integer l, X is contained in the set Ω_l of elements of order l in $\mathbb{R}/\mathbb{Z}$. Consequently $X = l^k X$ is contained in $l^k\Omega_l = \{0\}$.

Step 2. If X admits 0 as a limit-point, then $X = \mathbb{R}/\mathbb{Z}$. Indeed, let $(x_n)_{n \in \mathbb{N}_0}$ be an injective sequence tending to 0 in $\mathbb{R}/\mathbb{Z}$. Identify $\mathbb{R}/\mathbb{Z}$ with the interval [0, 1[; then (x_n) is a sequence in [0, 1[having at most two limit points, 0 and 1. Assume that 0 is such a limit-point; then there is a subsequence $(y_n)_{n \in \mathbb{N}_0}$ tending to 0. Denoting by $[x]$ the integer part of the real number x, we consider the following sequence:

$$(1) \qquad y_1, 2^k y_1, 3^k y_1, \ldots, \left[y_1^{-1/k}\right]^k y_1,\ y_2, 2^k y_2, \ldots, \left[y_2^{-1/k}\right]^k y_2,\ y_3, \ldots.$$

This sequence is clearly contained in X. Now, consider the sequence:

$$(2) \qquad y_1^{1/k}, 2y_1^{1/k}, 3y_1^{1/k}, \ldots, \left[y_1^{-1/k}\right] y_1^{1/k},\ y_2^{1/k}, 2y_2^{1/k}, \ldots, \left[y_2^{-1/k}\right] y_2^{1/k},$$
$$y_3^{1/k}, 2y_3^{1/k}, \ldots, \left[y_3^{-1/k}\right] y_3^{1/k},\ y_4^{1/k}, \ldots,$$

which is dense in [0, 1[since $(y_n^{1/k})$ tends to 0. But (2) is obtained by taking the kth root of each term in (1), so (1) itself is dense. All this shows that X is dense in $\mathbb{R}/\mathbb{Z}$, and this concludes our proof in the case where 0 is a limit-point of (x_n). The case where 1 is a limit-point can be reduced to the previous one by identifying $\mathbb{R}/\mathbb{Z}$ with $[-1, 0[$ instead of $[0, 1[$. This concludes the second step.

Step 3. We now prove the lemma itself. Assume that X contains some irrational number ϑ. Then X contains the sequence $(n^k\vartheta)_{n \in \mathbb{N}_0}$ which, by van der Corput's theorem (see [7, Theorem 3.2]), is uniformly distributed, hence dense, in $\mathbb{R}/\mathbb{Z}$. So $X = \mathbb{R}/Z$ in this case. It remains to show that if X is contained in $\mathbb{Q}/\mathbb{Z}$, then necessarily $X = \{0\}$. Assume the contrary; then, by step 1 of the proof, X is infinite, so by compactness X contains at least one limit-point x. Since X is rational, we find a positive integer l such that $0 = lx$ in $\mathbb{R}/\mathbb{Z}$. So $0 = l^k x$ is a limit-point of X as well, and by step 2 of the proof, we have $X = \mathbb{R}/\mathbb{Z}$, a contradiction. This proves Lemma 1 together with Proposition 1.

REMARK 1. Step 1 of the above proof is essentially Lemma 1 of [11]. This step suffices to show that, for any group G, the set $S(G)$ is contained in $n(G)$. More generally, for any field K and any group H of diagonalizable matrices in $\mathrm{GL}_n(K)$, the set $S(G)$ is contained in the kernel of any homomorphism $G \to H$.

We denote by $\mathrm{GL}(E)$ the group of bounded invertible operators on the Banach space E, and by $\mathrm{GL}_0(E)$ the subgroup of those operators that are scalar modulo compacts.

LEMMA 2. *Let E be a Banach space; if $T \in \mathrm{GL}(E)$ is such that $\operatorname{sp} T = \{1\}$ and $\sup \|T^n\| < \infty$, then $T = 1$.*

PROOF. There are two cases where this lemma is well known: if either E is a Hilbert space (then the representation of $\mathbb{Z}$ defined by T can be unitarized, by amenability of $\mathbb{Z}$), or T is an isometry on E (then the lemma is just [12, 8.1.11]). For the general case, we find by holomorphic functional calculus a quasi-nilpotent operator H such that $e^H = T$. Then a simple interpolation argument shows that the uniformly continuous representation $t \to e^{tH}$ of $\mathbb{R}$ is bounded; by Lemma 1 of [3], we have $H = 0$, i.e. $T = 1$.

