# 1 Introduction

In this chapter, we first provide some motivation for the type of modeling problems we address in this book. Then we provide an overview of the types of mathematical model used to describe the behavior of the classes of systems of interest. We also describe the types of uncertainty model adopted and how they fit into the mathematical models that describe system behavior. In addition, we provide a preview of the applications discussed throughout the book, mostly centered around electric power systems. We conclude the chapter by providing a brief summary of the content of subsequent chapters.

## 1.1 Motivation

Loosely speaking, an engineered system is a collection of hardware and software components assembled and interacting in a particular way so that they collectively fulfill some function. The interaction between components can be of physical nature, i.e., components can be electrically, mechanically, or thermally coupled, and thus may involve some exchange of energy. Components can also be coupled in the sense that they exchange information with each other. Because of phenomena external to the system, there is some uncertainty as to how the system will perform. This phenomenon can materialize as an external (time-varying) input that drives the system response, or as a change in the system structure. In both cases, these external phenomena will alter the system nominal response and might cause the system to fail to perform its function. While most systems are typically designed to withstand some structural and operational uncertainty, it is important to verify that this is the case before the system is deployed.

To illustrate these ideas, consider the power supply system in Fig. 1.1, whose function is to reliably provide electric power to a mission-critical computer load, labeled as IT load, at a certain voltage level. To this end, there are three sources of power: two utility feeders, labeled as feeder 1 and feeder 2, and a backup generator, labeled as genset. Having such a redundant arrangement ensures delivery of power to the IT load with high assurance. While not depicted in the figure, there is a computer-based control system in charge of monitoring and controlling the power sources and switchgear, which plays an important role in the analysis of the system.

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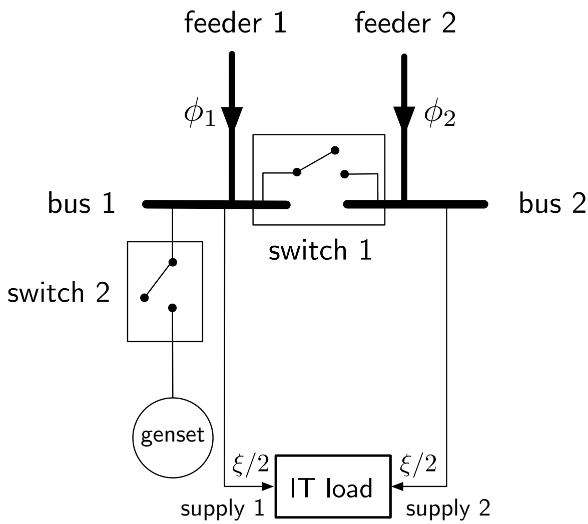


**Figure 1.1** Schematic of power supply system for mission-critical load, where $\phi_1$ and $\phi_2$ denote the active power flowing through feeder 1 and feeder 2, respectively, and $\xi$ denotes the total power delivered to the IT load.

Under normal operating conditions, both switch 1 and switch 2 are open (therefore genset is not initially used to supply power to the IT load). Then, feeder 1 supplies power to bus 1, which, in turn, serves half of the power demanded by the IT load via supply 1. Similarly, feeder 2 supplies bus 2, which in turn serves the other half of the power demanded by the IT load via supply 2. The total power demanded by the IT load is determined by the computer workload, which can vary according to external requests received by the computer and is a priori unknown. Thus, from the point of view of the power supply system, the computer workload is an external input that drives the power supply system response. Since the workload evolves over time and is unknown a priori, there is some uncertainty on how the power supply system will perform its function.

In the event that there is an outage in either feeder 1 or feeder 2, switch 1 will close and all power to the IT load will be supplied by the available feeder. In the event that there is an outage in both feeders (either sequential in time or simultaneous), switch 2 is closed and the genset will supply all the power to the IT load until either (i) one or both utility feeders are restored back to operation, at which point in time, switch 2 is open and the power to the IT load is once again supplied by one or both utility feeders, or (ii) there is an event that causes the genset to fail, at which point the IT load is shut down offline. Thus, the phenomena causing feeder outages and genset failure result in a change in the system structure in the sense of how power is routed from the available sources to the IT load.

