ON TRACE BILINEAR FORMS ON LIE-ALGEBRAS

by HANS ZASSENHAUS

(Received 17th October, 1958)

To what extent is the structure of a Lie-algebra L over a field F determined by the bilinear form

$$f(a, b) = (a, b)_\Delta \quad \ldots\ldots(1)$$

on L that is derived from a matrix representation

$$a \rightarrow \Delta(a) \quad (a \in L)$$

of L with finite degree $d(\Delta)$ by forming the trace of the matrix products

$$f(a, b) = \mathrm{tr}(\Delta a\, \Delta b) \quad (a, b \in L)? \quad \ldots\ldots(2)$$

Such a bilinear form is a function with two arguments in L, values in F and the properties :

$$\left.\begin{aligned} f(a_1+a_2, b) &= f(a_1, b)+f(a_2, b) \\ f(a, b_1+b_2) &= f(a, b_1)+f(a, b_2) \\ f(\lambda a, b) &= f(a, \lambda b) = \lambda f(a, b) \end{aligned}\right\} (\textit{bilinearity}) \quad \begin{aligned} &\ldots\ldots(3) \\ & \\ &\ldots\ldots(4) \end{aligned}$$

$$f(a, b) = f(b, a) \qquad (\textit{symmetry}) \quad \ldots\ldots(5)$$

$$f(ab, c) = f(a, bc) \qquad (\textit{invariance under } L) \quad \ldots(6)$$

$(\lambda \in F\,;\ a, a_1, b, b_1, c \in L)$.

It is clear from the definition that the trace bilinear form (1) depends only on the class of equivalent representations to which Δ belongs.

For any subset K of L, the set $K^\perp$ of all elements x of L satisfying $f(K, x) = 0$ † is a linear subspace of L, because of the bilinearity of f. This linear subspace is called the *orthogonal subspace of* K. It coincides with the orthogonal subspace of the linear subspace $\{FK\}$ generated by K. If $K_1 \subseteq K_2$ then $K_1^\perp \supseteq K_2^\perp$. By the symmetry of f we have $K \subseteq (K^\perp)^\perp$. If K is an ideal of L, then it follows from the invariance of f that the orthogonal subspace $K^\perp$ is also an ideal. The ideal $L^\perp = L^\perp(\Delta)$ is called the *radical of the representation* Δ. For any ideal A of L contained in $L^\perp$, a symmetric invariant bilinear form f^A is induced on the factor algebra L/A by setting

$$f^A(a/A, b/A) = f(a, b) \quad (a, b \in L). \quad \ldots\ldots(7)$$

We observe that the kernel of Δ, i.e. the ideal L_Δ of L formed by the elements x that are mapped onto 0 by Δ, lies in the radical of Δ. By the first isomorphism theorem, L/L_Δ is isomorphic to a Lie-subalgebra of the Lie-algebra formed by the matrices of degree $d(\Delta)$ over F. Hence L/L_Δ and *a fortiori* $L/L^\perp$ are finite-dimensional Lie-algebras.

It will be the aim of the investigation to determine the structure of the factor algebra $L/L^\perp$ in terms of simple algebras.

THEOREM 1. *If the characteristic of F is distinct from* 2 *and* 3, *then, for any solvable ideal A of L, the ideal LA is contained in the radical of any matrix representation Δ.*

† For any two subsets K_1, K_2 of L, denote by $f(K_1, K_2)$ the set of all values $f(x_1, x_2)$, where x_i denotes any element of K_i $(i = 1, 2)$. Hence $f(K, K^\perp) = f(K^\perp, K) = 0$.

Before we enter into the proof of Theorem 1, let us prove

LEMMA 1. *For any irreducible representation Δ of a Lie-algebra L over the field of reference F all of the irreducible components of the representation Δ^T obtained by restricting Δ to the sub-invariant subalgebra T are equivalent,*

and

LEMMA 2. *If the irreducible representation Δ of the Lie-algebra L over the field of reference F induces by restriction to the ideal A of L a nilrepresentation Δ^A of A, then Δ^A is a null representation of A.*

Proof of Lemma 1. By assumption there is a chain $L = L_0 \supseteq L_1 \supseteq \ldots \supseteq L_m = T$ of Lie-algebras over F from L to T such that L_i is an ideal of L_{i-1} $(i = 1, 2, \ldots, m)$. Let M be a representation space of Δ. Since it is of finite dimension over F, it must contain an irreducible L_1-F-subspace m. Also there is a maximal L_1-F-subspace M_1 of M such that $m \subseteq M_1$ and all irreducible components of the representation of L_1 with representation space M_1 are equivalent to the representation Γ of L_1 with representation space m. Let s be an element of L, x an element of L_1, u an element of M_1; then

$$x(su) = x(su) - s(xu) + s(xu) = (xs)u + s(xu). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Hence $x(su)$ is contained in $sM_1 + M_1$ and thus $sM_1 + M_1$ is an L_1-F-module such that the mapping of u onto su is an operator homomorphism of M_1 onto $(sM_1 + M_1)/M_1$. It follows that the irreducible components of the representation of L_1 with representation space $(sM_1 + M_1)/M_1$ are equivalent to Γ. By the Jordan-Hölder Theorem, the same applies to the irreducible components of the representation of L_1 with representation space $sM_1 + M_1$. Because of the maximality of M_1 we have $sM_1 + M_1 = M_1$, $sM_1 \subseteq M_1$, $LM_1 \subseteq M_1$. Since M is an irreducible L-F-space, it follows that $M_1 = M$ and thus every irreducible component of Δ^{L_1} is equivalent to Γ.

The proof of Lemma 1 can now be completed by induction on m and by an application of the Jordan-Hölder Theorem.