From Proposition 1 and Lemma 2, we deduce

THEOREM 1. *Let G be a group, and let π be a bounded representation on some Banach space. Then, for any g in $S(G) \setminus \operatorname{Ker} \pi$, we have $\operatorname{sp} \pi(g) = S^1$ and $\|\pi(g) - 1\| \geqslant 2$, with equality if π is unitary.*

PROOF. To prove the inequality, simply notice that

$$2 = r(\pi(g) - 1) \leqslant \|\pi(g) - 1\|$$

because $\operatorname{sp} \pi(g) = S^1$. The other statements are clear.

COROLLARY 1. *If π is a bounded representation of G whose image is contained in $\mathrm{GL}_0(E)$, then $S(G)$ is contained in $\operatorname{Ker} \pi$.*

This follows from Theorem 1 and from the fact that $\operatorname{sp} \pi(g)$ is countable.

COROLLARY 2. *Let A be a unital post-liminal C^*-algebra admitting a finite composition series $0 = I_0 \triangleleft I_1 \triangleleft \cdots \triangleleft I_n = A$ such that I_{i+1}/I_i is liminal for $i = 0, 1, \ldots, n-1$. Let $\mathscr{U}(A)$ be the unitary group of A. Then, for any group G, the set $S(G)$ is contained in the kernel of any homomorphism $G \to \mathscr{U}(A)$.*

PROOF. (For general background on post-liminal C^*-algebras and composition series, see [12].) We have to show that $S(\mathscr{U}(A)) = \{1\}$. Assume, by contradiction, that there exists some $u \neq 1$ in $S(\mathscr{U}(A))$. Denoting by $\tilde{I}_i$ the unital C^*-sub-algebra generated by I_i, we see that, for some $i > 0$, the unitary u belongs to $\tilde{I}_i$ but not to $\tilde{I}_{i-1}$; let $\dot{u}$ be the image of u in $\tilde{I}_i/I_{i-1}$ ($\dot{u} \neq 1$), and let π be an irreducible representation of $\tilde{I}_i/I_{i-1}$ on some Hilbert space $\mathscr{H}_\pi$, such that $\pi(\dot{u}) \neq 1$. Since I_i/I_{i-1} is liminal, $\pi(\dot{u})$ belongs to $\mathrm{GL}_0(\mathscr{H}_\pi)$, but this contradicts Corollary 1.

We conclude this section by giving some examples to which the preceding results apply.

PROPOSITION 2. *Let k be a field of characteristic 0, and let H be a subgroup of $\mathrm{GL}_m(k)$ admitting a cyclic vector x_0 in k^m, and such that, for any $n \in \mathbb{N}_0$, there is an $h_n \in H$ satisfying $h_n(x_0) = nx_0$. Let G be the semi-direct product $k^m \rtimes H$, and let π be a representation of G satisfying the assumptions of either Corollary 1 or Corollary 2. Then π factorizes through H.*

PROOF. It is enough to show that the subgroup generated by $S(G)$ contains k^m. Clearly, for any $\lambda \in k^\times$ and any $h \in H$, the element $h(\lambda x_0)$ is in $S(G)$; by cyclicity of x_0, we may select a basis $x_1, \ldots, x_m$ inside the orbit Hx_0, and the preceding argument shows that any linear combination of the x_i's, i.e. any element of k^m, belongs to the subgroup generated by $S(G)$.

An important application of this result is given by the "$ax + b$" group of k, i.e. the semi-direct product $k \rtimes k^\times$. In particular, any finite-dimensional unitary representation of this group factorizes through $k^\times$ (for a locally compact non-discrete k and a continuous representation, this follows immediately from Mackey's theory [8]).

THEOREM 2. *Let G be an isotropic simple algebraic group defined over some field k of characteristic 0. Let G_k be its group of k-rational points, and G_k^0 be the subgroup generated by all unipotent k-subgroups which are split over k. Let π be a representation of G_k satisfying the assumptions of either Corollary 1 or Corollary 2. Then π factorizes through G_k/G_k^0.*

The group G is *isotropic* if it admits a split torus of positive dimension (over k). In characteristic 0, G_k^0 may be defined more simply as the subgroup generated by all unipotent elements of G_k (see [1, 6.2]). The structure of G_k/G_k^0 is discussed in [17, 1.4]. A conjecture of Kneser and Tits asserts that this group is always abelian, and is trivial if G is simply connected over k (this is known if G admits a Borel subgroup defined over k [1, 6.6], [17, 1.4]). If G is a classical group, the structure of G_k/G_k^0 is given explicitly in [2]. In any case, G_k^0 is Zariski-dense in G ([17, 3.2(20)]).