In the context of the system above, one might be interested in quantifying the impact of workload variability on certain variables of interest, e.g., the flows of power through the wires connecting the buses and the IT load, and the magnitude of the voltage at bus 1 and bus 2 (when not connected together). This analysis is necessary to ensure that wires are sized correctly and protection equipment, e.g., under- and over-voltage protection relays, is calibrated appropriately. It is also necessary to ensure that after outages in one or both feeders occur, subsequent switching actions are correctly executed. In addition, one could be interested in quantifying the impact of equipment failure on the system ability to perform its function over some period of time.

The goal of this book is to develop analysis tools to perform the types of analysis described above. The applications and examples throughout the book draw heavily from electric power applications, including bulk power systems and microgrids, and linear and switched linear circuits encountered in power electronics applications. However, the modeling framework and techniques presented are general and can be applied to other engineering domains, including automotive and aerospace applications. For example, they can be used to assess the dynamic performance of an automotive steer-by-wire system and the lateral-directional control system of a fighter aircraft.

## 1.2 System Models

In a broad sense, one can think of an engineered system as an entity imposing constraints on certain *variables* associated with the system energy and information content. With this point of view, we can represent the behavior of the system by a set of mathematical relations between the aforementioned variables; this is what we refer to as the model of the system. These relations can be a result of physical laws, e.g., Kirchhoff's laws, energy conservation law, or moment conservation law. They can also arise from algorithms implemented in a digital computer, for modifying (controlling) the physical behavior of the system, e.g., the proportional-integral control scheme used in a bulk power system to automatically regulate frequency across the system. These mathematical relations will, in general, also include numerous *parameters*, i.e., quantities defining physical or information properties of the components comprising the system. Such system parameters can be constant or vary with time, and their value can be a priori unknown or uncertain. When modeling a system, the distinction between parameters and variables is typically clear because the values taken by the system parameters should not be affected by the values taken by the system variables or other parameters. Indeed, if the value of some parameter $p$ is actually affected by the values taken by the system variables or other parameters, then the model should reflect this dependence and instead of being treated as a parameter in the model, $p$ should be considered as an additional system variable and treated in the model as such. Next, we illustrate the ideas above via some examples.

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**Example 1.1** (Power supply system) Consider the system in Fig. 1.1 and assume there are no losses in any of its components. Let $\phi_1$ and $\phi_2$ denote the active power flowing through feeder 1 into bus 1 and the active power flowing through feeder 2 into bus 2, respectively. Let $p_g$ denote the active power supplied by the genset. Let $\xi$ denote the active power demanded by the IT load. Recall that under normal operating conditions, half of the power to the IT load is supplied by feeder 1, while the other half is supplied by feeder 2 (the genset does not supply any power). Then, since the active power flowing in and out of both bus 1 and bus 2 needs to be balanced, we have that

$$\phi_1 = \frac{\xi}{2}, \quad \phi_2 = \frac{\xi}{2}, \quad p_g = 0. \tag{1.1}$$

Now, recall that if there is an outage in feeder 1 (feeder 2), switch 1 will close and all the power to the IT load will be delivered by feeder 2 (feeder 1); thus,

$$\begin{aligned} &\phi_1 = 0, \quad \phi_2 = \xi, \quad p_g = 0, \quad \text{if outage in feeder 1 and feeder 2 in service,} \\ &\phi_1 = \xi, \quad \phi_2 = 0, \quad p_g = 0, \quad \text{if outage in feeder 2 and feeder 1 in service.} \end{aligned} \tag{1.2}$$

Also, recall that if there is an outage in both feeders, the genset will supply all the power to the load; thus,

$$\phi_1 = 0, \quad \phi_2 = 0, \quad p_g = \xi. \tag{1.3}$$

Finally, for the case when there is an outage in both feeders and a failure in the genset, we have that

$$\phi_1 = 0, \quad \phi_2 = 0, \quad p_g = 0, \tag{1.4}$$

and the IT load is shut down offline; thus, $\xi = 0$. Note that the relations in (1.1–1.4) only involve $\xi$, $\phi_1$, $\phi_2$, and $p_g$, which in this case are the system variables, i.e., the model here does not involve any parameters.

