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Q-DIVISIBLE MODULES

BY

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1. **Introduction.** Let R be a ring with 1 and let Q denote the maximal left quotient ring of R [6]. In a recent paper [12], Wei called a (left) R-module M *divisible* in case $\mathrm{Hom}_R(Q, N) \neq 0$ for each nonzero factor module N of M. Modifying the terminology slightly we call such an R-module a *Q-divisible* R-module. As shown in [12], the class D of all Q-divisible modules is closed under factor modules, extensions, and direct sums and thus is a torsion class in the sense of Dickson [5]. It follows that every R-module M contains a (unique) maximal Q-divisible submodule $D(M)$ such that $M/D(M)$ contains no nonzero Q-divisible submodule. Moreover, the class D contains all injective R-modules and hence contains the torsion class D_0 generated by the injective R-modules. In general D and D_0 are distinct, but in some instances coincidence of these classes occurs. In this note we examine some of these situations as well as some relationship between the class D and the class of R-modules with zero singular submodule. (As in [9], we call modules with zero singular submodule *nonsingular* and if the (left) singular ideal of R is zero then R is a *nonsingular ring*.) In §2 we characterize rings for which every Q-divisible module is injective, nonsingular rings for which every nonsingular Q-divisible module is injective, and finite-dimensional nonsingular rings for which every Q-divisible R-module is a factor of an injective R-module. In §3, some examples are given related to the classes D and D_0.

2. **Main results.** We first consider the case when all Q-divisible R-modules are injective.

PROPOSITION 2.1. *For a ring R the following conditions are equivalent*:

(a) *Every Q-divisible R-module is injective.*

(b) *The injective R-modules form a torsion class.*

(c) *R is left hereditary and left Noetherian.*

Proof. We show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). It is clear that (a) $\Rightarrow$ (b) since every injective R-module is Q-divisible. Assuming (b), then by [5] direct sums of injectives are injective so by a theorem of Bass [4], R is left Noetherian; also factors of injectives are injective so R is left hereditary [3]. Thus (b) $\Rightarrow$ (c). Now assume (c) holds and let M be Q-divisible. Since R is left hereditary, its (left) singular ideal is zero. But for any nonsingular ring the maximal left quotient ring is an injective R-module [6], thus Q is injective. Let $B = \sum \mathrm{Im}\, \beta$ where β varies over $\mathrm{Hom}_R(Q, M)$; then B is a factor of a direct sum of copies of Q and so B is injective since R is left

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Noetherian and left hereditary. It follows that $M=B$ since M is Q-divisible, completing the proof.

As noted in the previous proof, Q is an injective R-module whenever R is a nonsingular ring. We will make repeated use of this fact as well as of the following well-known property:

(*) If A is an injective R-module, B is a nonsingular R-module and $\alpha \in \mathrm{Hom}_R(A, B)$ then $\mathrm{Im}\, \alpha$ is injective.

Indeed, $\mathrm{Ker}\, \alpha$ can have no essential extension in A since B is nonsingular and hence $\mathrm{Ker}\, \alpha$ is a direct summand of A.

The following characterizes nonsingular rings for which every nonsingular Q-divisible module is injective.

THEOREM 2.2. *Let R be a nonsingular ring. Then every nonsingular Q-divisible R-module is injective if and only if R is a finite-dimensional R-module.*

Proof. Suppose first that R is a finite-dimensional R-module. Then by [1, Theorem 1], every nonsingular R-module contains a unique maximal injective submodule. Thus if A is nonsingular and Q-divisible then $A=B\oplus C$ with B injective and C containing no nonzero injective submodules. If $C\neq 0$ then since A is Q-divisible, $\mathrm{Hom}_R(Q, C)\neq 0$ and so by (*) C contains a nonzero injective submodule, a contradiction. Thus $C=0$ and so $A=B$ is injective. For the converse note that Q is nonsingular hence any direct sum of copies of Q being nonsingular and Q-divisible is injective. If $\{U_i \mid i\in I\}$ is a family of left ideals of R whose sum is direct then $B=\oplus\sum_{i\in I} Q_i$, $Q_i=Q$ for all $i\in I$, is injective and there is a monomorphism $\alpha: \oplus\sum_{i\in I} U_i \to B$. Then α can be extended to $\beta: R\to B$. Since $\mathrm{Im}\, \beta$ is cyclic it lies in a finitely generated summand of B and hence so does $\mathrm{Im}\, \alpha$. This implies I is a finite set and so R is a finite dimensional R-module.

As an immediate consequence we have the

COROLLARY. *If R is any integral domain, then every torsion-free Q-divisible R-module is injective if and only if R is a (left) Ore domain.*

When R is nonsingular and finite-dimensional, Theorem 2.2 states that the nonsingular modules in D coincide with the nonsingular modules in D_0. This situation occurs also if every Q-divisible module is a factor of an injective (and so D coincides with D_0). We examine this condition next for nonsingular finite-dimensional rings, obtaining a result related to Theorem 1.2 of [7]. We remark that by (*) the condition in Theorem 2.2 that every nonsingular Q-divisible module is injective is equivalent to every nonsingular Q-divisible module is a factor of an injective.