Following Howe and Moore [5], we say that a *one-parameter subgroup* in G_k is a non-trivial algebraic homomorphism β: $k_{\mathrm{add}} \to G_k$, and that a one-parameter subgroup β is of *Jacobson-Morosov type* if there exists an homomorphism $\mathrm{SL}_2(k) \to G_k$ which coincides with β when restricted to the subgroup k_{add} of upper strictly triangular matrices (possibly after a reparametrization of β). It is clear from formula (*) in the introduction that, in characteristic 0, the union of all one-parameter subgroups of Jacobson-Morosov type is contained in $S(G_k)$.

PROOF OF THEOREM 2. It suffices to show that the (normal) subgroup of G_k generated by $S(G_k)$ contains G_k^0. But, according to a theorem of Tits [17, 1.1], the group G_k^0 is simple modulo its centre. So it is enough to prove that G_k^0 contains one-parameter subgroups of Jacobson-Morosov type. This follows from root theory: to any restricted root α of G (such an α does exist, for G is isotropic), one associates a non-trivial homomorphism β: $\mathrm{SL}_2(k) \to G_k^0$ (see [17, 3.1(13) and 3.3]). This concludes the proof.

REMARK 2. As a consequence, we see that any finite-dimensional unitary representation of G_k factorizes through G_k/G_k^0. Several variants of this result can be found in the literature, especially in Borel and Tits' paper [1] (for example, this is a consequence of [1, 10.3] in the case where k is not a subfield of $\mathbb{C}$; also, a very particular case of [1, Theorem A] shows that there is no homomorphism $G_k^0 \to \mathrm{SU}(n)$ with Zariski-dense image).

REMARK 3. Theorem 2 implies the fact that connected simple non-compact Lie groups are m.a.p. in the discrete topology. Taking Remark 1 into account, we see that a finite-dimensional representation of such a group by *normal* operators is trivial as well. This is a particular case of a result of Sherman [15].

REMARK 4. As a consequence of Theorem 1, we see that, for any g belonging to a one-parameter subgroup of Jacobson-Morosov type in G_k, and any unitary representation π of G_k such that $\pi(g) \neq 1$, the spectrum of $\pi(g)$ is S^1. For $k = \mathbb{R}$ and strongly continuous representations π not containing the trivial representation, there is a much more precise result due to Moore [9, Theorem 2], which completely classifies the unitary type of $\pi(g)$.

2. Elements approximately conjugate to 1

We say that a topological group G *belongs to the class* $\mathscr{C}$ if there exists a function φ on G which is central (i.e. constant on conjugacy classes), continuous at 1, and such that $\varphi(g) = 0$ if and only if $g = 1$.

The following proposition is an immediate consequence of the definition.

PROPOSITION 3. *Let* G, H *be topological groups, and let* β: $G \to H$ *be a continuous homomorphism. If* H *belongs to class* $\mathscr{C}$, *then the set of elements approximately conjugate to* 1 *in* G *is contained in* $\operatorname{Ker}\beta$.

This proposition is exemplified by the following result.

PROPOSITION 4. *The following groups belong to class $\mathscr{C}$.*

(i) *The unitary group of a finite von Neumann algebra M, endowed with the strong topology.*

(ii) *Any bounded subgroup of* $\mathrm{GL}(E)$, *endowed with the norm topology (where E is a Banach space).*

PROOF. (i) Let τ be any faithful, finite, normal trace on M; for g in $\mathscr{U}(M)$, define $\varphi(g) = \tau(1 - g)$; then φ is central, it is strongly continuous (by [12, 3.6.4]), and the third condition is proved as in [10] (using the fact that, if $\mathrm{Re}(g)$ is the real part of g, then $1 - \mathrm{Re}(g)$ is a positive element).

(ii) Define $\varphi(g) = r(g - 1)$; the conclusion follows from Lemma 2.