**Example 1.2** (Linear circuit) Consider the circuit in Fig. 1.2 and assume that $c(t) = c,\ t \geq 0$, where $c$ is a positive scalar. First, note that

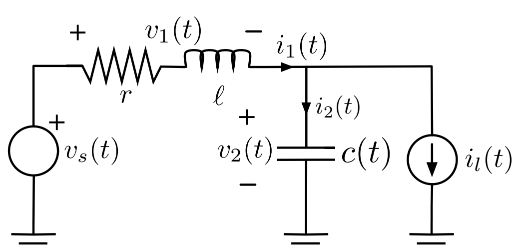


**Figure 1.2** Linear circuit.

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$$v_1(t) = r\, i_1(t) + \ell \frac{di_1(t)}{dt},$$
$$i_2(t) = c \frac{dv_2(t)}{dt}. \tag{1.5}$$

Then, by using Kirchhoff's laws, we can obtain the following model describing the relation between the currents $i_1(t)$ and $i_l(t)$, and the voltages $v_s(t)$ and $v_2(t)$:

$$0 = i_1(t) - c\frac{dv_2(t)}{dt} - i_l(t),$$
$$0 = v_s(t) - r\, i_1(t) - \ell \frac{di_1(t)}{dt} - v_2(t), \tag{1.6}$$

where $r$, $\ell$, and $c$ are positive constants. Here, $i_1(t)$, $i_2(t)$, $v_s(t)$, and $v_2(t)$ are variables, whereas $r$, $\ell$, and $c$ are parameters. Now, assume that $c(t)$ is known to evolve according to

$$\frac{dc(t)}{dt} = -c(t) + \alpha u(t),$$

where $\alpha$ is a positive scalar; thus,

$$i_2(t) = c(t)\frac{dv_2(t)}{dt} + v_2(t)\big(-c(t) + \alpha u(t)\big).$$

Then, the model describing the circuit behavior is as follows:

$$0 = i_1(t) - c(t)\frac{dv_2(t)}{dt} - v_2(t)\big(-c(t) + \alpha u(t)\big) - i_l(t),$$
$$0 = v_s(t) - r i_1(t) - \ell\frac{di_1(t)}{dt} - v_2(t),$$
$$0 = \frac{dc(t)}{dt} + c(t) - \alpha u(t); \tag{1.7}$$

thus, in this model, the capacitance, $c(t)$, is no longer a parameter but a variable.

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There are some fundamental differences between the models in (1.1–1.4), (1.6), and (1.7). First, in the models in (1.1–1.4) and (1.6), the relation between the variables is linear, whereas in the model in (1.7), the relation between the variables is nonlinear. Second, in the model in (1.1–1.4), the constraints imposed on the system variables are in the form of a system of algebraic equations, whereas in the models in (1.6) and (1.7), the constraints imposed on the system variables are in the form of a set of ordinary differential equations (ODEs). In this book, we refer to systems whose behavior can be described by a set of algebraic equations as *static systems*, whereas systems whose behavior can be described by a set of ODEs are referred to as *continuous-time dynamical systems*. Furthermore, we refer to systems whose behavior can be described iteratively by a set of recurrent relations as *discrete-time dynamical systems*.