Proof of Lemma 2. Without restricting the generality we can assume that Δ is a faithful representation. Hence Δ^A is faithful. By [4, p. 34, Satz 11], the Lie-algebra A is nilpotent. By [4, p. 29], every irreducible component of Δ^A is a null representation. Let M be a representation space of Δ. It contains a minimal A-F-subspace $\neq 0$, say m. Hence $Am = 0$. Let M_1 be the linear subspace of M consisting of all elements u of M satisfying $Au = 0$. Applying (8) for s of L, x of A, u of M_1, we find that su belongs to M_1. Hence M_1 is a non-vanishing invariant subspace of the L-F-space M. Since M is irreducible, it follows that $M_1 = M$, $AM = 0$ and this proves Lemma 2.

Proof of Theorem 1. (1) Let F be algebraically losed, $L^\perp \neq L$, Δ be irreducible and faithful and $A(AA) = 0$. By Lemma 1, the irreducible representation Δ induces on A a representation Δ^A all of whose irreducible constituents are equivalent. Since A is nilpotent, it follows from [4, p. 29] that each irreducible representation of A maps each element of A onto a matrix with only one characteristic root (of maximal multiplicity). Hence, for any element a of A, the matrix $\Delta(a)$ has only one characteristic root, say $\alpha(a)$, of maximal multiplicity $d(\Delta)$.

If the characteristic of F is 0, then by the trace argument we have

$$\alpha(a+b) = \alpha(a) + \alpha(b). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

If the characteristic of F does not vanish, then it is by assumption greater than 3 and,

https://doi.org/10.1017/S204061850003389X Published online by Cambridge University Press

since $A(AA) = 0$, it follows that (9) again holds by [**4**, p. 95, formula (66)]. We observe also that

$$\Delta(\lambda a) = \lambda\Delta(a) \quad (\lambda \in F, a \in A), \quad \ldots\ldots(10)$$

so that α is a linear form on A.

As a next step we want to show that, for any element x of L,

$$\alpha(xA) = 0. \quad \ldots\ldots(11)$$

It suffices to show (11) under the additional assumption that

$$(x, x)_\Delta \neq 0. \quad \ldots\ldots(12)$$

Indeed, we know that there are elements y, z of L for which $(y, z)_\Delta \neq 0$, and from the identity

$$(y+z, y+z)_\Delta = (y, y)_\Delta + 2(y, z)_\Delta + (z, z)_\Delta$$

it follows, in view of the assumption that the characteristic of F is not 2, that at least one of the three elements $(y+z, y+z)_\Delta$, $(y, y)_\Delta$, $(z, z)_\Delta$ does not vanish. Hence there is an element x_0 of L satisfying $(x_0, x_0)_\Delta \neq 0$. For any element x of L we have the identity

$$(x, x)_\Delta + (x_0, x_0)_\Delta = \tfrac{1}{2}\left((x+x_0, x+x_0)_\Delta + (x-x_0, x-x_0)_\Delta\right),$$

so that at least one of the three elements $(x, x)_\Delta$, $(x+x_0, x+x_0)_\Delta$, $(x-x_0, x-x_0)_\Delta$ does not vanish. Therefore, if we have shown already that $\alpha(x_0A) = 0$ and that at least one of the three conditions $\alpha(xA) = 0$, $\alpha((x+x_0)A) = 0$, $\alpha((x-x_0)A) = 0$ is satisfied, it follows from the linearity of α that (11) is true without restrictions on the element x of L.

Now let us assume (12).

We want to show that for any subalgebra U of A satisfying $xU \subseteq U$ we have $\alpha(xU) = 0$. We observe that $V = Fx + U$ is a subalgebra of L containing U as an ideal. The representation Δ induces a representation Δ^V on V. Let Γ be an irreducible constituent of Δ^V with representation space m. Since $(x, x)_\Delta$ is the trace of $(\Delta x)^2$, which can be formed by adding up the traces of $(\Gamma x)^2$ over all irreducible constituents of Δ^V, it follows from (12) that Γ may be chosen in such a way that

$$(x, x)_\Gamma \neq 0. \quad \ldots\ldots(13)$$

(*a*) If V is nilpotent then, by [**4**, p. 29], the matrix $\Gamma(x)$ has only one characteristic root ξ, so that $(x, x)_\Gamma = d(\Gamma)\xi^2$ and thus, by (13), we have $d(\Gamma) \neq 0$ in F, $\xi^2 \neq 0$. From [**4**, p. 97, Satz 12] it follows that $d(\Gamma) = 1$, $\Gamma(xU) = 0$, $\alpha(xU) = 0$.

(*b*) If $U = Fu$ and

$$xu = \lambda u \quad (\lambda \neq 0), \quad \ldots\ldots(14)$$

then there is a characteristic root ξ of $\Gamma(x)$ and an element $v \neq 0$ of m such that

$$xv = \xi v. \quad \ldots\ldots(15)$$

Set $v_0 = v$ and $v_{i+1} = uv_i$ for $i = 0, 1, 2, \ldots$. It follows by induction that

$$xv_i = (\xi + i\lambda)v_i \quad (i = 0, 1, 2 \ldots). \quad \ldots\ldots(16)$$

Indeed (15) is (16) for $i = 0$. Let (16) be proved for some subscript i; then it follows from (14) that

$$xv_{i+1} = x(uv_i) = (xu)v_i + u(xv_i) = uv_i + u(\xi + i\lambda)v_i = \lambda v_{i+1} + (\xi + i\lambda)v_{i+1} = (\xi + (i+1)\lambda)v_{i+1}.$$

Since m is finite-dimensional, it follows that there is a first element among the elements

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$v_0, v_1, \ldots$ that is linearly dependent on the preceding elements, say v_g. Hence the linearly independent elements $v_0, v_1, \ldots, v_{g-1}$ span a linear subspace of m which is invariant under V. Since m is irreducible, it follows that the g elements $v_0, \ldots, v_{g-1}$ form a basis of m. Hence

$$(x, x)_\Gamma = \operatorname{tr}((\Gamma x)^2) = \sum_{i=0}^{g-1} (\xi + i\lambda)^2$$
$$= g\xi^2 + g(g-1)\xi\lambda + \frac{g(g-1)(2g-1)}{6}\lambda^2$$
$$= g\left(\xi^2 + (g-1)\xi\lambda + \frac{(g-1)(2g-1)}{6}\lambda^2\right),$$

since the characteristic of F is different from 2 and 3.