Before proceeding we introduce the following notation. For any R-module M let $q(M)=\sum \mathrm{Im}\, \beta$, where β varies over $\mathrm{Hom}_R(Q, M)$. We now define a (transfinite) sequence of submodules $q_\lambda(M)$ of M by letting $q_1(M)=q(M)$ and, for any ordinal

$\lambda \geq 1$, letting: $q_\lambda(M) = \bigcup_{\alpha<\lambda} q_\alpha(M)$, if λ is a limit ordinal;

$$q_\lambda(M)/q_{\lambda-1}(M) = q(M/q_{\lambda-1}/(M)), \quad \text{if } \lambda-1 \text{ exists.}$$

The least ordinal τ for which $q_\tau(M) = q_{\tau+1}(M)$ will be called the *q-length of M*. It is readily verified that $q_\tau(M) = M$ if and only if M is Q-divisible.

THEOREM 2.3. *Let R be a finite-dimensional nonsingular ring. The following conditions are equivalent*:

(a) *Every Q-divisible R-module is a factor of an injective R-module.*

(b) *The singular submodule of every Q-divisible R-module is a direct summand.*

(c) $\text{hd}_R(Q) \leq 1$.

Proof. (a) ⇒ (b) is a consequence of [8, Theorem 2.10], while (b) ⇒ (c) can be obtained by a modification of the proof of [7, Theorem 1.2], replacing "torsion" by "singular" and "quotient field" by "maximal left quotient ring". For (c) ⇒ (a), assume that $\text{hd}_R(Q) = 0$; i.e. Q is a projective R-module. In this case the q-length of any Q-divisible R-module is 1 by [12, Corollary, Proposition 7*]. Since R is nonsingular and finite-dimensional, any direct sum of copies of Q is injective [11, Theorem 2.1], and so every Q-divisible R-module is a factor of an injective R-module. Now assume $\text{hd}_R(Q) = 1$, and let M be any Q-divisible R-module. We induct on the q-length of M, the result being true if the q-length of M is 1 exactly as in the case when Q is projective. So suppose the q-length of $M = \tau > 1$. If τ is a limit ordinal then $M = \bigcup_{\alpha<\tau} q_\alpha(M)$ and each $q_\alpha(M)$ is a factor of an injective R-module. Since R is nonsingular and finite-dimensional we may assume that there exist nonsingular injectives Q_α and epimorphisms $f_\alpha: Q_\alpha \to q_\alpha(M)$. Then there is an epimorphism $f: \bigoplus\sum_{\alpha<\tau} Q_\alpha \to M$ and $\bigoplus\sum_{\alpha<\tau} Q_\alpha$ is an injective R-module. If $\tau = \alpha+1$ there is an exact sequence $0 \to K \to M \to N \to 0$ with q-length of $M = \alpha$ and the q-length of $N = 1$. This gives the exact sequence

$$\text{Ext}^1_R(Q/R, K) \to \text{Ext}^1_R(Q/R, M) \to \text{Ext}^1_R(Q/R, N).$$

Now it can be verified that [7, Proposition 2.1] is valid in our situation hence the two end modules are zero and thus also $\text{Ext}^1_R(Q/R, M)$. It follows that M is a factor of an injective R-module.

3. **Some examples.** The class D_0 consists of all R-modules M for which every nonzero factor of M contains a nonzero factor of an injective R-module. Thus it follows that if Q is an injective R-module $D = D_0$. In particular, if R is self-injective, $D = D_0$ and in fact D consists of all R-modules.

EXAMPLE 3.1. Let R be a commutative semiprimary ring which is not self-injective. Then every proper ideal of R has nonzero annihilator and so $R = Q$. By [2, Theorem 6.3] every simple R-module is a factor of an injective R-module. Since nonzero modules contain nonzero simples, every R-module is in D_0. Thus $D_0 = D$ but R need not be self-injective and Q need not be injective.

EXAMPLE 3.2. The following is an example of ring R for which $D \neq D_0$. Let K be any field and let R consist of all 3×3 matrices over K of the form

$$\begin{pmatrix} a & b & d \\ 0 & a & 0 \\ 0 & c & e \end{pmatrix}.$$

As noted in [10], $R=Q$; moreover R is left Artinian and the right ideal A of all matrices of the form

$$\begin{pmatrix} a & b & d \\ 0 & a & 0 \\ 0 & c & 0 \end{pmatrix}$$

has zero left annihilator. By [2, Theorem 6.3], R has a simple left-R-module S which is not a factor of an injective R-module, hence $S \notin D_0$.

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