So far, we have not discussed the nature of the variables describing the energy and information state of a system; in general, we will categorize them as either inputs or states. By inputs, we refer to variables that are set and can be varied extraneously, whereas by states, we refer to variables that result from the constraints describing the system behavior and the values the inputs take. With this categorization, we can rewrite the static system in (1.1–1.4) as follows:

$$x = H_i \xi, \quad i = 1, 2, 3, \tag{1.8}$$

where $x = [\phi_1, \phi_2, p_g]^\top$, $\xi \geq 0$, and

$$H_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{1.9}$$

and

$$x = H_4 \xi, \tag{1.10}$$

where $\xi = 0$, and

$$H_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.11}$$

More generally, in this book we will consider static systems of the form

$$x = h_i(w), \quad i \in \mathcal{Q}, \tag{1.12}$$

where $x \in \mathbb{R}^n$ is referred to as the state vector, $w \in \mathbb{R}^m$ is referred to as the input vector, $\mathcal{Q}$ takes values in some finite set, and $h_i \colon \mathbb{R}^m \to \mathbb{R}^n$, which we refer to as the system input-to-state mapping, is defined by the relations between the state variables (i.e., the entries of the state vector), the inputs (i.e., the entries of the input vector), and the system parameters.

By using the same categorization of variables as inputs or states, we can rewrite the model in (1.6) in state-space form as follows:

$$\frac{d}{dt} x(t) = Ax(t) + Bw(t), \quad t \geq 0, \tag{1.13}$$

where $x(t) = \big[v_2(t), i_1(t)\big]^\top$ is referred to as the state vector, $w(t) = \big[v_s(t), i_l(t)\big]^\top$ is referred to as the input vector, and

$$A = \begin{bmatrix} 0 & \frac{1}{c} \\ -\frac{1}{\ell} & -\frac{r}{\ell} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -\frac{1}{c} \\ \frac{1}{\ell} & 0 \end{bmatrix}. \tag{1.14}$$

More generally, we will consider continuous-time dynamical systems of the form

$$\frac{d}{dt} x(t) = f\big(t, x(t), w(t)\big), \quad t \geq 0, \tag{1.15}$$

where $f \colon [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, is defined by the relations between the state variables (i.e., the entries of the state vector, $x \in \mathbb{R}^n$); and the inputs (i.e., the entries of the input vector, $w(t) \in \mathbb{R}^m$). For example, the system in (1.7) can be written as

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$$\frac{d}{dt}x(t) = f\big(x(t), w(t)\big), \tag{1.16}$$

where $x(t) = \big[v_2(t), i_1(t), c(t)\big]^\top$, $w(t) = \big[v_s(t), i_l(t), u(t)\big]^\top$, and

$$f\big(x(t), w(t)\big) = \begin{bmatrix} \frac{1}{c(t)}\big(v_2(t)c(t) - \alpha v_2(t)u(t) + i_1(t) - i_l(t)\big) \\ \frac{1}{\ell}\big( - v_2(t) - r\, i_1(t) + v_s(t)\big) \\ -c(t) + \alpha u(t) \end{bmatrix}. \tag{1.17}$$

Finally, in this book we also consider discrete-time dynamical systems of the form

$$x_{k+1} = h_k(x_k, w_k), \quad k = 0, 1, 2, \ldots, \tag{1.18}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^m$, and $h_k \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ k = 0, 1, 2, \ldots,$ is defined by the relations between the state variables and inputs of the particular system under consideration.

## 1.3 Uncertainty Models

Here, we provide an overview of the different uncertainty models considered in subsequent chapters for both static and dynamical systems. In the process, we also state the analysis objectives for both classes of systems under each uncertainty model considered.

### 1.3.1 Static Systems

Consider a static system of the form:

$$x = h_i(w), \quad i \in \mathcal{Q}. \tag{1.19}$$

If the value that $w$ takes is uncertain, we say the system is subject to input uncertainty, whereas if the value that $i$ takes is uncertain, we say the system is subject to structural uncertainty. In terms of formally describing input uncertainty, we will consider both probabilistic models and set-theoretic models. In terms of formally describing structural uncertainty, we will only consider a probabilistic model. Each of these models is briefly described next.

**Probabilistic Input Uncertainty Model:** We assume the values that $w$ can take are described by a random vector with known first and second moments or known probability density function (pdf). Then, for each $h_i(\cdot),\ i \in \mathcal{Q}$, the values that $x = h_i(w)$ can take will be also random and described by some random vector, and the objective is to characterize the first and second moments (or the pdf) of this random vector.