From (13) it follows that $g \neq 0$ in F. Hence

$$\operatorname{tr}(\Gamma(xu)) = g\alpha(xu) = \operatorname{tr}(\Gamma x\Gamma u - \Gamma u\Gamma x) = 0, \quad \alpha(xu) = 0, \quad \alpha(xU) = 0.$$

(c) If $UU = 0$ and if there is a basis $u_1, u_2, \ldots, u_\mu$ of U over F such that $xu_i = \lambda u_i + u_{i+1}$ ($\lambda \neq 0$, $i = 1, 2, \ldots, \mu$; $u_{\mu+1} = 0$), and if we have shown already that $\alpha(xu_i) = 0$ for $i = k, k+1, \ldots, \mu+1$, then we find that the linear form α vanishes on the ideal $Fu_k + Fu_{k+1} + \ldots + Fu_{\mu+1}$ of V, so that Γ induces on this ideal a nil representation. By Lemma 2 this nil representation is a null representation. If $k > 1$, then we can apply (b) to the Lie-algebra $\Gamma(Fx) + \Gamma(Fu_{k-1})$, substituting $\Gamma(x)$ for x and $\Gamma(u_{k-1})$ for u, and obtain $\alpha(u_{k-1}) = 0$. Hence, by induction, $\alpha(u_1) = \alpha(u_2) = \ldots = \alpha(u_\mu) = \alpha(u_{\mu+1}) = 0$, $\alpha(xU) = 0$.

(d) If $UU = 0$, then let us consider a decomposition

$$U = \dot{\sum}_{j=1}^{s} U_j$$

of U into the direct sum of linear subspaces U_j, invariant under the linear transformation $\begin{pmatrix} u \\ xu \end{pmatrix}$ of U, that cannot be further decomposed into invariant subspaces. To each of the subalgebras $Fx + U_j$, either (a) or (c) is applicable and thus we have $\alpha(xU_j) = 0$; moreover $\alpha(xU) = 0$ because of the linearity of α.

We may set $U = AA$ and in this event we find that $\alpha(x(AA)) = 0$. As had been shown before, it follows that $\alpha(L(AA)) = 0$. Hence the irreducible representation Δ induces on the ideal $L(AA)$ of L a nil representation and this nil representation is a null representation by Lemma 2. Since it is faithful by assumption, it follows that

$$L(AA) = 0. \quad \ldots\ldots(17)$$

(e) Denoting by x^* the linear transformation $\begin{pmatrix} a \\ xa \end{pmatrix}$ of A and by S the set of the characteristic roots of x^*, it follows that there is a decomposition $A = \dot{\sum}_{k \in S} A_k$ of A into the direct sum of the characteristic subspaces A_k of x^* consisting of all elements a of A satisfying an equation $(x^* - k)^\mu a = 0$ for some exponent μ. Moreover, by [4, p. 32], we have $A_j A_k \subseteq A_{j+k}$, where we set $A_h = 0$ if h is not a characteristic root of x^*. From (17) it follows that AA is contained in A_0. Since the characteristic of F is distinct from 2, it follows that $A_k A_k \subseteq AA \cap A_{2k} \subseteq A_0 \cap A_{2k} = 0$ if $k \neq 0$; hence A_k is an abelian subalgebra of A. In this event A_k admits a decomposition into the direct sum of abelian subalgebras of A to which (c) is applicable, so that $\alpha(xA_k) = 0$ if $k \neq 0$. If $k = 0$, then (a) is applicable and we find again that $\alpha(xA_0) = 0$. Hence $\alpha(xU_k) = 0$ for all k of S and hence $\alpha(xA) = 0$ because of the

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linearity of α.

It now follows that $\alpha(LA) = 0$, as has been shown above. The irreducible representation Δ induces a nil representation on the ideal LA. By Lemma 2, this nil representation is a null representation and, since Δ is faithful, it follows that $LA = 0$.

Let B be any solvable ideal of L so that $D^k B = 0$ for some exponent k. There is the chain of ideals

$$B \supseteq DB = BB \supseteq D^2B \supseteq \dots \supseteq D^kB = 0.$$

If $k > 0$, then $D^{k-1}B$ is an abelian ideal of L and then it follows that $LD^{k-1}B = 0$, as was shown above. If $k > 1$, then the ideal $A = D^{k-2}B$ satisfies the condition $A(AA) = 0$, so that $LA = 0$, as was shown above. Since $D^{k-1}B = AA \subseteq LA = 0$, it follows that $D^{k-1}B = 0$. Hence $LB = 0$. $LB \subseteq L^\perp$.

(2) Let F be algebraically closed and Δ be irreducible. If $L^\perp = L$, then it is obvious that $LA \subseteq L^\perp$. Let $L^\perp \neq L$. The representation Δ induces a faithful irreducible representation of the Lie-algebra ΔL. We denote the Lie-multiplication in ΔL by $X \circ Y = XY - YX$. Since A is a solvable ideal of L, it follows that ΔA is a solvable ideal of ΔL and hence it follows, as was shown at the close of (1), that $\Delta L \circ \Delta A \subseteq (\Delta L)^\perp$. But $\Delta L \circ \Delta A = \Delta(LA)$ and $(\Delta L)^\perp = \Delta(L^\perp)$; hence $\Delta(LA) \subseteq \Delta(L^\perp)$, $LA \subseteq L_\Delta + L^\perp = L^\perp$.