From this, we immediately deduce

COROLLARY 3. *For any topological group G, the set of elements approximately conjugate to* 1 *in G is contained in $n(G)$.*

Let us give examples where these results apply

PROPOSITION 5. *Let k be a non-discrete locally compact field of any characteristic, and let H be a closed subgroup of $\mathrm{GL}_m(k)$. Assume that there exist a vector x_0 in k^m, cyclic for H, and for any $\lambda \in k^\times$, an element $h_\lambda \in H$ such that $h_\lambda(x_0) = \lambda x_0$. Then any continuous homomorphism from the semi-direct product $G = k^m \rtimes H$ to a group of class $\mathscr{C}$ factorizes through H.*

This is proved exactly like Proposition 2. In particular, it applies to the "$ax + b$" group of k.

THEOREM 3. *Let k be a non-discrete locally compact field, and let G, G_k, G_k^0 be as in Theorem 2. Endow G_k with its natural locally compact topology. Any continuous homomorphism from G_k to a group of class $\mathscr{C}$ factorizes through G_k/G_k^0, a compact abelian group.*

Note that G is isotropic if and only if G_k is non-compact in its locally compact topology. Results of Borel-Tits [1, 6.14–15] assert that G_k^0 is closed in G_k, and that G_k/G_k^0 is compact abelian. Moreover, if $k = \mathbb{R}$, G_k^0 coincides with the topological connected component of the identity in G_k.

PROOF OF THEOREM 3. We begin with $G = SL_2$; but we saw in the introduction that $SL_2(k)$ is generated by elements approximately conjugate to 1. For a general G, as in Theorem 2 we associate to any restricted root a non-trivial continuous homomorphism $SL_2(k) \to G_k^0$. So G_k^0 contains non-central elements which are approximately conjugate to 1, and the simplicity of G_k^0 modulo its centre allows one to conclude.

REMARK 5. The preceding result shows that any uniformly continuous bounded representation of G_k factorizes through G_k/G_k^0 (and the proof shows that $G_k^0$is m.a.p.). In characteristic 0, it is possible to deduce these results from Theorem 1; indeed, let β: $k_{\text{add}} \to G_k^0$ be a continuous one-parameter subgroup of Jacobson-Morosov type. If π is a non-trivial bounded representation of G_k^0, then by Theorem 1, for any $s \in k^\times$: $2 \leqslant \|1 - \pi(\beta(s))\|$. So, letting s tend to 0 in k, we see that π cannot be uniformly continuous. Note that for $k = \mathbb{R}, \mathbb{C}$, one might also reduce the whole problem to a finite-dimensional situation by using the fact that any uniformly continuous representation of G_k^0 is the direct sum of finite-dimensional irreducible representations (see [4, Proposition 4]; this is proved using Weyl's unitary trick).

COROLLARY 4. *If $k \neq \mathbb{R}, \mathbb{C}$, and if G, G_k, G_k^0 are as above, then any uniformly continuous representation of G_k factorizes through G_k/G_k^0 (without any boundedness assumption).*

PROOF. As in Theorem 3, it is enough to give the proof for $G = SL_2$. But since k is distinct from $\mathbb{R}$ and $\mathbb{C}$, k contains a valuation ring ω, and $SL_2(\omega)$ is a maximal compact subgroup of $SL_2(k)$; so the restriction of any uniformly continuous representation π to $SL_2(\omega)$ is bounded, and since any element $g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \in \omega$, is approximately conjugate to 1 in $SL_2(k)$, we see by Proposition 3 and 4 that g belongs to $\operatorname{Ker}\pi$ (see [4, Proposition 5] for a different proof).

Concerning finite-dimensional representations, the preceding corollary has some overlap with Théorème (A) in Borel-Tits [1]. Indeed, it follows from this result that G_k^0 has no embedding in $GL_n(\mathbb{C})$ if $\operatorname{char} k \neq 0$, even if there is no continuity condition. On the other hand, in characteristic 0, our corollary is in a certain sense "best possible", since there are plenty of embeddings of p-adic fields into $\mathbb{C}$ giving rise to discontinuous representations of G_k^0.

To conclude, we mention that, for finite-dimensional unitary representations of G_k ($\operatorname{char} k = 0$), our Theorem 2 seems to be stronger than Theorem 3, since there are no continuity assumptions involved in Theorem 2. However, another result of Borel-Tits [1, 9.1] shows that any homomorphism from G_k to a compact group is necessarily continuous; this extends an old result of van der Waerden [18] in the case $k = \mathbb{R}$.

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