**Set-Theoretic Input Uncertainty Model:** We assume the values that the input vector, $w$, can take belong to some convex set. While in general this set can have any shape, in this book we restrict our analysis to two particular classes of

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closed, convex sets, namely ellipsoids and zonotopes. Then, for each $h_i(\cdot)$, $i \in \mathcal{Q}$, the possible values that the state, $x = h_i(w)$, can take will belong to some set, and the objective is to characterize such a set.

**Structural Uncertainty Model:** We assume that the input-to-state mapping, $h_i(\cdot)$, evolves with time according to a Markov chain with known transition probabilities. Thus, for a given $w$, the value that $x = h_i(w)$ takes will evolve according to the aforementioned Markov process. Here, the value that $w$ can take is either a known constant or can be described by a probabilistic model like the one above. Then, the objective is to characterize the state vector statistics.

### 1.3.2 Dynamical Systems

Consider continuous-time systems of the form

$$\frac{d}{dt}x(t) = f\big(t, x(t), w(t)\big), \tag{1.20}$$

and discrete-time systems of the form

$$x_{k+1} = h_k(x_k, w_k). \tag{1.21}$$

Here we will only analyze the system behavior under input uncertainty, i.e., the values that $w(t)$ and $w_k$ can take are not a priori known, and will use both probabilistic and set-theoretic models to describe them.

**Probabilistic Input Uncertainty Model:** For continuous-time dynamical systems, we consider the case when the function $f(\cdot,\cdot,\cdot)$ is defined as follows:

$$f(t, x, w) = \alpha\big(t, x\big) + \beta(t, x)w, \tag{1.22}$$

whereas, for discrete-time dynamical systems, we do not impose any restrictions on $h_k(\cdot,\cdot)$. We assume that the values that $w(t)$ and $w_k$ take are random and governed by a "white noise" process; thus, the values that $x(t)$ and $x_k$ take are also random and governed by some stochastic process. We first consider the case when we only know the mean and covariance functions of the "white noise" input process and characterize the mean and covariance functions of the stochastic process describing the evolution of $x(t)$ and $x_k$. Then, we further assume that the "white noise" input process is Gaussian and provide the complete probabilistic characterization of the stochastic process describing the evolution of $x(t)$ and $x_k$.

**Set-Theoretic Input Uncertainty Model:** We consider general functions $f(\cdot,\cdot,\cdot)$ and $h_k(\cdot,\cdot)$, and assume the values that $w(t)$ and $w_k$ can take are known to belong to a convex set, namely an ellipsoid. Then, the values that $x(t)$ and $x_k$ can take also belong to some set (not necessarily an ellipsoid), and the objective is to characterize such a set. Providing an exact characterization of such a set is difficult in general (even if $f(\cdot,\cdot,\cdot)$ and $h_k(\cdot,\cdot)$ are affine functions); thus, we settle for obtaining ellipsoidal upper bounds.

## 1.4 Application Examples

Most of the techniques presented in the book are illustrated by using examples from circuit theory, electric power systems, and power electronics. For example, we utilize a simplified formulation of the power flow problem in AC power systems to illustrate the techniques developed for analyzing static systems subject to input uncertainty. Also, in order to illustrate the analysis techniques for dynamical systems subject to input uncertainty, we utilize a simplified model of the dynamics of an inertia-less AC microgrid, i.e., a small AC power system whose generators and loads are interfaced with the network via power electronics.

### 1.4.1 Power Flow Analysis under Active Power Injection Uncertainty

Consider a three-phase power system comprising $n$ buses ($n > 1$) indexed by the elements in $\mathcal{V} = \{1, 2, \dots, n\}$, and $l$ transmission lines $\big(n-1 \leq l \leq n(n-1)/2\big)$ indexed by the elements in the set $\mathcal{L} = \{1, 2, \dots, l\}$, and assume the following hold:

**A1.** The system is balanced and operating in sinusoidal regime.
**A2.** There is at most one transmission line connecting each pair of buses.
**A3.** Each transmission line is short and lossless.
**A4.** The voltage magnitude at each bus is fixed by some control mechanism.