(3) Let F be algebraically closed. Let

$$\Delta \sim \begin{pmatrix} \Delta_1 & * & \cdot & \cdot & * \\ & \Delta_2 & \cdot & & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & * \\ & & & & \Delta_r \end{pmatrix} \qquad \text{.......(18)}$$

be a complete reduction of the representation Δ with irreducible constitutents $\Delta_1, \dots, \Delta_r$. We have

$$\operatorname{tr}(\Delta a \Delta b) = \textstyle\sum_{i=1}^r \operatorname{tr}(\Delta_i a \Delta_i b),$$

$$(a, b)_\Delta = \textstyle\sum_{i=1}^r (a, b)_{\Delta i}; \qquad \text{.......(19)}$$

hence

$$L^\perp(\Delta) \subseteq \bigcap_{i=1}^r L^\perp(\Delta_i). \qquad \text{.......(20)}$$

Since it was shown in (2) that $LA \subseteq L^\perp(\Delta_i)$, it follows from (20) that $LA \subseteq L^\perp(\Delta)$.

(4) Let E be an algebraically closed extension of the field of reference. The product algebra $L_E = L \times E$ over F is a Lie algebra over E such that any F-basis B of L is an E-basis of L_E. The representation Δ can be uniquely extended to a representation Δ^E of L_E by setting $\Delta^E(\sum_{b \in B} \lambda(b)b) = \sum_{b \in B} \lambda(b)b$ with coefficients $\lambda(b)$ in E. The product algebra $A_E = A \times E$ over F is a solvable ideal of L_E; hence it follows from (3) that $L_E A_E \subseteq L_E^\perp$ and thus $LA \subseteq L_E^\perp \cap L = L^\perp$.

From the proof of Theorem 1 and another application of Lemma 2 we derive the

Corollary of Theorem 1. *Under the same assumptions, for an irreducible representation Δ of L either the radical of Δ coincides with L or the radical of Δ does not coincide with L and LA lies in the kernel of Δ.*

The example of the solvable linear Lie-algebras formed by all 2×2-matrices over any field of characteristic 2 shows that Theorem 1 does not hold for fields of characteristic 2. The example of the solvable linear Lie-algebras formed by the linear combinations of the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

over any field of reference of characteristic 3 shows that the corollary of Theorem 1 does not hold any longer.

The following theorem states that, as far as the structure of $L/L^{\perp}$ and the non-degenerate symmetric invariant bilinear form induced on $L/L^{\perp}$ is concerned, it suffices to assume that Δ is fully reducible and faithful, that $L^{\perp}$ lies in the centre of L and that every solvable ideal of L lies in the centre.

THEOREM 2. *If the characteristic of the field of reference is distinct from 2 and 3, then for any Lie-algebra L with a matrix representation Δ there is a subalgebra U with a fully reducible representation Ψ and kernel U_{Ψ} such that*

$$U + L^{\perp} = L, \quad \text{.......................................(21)}$$

$$(a, b)_{\Psi} = (a, b)_{\Delta} \quad \text{for } a, b \in U, \quad \text{................................(22)}$$

$$UU^{\perp}(\Psi) \subseteq U_{\Psi} \subseteq U^{\perp}(\Psi), \quad \text{...................................(23)}$$

$$UA \subseteq U_{\Psi} \quad \textit{for any ideal } A \textit{ of } U \textit{ for which } \Psi A \textit{ is solvable.} \quad \text{..............(24)}$$

For the proof of Theorem 2 we need the following

LEMMA 3. *For any ideal A of a finite-dimensional Lie-algebra L over the field of reference F, there is a subalgebra U of L such that $U + A = L$ and $U \cap A$ is nilpotent. If L/A is nilpotent, then U can be chosen as a nilpotent subalgebra* (cf. [**3**, Theorem 4]).

Proof of Lemma 3. If $L = 0$, then Lemma 3 is clear. Let $L \neq 0$ and the theorem be proved already for Lie-algebras of dimension less than $\dim_F L$. For any element a of A we form the adjoint linear transformation $\operatorname{ad}(a) = \begin{pmatrix} x \\ ax \end{pmatrix}$ of L. The set of all elements x of L that are annihilated by some power of $\operatorname{ad}(a)$ forms a subalgebra L_0, by [**4**, p. 31]; moreover, L is the direct sum of L_0 and another linear subspace $\hat{L}_0$ such that $\operatorname{ad}(a)(\hat{L}_0) = \hat{L}_0$. Now let a be an element of L for which $\operatorname{ad}(a)$ induces a nilpotent linear transformation of L/A (e.g. an element of A). Then it follows that $\hat{L}_0 = [\operatorname{ad}(a)]^r \hat{L}_0 \subseteq [\operatorname{ad}(a)]^r L = A$, if r is large enough; hence $L_0 + A = L$. If $\dim_F L_0 < \dim_F L$, then, by the induction assumption, it follows that there is a subalgebra U of L_0 such that $U + L_0 \cap A = L_0$ and $U \cap (L_0 \cap A) = U \cap A$ is nilpotent. But $U + A = U + (L_0 \cap A) + A = L_0 + A = L$. Moreover, if L/A is nilpotent, then, since by the second isomorphism theorem $L_0/(L_0 \cap A)$ is isomorphic to L/A, it follows that $L_0/(L_0 \cap A)$ is nilpotent, so that it can be assumed that U is nilpotent.

If the subalgebra L_0 always coincides with L, then the adjoint representation of L induces a nil representation of A. The adjoint representation of A is a constituent of a nil representation; hence it is itself a nil representation and hence A is nilpotent, by Engel's Theorem. In this case we may set $U = L$, if L/A is not nilpotent. If L/A is nilpotent, then for every element a of L the adjoint linear transformation induces a nilpotent linear transformation of L/A. Thus by assumption the adjoint representation of L is a nil representation and by Engel's Theorem it follows that L is nilpotent. In this case we set $U = L$.

Proof of Theorem 2. By Lemma 3 there is a subalgebra U of L satisfying (21) such that $U \frown L^{\perp}$ is nilpotent. The representation Δ^U induced by Δ by restriction to U has a complete reduction

$$\Delta^U \sim \begin{pmatrix} \Delta_1 & * & \cdot & \cdot & * \\ & \Delta_2 & \cdot & & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & * \\ & & & & \Delta_r \end{pmatrix}$$

with irreducible constituents $\Delta_1, \Delta_2, \ldots, \Delta_r$. For the fully reducible representation Ψ that is obtained by adding only those irreducible constitutents Δ_i for which the Δ_i-radical does not coincide with L, we clearly obtain (22). Since $U^{\perp}(\Psi) = U \frown L^{\perp}$ is a nilpotent ideal and therefore $U^{\perp} = U^{\perp}(\Psi)$ is a solvable ideal of U, (23) follows by an application of the corollary of Theorem 1 ; (24) is proved similarly.

After these preparations we have the following

STRUCTURE THEOREM (THEOREM 3). (*a*) *For any Lie-algebra L over a field F of characteristic distinct from* 2 *and* 3 *and for any matrix representation Δ of L, the factor algebra $\bar{L}$ of L over the Δ-radical of L permits a decomposition*

$$\bar{L} = \dot{\sum}_{i=1}^{r} \bar{L}_i \qquad \text{.......................................(25)}$$

into the direct sum of mutually orthogonal and indecomposable ideals $\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_r$ distinct from 0.

(*b*) *The ideals $\bar{L}_i\bar{L}_i$ are perfect ideals and uniquely determined up to the order. The centre $z(\bar{L}_i)$ of $\bar{L}_i$ is of the same dimension over the field of reference as the factor algebra $\bar{L}_i/\bar{L}_i^2$ of $\bar{L}_i$ over $\bar{L}_i^2$.*

(*c*) *If the ideal $\bar{L}_i$ is abelian, then it is one-dimensional.*

(*d*) *If the centre of $\bar{L}_i$ vanishes, then $\bar{L}_i = \bar{L}_{i1}$ is simple non-abelian.*

(*e*) *Only if the characteristic of F does not vanish can there be non-abelian components $\bar{L}_i$ with non-vanishing centre $z(\bar{L}_i)$. In this event the ideal $\bar{L}_i^2$ is the sum of the minimal non-vanishing perfect ideals $\bar{L}_{i1}, \ldots, \bar{L}_{im_i}$ of $\bar{L}$ contained in $\bar{L}_i$. The algebra $\bar{L}_i^2$ is directly indecomposable but there is the decomposition*

$$\bar{L}_i^2/z(\bar{L}_i) = \dot{\sum}_{j=1}^{m_i}(\bar{L}_{ij} + z(\bar{L}_i))/z(\bar{L}_i)$$

of the factor algebra $\bar{L}_i^2/z(\bar{L}_i)$ into the direct sum of its minimal non-vanishing ideals, each of which is simple non-abelian

(*f*) *Every minimal non-vanishing perfect ideal of $\bar{L}$ coincides with one of the ideals $\bar{L}_{ij}$. If and only if its centre vanishes, we have $\bar{L}_{ij} = \bar{L}_i$. The minimal non-vanishing perfect ideals are mutually orthogonal.*

Proof of Theorem 3. From the definition of $\bar{L}$ it follows that the trace bilinear form of Δ induces on $\bar{L}$ a symmetric invariant bilinear form such that the orthogonal space of $\bar{L}$ vanishes, i.e. a non-degenerate bilinear form. Hence, for every linear subspace $\bar{X}$ of $\bar{L}$, the dimension of $\bar{X}$ plus the dimension of the orthogonal subspace $\bar{X}^{\perp}$ is equal to the dimension of $\bar{L}$. Hence

$(\bar{X}^{\perp})^{\perp} = \bar{X}$. If $\bar{X}$ is non-degenerate, i.e. if $\bar{X} \cap \bar{X}^{\perp} = 0$, then we have in any event the direct decomposition $\bar{L} = \bar{X} \dotplus \bar{X}^{\perp}$. Thus there is a decomposition (25) of the finite-dimensional Lie-algebra $\bar{L}$ into the direct sum of r mutually orthogonal non-vanishing ideals $\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_r$, such that there is no further decomposition of $\bar{L}_i$ into the direct sum of mutually orthogonal non-vanishing ideals ($i = 1, 2, \ldots, r$). Note that every ideal of $\bar{L}_i$ is also an ideal of $\bar{L}$ and that the trace bilinear form of Δ induces on $\bar{L}_i$ a non-degenerate symmetric invariant bilinear form.

If $\bar{L}_i$ is abelian, then, since the characteristic of F is distinct from 2, it follows that there is an element $\bar{x}$ of $\bar{L}_i$ for which $(\bar{x}, \bar{x})_\Delta \neq 0$, so that $\bar{L}_i$ is orthogonally decomposable into the direct sum of the ideal Fx and the orthogonal complement $(Fx)^{\perp} \cap \bar{L}_i$, and this implies that $\bar{L}_i = F\bar{x}$. Note that $\bar{L}_i^2 = 0$ implies that $\bar{L}_i^2$ is a perfect ideal.

Let $\bar{L}_i^2 \neq 0$. For the Lie-algebra $M = \bar{L}_i$ with non-degenerate bilinear form f satisfying (2)–(5), we find that

$$f(M^2, z(M)) = f(M, Mz(M)) = f(M, 0) = 0.$$

Conversely, if $f(M^2, x) = 0$ for the element x of M, then $f(M^2, x) = f(M, Mx) = 0$, $Mx = 0$, x lies in $z(M)$; hence $z(M) = (M^2)^{\perp}$, $z(M)^{\perp} = M^2$. If for an element $\bar{x}$ of the centre of $\bar{L}_i$ we have $(\bar{x}, \bar{x})_\Delta \neq 0$, then there is the orthogonal decomposition of $\bar{L}_i$ into the ideal $F\bar{x}$ and its orthogonal complement. Since this is impossible and since the characteristic of the field of reference is distinct from 2, it follows that $z(\bar{L}_i)$ is contained in $(z(\bar{L}_i))^{\perp} = \bar{L}_i^2$. The dimensions of $z(\bar{L}_i)$ and of $\bar{L}_i^2$ add up to the dimension of $\bar{L}_i$, so that $z(\bar{L}_i)$ is isomorphic to the factor algebra of $\bar{L}_i$ over $\bar{L}_i^2$.

By Theorem 1 every solvable ideal of $\bar{L}$ lies in $z(\bar{L})$. For every solvable ideal $\bar{A}$ of $\bar{L}_i^2$, it follows from Theorem 1 that $\bar{L}_i^2\bar{A} \subseteq (L_i^2)^{\perp} \cap \bar{L}_i = z(\bar{L}_i)$; hence $\bar{A}$ lies in the second centre of $\bar{L}_i^2$, a solvable ideal of $\bar{L}$, and hence $\bar{A}$ lies in $z(\bar{L}_i)$. It follows that the factor algebra $\bar{L}_i^2/z(\bar{L}_i)$ contains no abelian ideal $\neq 0$. Moreover $\bar{L}_i^2/z(\bar{L}_i) \neq 0$. The trace bilinear form of Δ induces a non-degenerate symmetric invariant bilinear form f^* on $L_i^* = \bar{L}_i^2/z(\bar{L}_i)$.

There is a decomposition

$$L_i^* = \dot{\sum}_{j=1}^{m_i} L_{ij}^*$$

of L_i^* into the direct sum of mutually orthogonal ideals L_{ij}^* which permit no further proper orthogonal decomposition. For an ideal A^* of L_{ij}^*, set $B^* = A^{*\perp} \cap L_{ij}^*$, so that

$$f^*((A^* \cap B^*)^2, L_{ij}^*) = f^*(A^* \cap B^*, (A^* \cap B^*)L_{ij}^*) \subseteq f^*(A^*, B^*) = 0, \quad (A^* \cap B^*)^2 = 0.$$

Thus $A^* \cap B^*$ is an abelian ideal of L_{ij}^* and therefore of L_i^*. Hence $A^* \cap B^* = 0$, $L_{ij}^* = A^* + B^*$, so that, by assumption, $A^* = L_{ij}^*$, and therefore L_{ij}^* is simple non-abelian. If X^* is any minimal non-vanishing ideal of L_i^* then, as shown above, $X^{*2} \neq 0$; hence $X^*L_i^* \neq 0$, $X^*L_{ij}^* \neq 0$ for some index j, $X^*L_{ij}^* \subseteq X^* \cap L_{ij}^*$, $X^* \cap L_{ij}^* \neq 0$, $X^* \cap L_{ij}^* = X^* = L_{ij}^*$. It follows that the components L_{ij}^* are simple non-abelian ideals characterized as the minimal non-vanishing ideals of L_i^*†.

The ideal $\bar{L}_{ij}^*$ of $\bar{L}_i^2$ formed by the cosets in L_{ij}^* contains a minimal perfect ideal $\bar{L}_{ij} \neq 0$ of $\bar{L}_i^2$. It is clear that $L_{ij}^* \supseteq (\bar{L}_{ij} + z(\bar{L}_i))/z(\bar{L}_i)$ and hence

$$(\bar{L}_{ij} + z(\bar{L}_i))/z(\bar{L}_i) = L_{ij}^*, \qquad \bar{L}_{ij}^* = \bar{L}_{ij} + z(\bar{L}_i), \qquad (\bar{L}_{ij}^*)^2 = (\bar{L}_{ij})^2 = \bar{L}_{ij}.$$

† Compare [1], [2].

Thus $\bar{L}_{ij}$ is uniquely determined by L_{ij}^* as the derived algebra of the algebra $\bar{L}_{ij}^*$ formed by the cosets modulo $z(\bar{L}_i)$ belonging to L_{ij}^*.

Conversely, if $\bar{A}$ is a minimal perfect ideal $\neq 0$ of $\bar{L}$ then, because $\bar{A}\bar{A} = \bar{A}$, we find that the i-th component ideal $\bar{A}_i = (\bar{A} + \sum_{j \neq i} \bar{L}_j) \cap \bar{L}_i$ lies in $\bar{L}_i^2$ and is homomorphic to $\bar{A}$. Hence, if $\bar{A}_i \neq 0$, then A_i is a minimal perfect ideal $\neq 0$ of $\bar{L}_i$. Thus $\bar{A}_i = \bar{L}_{ij}$ for some j, $\bar{A}_i\bar{A}_i = \bar{A}_i \subseteq \bar{A}_i\bar{A} \subseteq \bar{A}_i$, $\bar{A}_i\bar{A} = \bar{A}_i$, $\bar{A}_i \subseteq \bar{A}$. Since $\bar{A}$ is itself a minimal perfect ideal $\neq 0$ of $\bar{L}$, it follows that $\bar{A} = \bar{A}_i = \bar{L}_{ij}$.

Since the trace bilinear form of Δ induces on $\bar{L}_i^2/z(\bar{L}_i)$ a non-degenerate bilinear form, it follows by an argument similar to an earlier one that

$$0 = (D^2\bar{L}_i, \bar{L}_i \cap (D^2\bar{L}_i)^\perp) = (D\bar{L}_i, D\bar{L}_i(\bar{L}_i \cap (D^2\bar{L}_i)^\perp)),$$

$$D\bar{L}_i(\bar{L}_i \cap (D^2L_i)^\perp) \subseteq \bar{L}_i \cap (D\bar{L}_i)^\perp = z(\bar{L}_i),$$

$$\bar{L}_i \cap (D^2\bar{L}_i)^\perp \text{ is solvable, } \bar{L}_i \cap (D^2\bar{L}_i)^\perp \subseteq z(\bar{L}_i),$$

$$\bar{L}_i \cap (D^2\bar{L}_i)^\perp = z(\bar{L}_i) = \bar{L}_i \cap (D\bar{L}_i)^\perp,$$

$D^2\bar{L}_i^\perp = D\bar{L}_i^\perp$, $D^2\bar{L}_i = D\bar{L}_i$. For the perfect ideal $D\bar{L}_i$ we find that

$$D\bar{L}_i = z(\bar{L}_i) + \textstyle\sum_{j=1}^{m_i} \bar{L}_{ij} = D^2\bar{L}_i = \sum_{j=1}^{m_i} \bar{L}_{ij}.$$

By Theorem 2, for the purpose of the structural investigation of $\bar{L}$ we can assume that every solvable ideal of L and also $L^\perp$ are contained in the centre of L. Let L_i be the ideal of L consisting of the cosets of $\bar{L}_i$ modulo $L^\perp$. The elements of the cosets of $z(\bar{L}_i)$ modulo $L^\perp$ form the centre $z(L_i)$ of L_i. Since $D\bar{L}_i = \bar{L}_i^2$ is perfect, it follows that $D^kL_i + z(L_i) = DL_i + z(L_i)$; hence $D^3L_i = (D^2L_i)^2 = (z(L_i) + D^2L_i)^2 = (z(L_i) + DL_i)^2 = (DL_i)^2 = D^2L_i$, so that D^2L_i is a perfect ideal.

Let E be an algebraically closed extension of F, let L_E, Δ^E be the extensions of L, Δ respectively over E. If $0 \subset z(L_i) \subset L_i$, then there is an element z of $z(D^2L_i)$ that is not contained in $\bar{L}^\perp$ and an irreducible constituent Γ of Δ^E for which $\Gamma(z) \neq 0$. Hence, by Schur's Lemma, $\Gamma(z) = \zeta I$, $0 \neq \zeta \in E$. If the degree $d(\Gamma)$ of Γ is not divisible by the characteristic of F, then $(z, z)_\Gamma = \operatorname{tr}(\Gamma(z)\Gamma(z)) = d(\Gamma)\zeta^2 \neq 0$. Hence D^2L_i is the direct sum of the ideal Fz and the ideal $(Fz)^\perp(\Gamma) \cap D^2L_i$, and therefore $D^3L_i \subseteq (Fz)^\perp(\Gamma) \cap D^2L_i \subset D^2L_i$, a contradiction. It follows that $0 \subset z(\bar{L}_i) \subset \bar{L}_i$ implies that the characteristic of the field of reference is not zero.

If DL_i is not decomposable and if there is a decomposition $L_i = A \dotplus B$ of L_i into the direct sum of the two ideals A, B, then there is the direct decomposition $L_i^2 = A^2 \dotplus B^2$ of L_i^2. It follows that either A or B is abelian, say A is abelian. Hence $A \subseteq z(L_i) \subseteq L_i^2 = (A + B)^2 = B^2 \subseteq B$, $A = 0$. Hence L_i is indecomposable.

It remains to show that L_i^2 is indecomposable. For this purpose we need

Lemma 4. *Let L be a fully reducible linear Lie-algebra over a field of reference F that is not of characteristic 2, such that the radical $L^\perp$ of L with respect to its natural representation Δ is contained in the centre $z(L)$ of L, and for every irreducible constituent Δ_i of Δ the Δ_i-radical of L does not coincide with L. Then every Cartan subalgebra of L is abelian.*

Proof of Lemma 4. Let H be a nilpotent subalgebra of L that is its own normalizer. It follows that $L^\perp \subseteq z(L) \subseteq H$. Let Δ^H be the representation of H obtained by restriction of Δ. Then†

$$H^{\perp}(\Delta^{H}) = H \cap L^{\perp}(\Delta). \qquad (26)$$

Let Γ be an absolutely irreducible constituent of Δ^{H}. Then for any element z of $z(H) \cap H^2$ we have, by Schur's Lemma, $\Gamma z = \zeta I$ for some element ζ of an extension of F. By [4, p. 29], for any element h of H the matrix $\Gamma(h)$ has only one characteristic root, say $\lambda(h)$, of maximal multiplicity $d(\Gamma)$, so that

$$(z, h)_{\Gamma} = \operatorname{tr}(\Gamma z \Gamma h) = \zeta \operatorname{tr}(\Gamma(h)) = d(\Gamma)\zeta\lambda(h).$$

Here either the degree of Γ is divisible by the characteristic of F or $d(\Gamma) = 1$, $\Gamma(H^2) = 0$, $\Gamma(z) = 0, \zeta = 0$. At any rate $(z, h)_{\Gamma} = 0$. Hence $(z, h)_{\Delta} = 0, z \subseteq H^{\perp}(\Delta^{H}), z \subseteq L^{\perp}(\Delta) \subseteq z(L)$. By assumption, for each irreducible constituent Δ_i of Δ we have $L^{\perp}(\Delta_i) \subset L$; hence $H^{\perp}(\Delta_i^{H}) \subset H$. Since the characteristic of F is not 2, it follows that there is an element h of H such that $(h, h)_{\Delta_i} \neq 0$. There is an absolutely irreducible constituent Γ of Δ_i^{H} for which $(h, h)_{\Gamma} \neq 0$. On the other hand we know that the matrix $\Gamma(h)$ has only one characteristic root $\lambda(h)$ of multiplicity $d(\Gamma)$, so that $0 \neq (h, h)_{\Gamma} = \operatorname{tr}((\Gamma h)^2) = d(\Gamma)\lambda(h)^2$, $d(\Gamma)$ is not divisible by the characteristic of F, $d(\Gamma) = 1$, by [4, p. 97, Satz 12]. Hence $\Gamma(z) = 0$, $\Delta_i(z)$ is a singular matrix. Hence, by Schur's Lemma, $\Delta_i(z)$ is a nilpotent matrix, Δ_i induces a nil representation of the ideal Fz of L, $\Delta_i z = 0$, by Lemma 2. Since L is fully reducible, it follows that $\Delta z = 0$, $z = 0$, $H^2 \cap z(H) = 0$, $H^2 = 0$, q.e.d.

Proof of the remainder of Theorem 3. By Theorem 2 and its proof we can assure that L satisfies the assumption of Lemma 4. Moreover we can assume that $0 \subset z(\bar{L}) \subset \bar{L}^2 \subset \bar{L} = \bar{L}_i$.

If there is a Cartan subalgebra H of L then, by Lemma 4, it is abelian. Since H is nilpotent and its own normalizer, it follows from [4, pp. 28–29] that there is a decomposition $L = H \dotplus \hat{H}$ of L into the direct sum of H and another linear subspace $\hat{H}$ such that $H\hat{H} = \hat{H}$. Hence $H + L^2 = L$. Let $\bar{H} = H/L^{\perp}$, so that $\bar{H} + \bar{L}^2 = \bar{L}$ and $\bar{H}$ is abelian. If there is a decomposition $\bar{L}^2 = \bar{A} \dotplus \bar{B}$ of $\bar{L}^2$ into the direct sum of the two ideals $\bar{A}$, $\bar{B}$ of $\bar{L}^2$, then it follows from $D\bar{L}^2 = \bar{L}^2$ that $D\bar{A} = \bar{A}$, $D\bar{B} = \bar{B}$; hence $\bar{A}$, $\bar{B}$ are ideals of $\bar{L}$. Moreover it follows from the relations $\bar{A} \cap \bar{B} = 0$, $\bar{A} + \bar{B} = \bar{L}^2$ that $\bar{A}^{\perp} + \bar{B}^{\perp} = \bar{L}$, $\bar{A}^{\perp} \cap \bar{B}^{\perp} = (\bar{L}^2)^{\perp} = z(\bar{L})$, so that $\bar{A}^{\perp} = \bar{B}_1 \dotplus \bar{A}^{\perp} \cap \bar{L}^2$, $\bar{B}^{\perp} = \bar{A}_1 \dotplus \bar{B}^{\perp} \cap \bar{L}^2$, where $\bar{A}_1$, $\bar{B}_1$ are linear subspaces of $\bar{H}$. Hence $\bar{A}_1 \cap \bar{B}_1 = 0$, $\bar{A}_1 \dotplus \bar{B}_1 \dotplus \bar{L}^2 = \bar{L}$, and since $\bar{H}$ is abelian, it follows that $\bar{L}$ is the direct sum of the orthogonal ideals $\bar{A} + \bar{A}_1$, $\bar{B} + \bar{B}_1$. Since $\bar{L}$ is orthogonally indecomposable, it follows that either $\bar{A}$ or $\bar{B}$ vanishes. Hence $\bar{L}^2$ is indecomposable.

If there is no Cartan subalgebra of L then, by [4, pp. 32–33], it follows that the field of reference is finite. Let $\mathscr{E}(\bar{L}^2)$ be the associative algebra over F that is generated by the adjoint linear transformations of $\bar{L}^2$. Let $\mathscr{C}(\bar{L}^2)$ be the linear associative algebra consisting of all linear transformations of $\bar{L}^2$ that are elementwise permutable with $\mathscr{E}(\bar{L}^2)$. Since $\bar{L}^2$ is perfect, it follows that there is, up to the order of the components, only one decomposition $\bar{L}^2 = \sum_{i=1}^{t} \bar{A}_i$ of $\bar{L}^2$ into the direct sum of indecomposable ideals $\neq 0$. Hence the factor algebra of $\mathscr{C}(\bar{L}^2)$ over its radical is isomorphic to a ring sum of finitely many division algebras E_1, E_2, ..., E_s of finite dimension over F. By a theorem of Maclagan-Wedderburn, all the E_i's

† From [4, pp. 28–29] it follows that there is a decomposition $L = H + \hat{H}$ of L into the direct sum of H and another linear subspace $\hat{H}$ such that $H\hat{H} = \hat{H}$. For every invariant bilinear form f we find that

$$f(H, \hat{H}) = f(H, H\hat{H}) = f(H^2, \hat{H}) = f(H^2, H\hat{H}) = f(H^3, \hat{H}) = \dots = f(H^{c+1}, \hat{H}) = 0$$

and hence (26) is satisfied.

are finite extensions of F. Since the numbers prime to the product P of the degrees of the extensions E_i over F are unbounded, it follows from [4, pp. 32–34] that there is an extension E of F of degree prime to P, such that the extended Lie-algebra L_E over E contains a Cartan subalgebra. By the method of the construction of E, there is, up to the order of the components, only one decomposition of $\bar{L}^2_E$ into the direct sum of indecomposable ideals $\neq 0$, viz., the decomposition $(\bar{L}^2)_E = \dot{\sum}_{i=1}^{t} (\bar{A}_i)_E$. As we have seen before, there is a decomposition $\bar{L}_E = \dot{\sum}_{i=1}^{t} \bar{B}_i$ of $\bar{L}_E$ into the direct sum of the mutually orthogonal ideals $\bar{B}_i$ such that $(\bar{A}_i)_E$ is contained in $\bar{B}_i$, for $i = 1, 2, \ldots, s$. We have $(\sum_{i=2}^{t} (\bar{A}_i)_E)^{\perp} = \bar{B}_1 + z(\bar{L}_E) = ((\sum_{i=2}^{t} \bar{A}_i)^{\perp})_E$ and there is a linear subspace $\bar{X}$ of $(\sum_{i=2}^{t} \bar{A}_i)^{\perp}$ such that $\bar{B}_1 + z(\bar{L}_E) = (\bar{A}_1)_E + z(\bar{L}_E)) \dot{+} \bar{X}_E$, $(\bar{A}_1)_E + \bar{X}_E$ is an ideal of $\bar{L}_E$ and $((\bar{A}_1)_E + \bar{X}_E)^{\perp} \frown ((\bar{A}_1)_E + \bar{X}_E) = (\bar{A}_1^{\perp})_E \frown (\bar{X}^{\perp})_E \frown ((\bar{A}_1)_E + \bar{X}_E) = 0$; hence $\bar{B} = \bar{A}_1 + \bar{X}$ is an ideal of $\bar{L}$ such that $\bar{B}^{\perp} \frown \bar{B} = 0$ and therefore there is the orthogonal decomposition $\bar{L} = \bar{B} \dot{+} \bar{B}^{\perp}$ of $\bar{L}$. It follows that $t = 1$, $\bar{L}^2$ is indecomposable, q.e.d.

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University of Notre Dame,
Notre Dame, Indiana, U. S. A.

https://doi.org/10.1017/S204061850003389X Published online by Cambridge University Press