Let $p_i$ denote the active power injected into the system network via bus $i$, and let $\phi_e$ denote the active power flowing on transmission line $e$, $e = 1, 2, \dots, l$. Assume that

$$p_i = \xi_i, \quad i = 1, 2, \dots, n-1,$$

where $\xi_i$ is extraneously set and a priori unknown. Then, since the system is lossless, the injection into bus $n$ must be such that $\sum_{j=1}^{n} p_j = 0$; thus,

$$p_n = -\sum_{j=1}^{n-1} \xi_j.$$

Define $\xi = [\xi_1, \xi_2, \dots, \xi_{n-1}]^\top$ and $\phi = [\phi_1, \phi_2, \dots, \phi_l]^\top$. Then, by imposing some conditions on the values that $\xi$ can take, there exists a function $f\colon \mathbb{R}^l \to \mathbb{R}^l$ encapsulating the system network topology and transmission line parameters such that

$$w = f(\phi), \tag{1.23}$$

where $w = \left[\xi^\top, \mathbf{0}_{l-n+1}^\top\right]^\top$. The formulation above can be generalized to the case when there are $m$, $1 \leq m \leq n-1$, power injections being extraneously set with the remaining power injections being adjusted so that $\sum_{j=1}^{n} p_j = 0$, which

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is a necessary condition that the power injections need to satisfy because of the assumption on the transmission lines in the system being lossless.

If the network is a tree, then we have that $l = n - 1$ and

$$\xi = \widetilde{M}\phi,$$

where $\widetilde{M} \in \mathbb{R}^{(n-1)\times(n-1)}$ is invertible; thus, we can write

$$\phi = \widetilde{M}^{-1}\xi. \tag{1.24}$$

For the case when the network is not a tree, because of the assumptions made on the values that $\xi$ can take, we can ensure that there exists $f^{-1}\colon \mathbb{R}^l \to \mathbb{R}^l$ such that

$$\phi = f^{-1}(w). \tag{1.25}$$

Then, given either a probabilistic model or a set-theoretic model describing the values that the vector of extraneous power injections, $\xi$, can take, the problem is to characterize the values that the vector of line flows, $\phi$, can take. We explore such settings in detail in Chapter 3 and Chapter 7.

### 1.4.2 Analysis of Inertia-less AC Microgrids under Power Injection Uncertainty

Consider a three-phase microgrid comprising $n$ buses $(n > 1)$ indexed by the elements in $\mathcal{V} = \{1, 2, \ldots, n\}$, and $l$ transmission lines $\big(n - 1 \leq l \leq n(n-1)/2\big)$ and assume the following hold:

**B1.** The microgrid is balanced and operating in sinusoidal regime.
**B2.** There is at most one transmission line connecting each pair of buses.
**B3.** Each transmission line is short and lossless.
**B4.** Connected to each bus there is either a generating- or a load-type resource interfaced via a three-phase voltage source inverter.
**B5.** The reactance of each voltage source inverter output filter is small when compared to the reactance values of the network transmission lines.
**B6.** The phase angle of the inverter connected to each bus is regulated via a droop control scheme updated at discrete time instants indexed by $k = 0, 1, 2, \ldots$.
**B7.** The inverter outer voltage and inner current control loops hold the inverter output voltage magnitude constant throughout time.

Consider the case when the frequency-droop control setpoints of the inverters at buses $1, 2, \ldots, m$ change randomly (this is typically the case in photovoltaic installations endowed with maximum power point tracking). Let $\xi_k$ denote an $m$-dimensional vector whose entries correspond to the values taken at instant $k$ of said setpoints. Assume that in order to mitigate the effect of these random fluctuations, the frequency-droop control setpoints of the inverters at buses $m + 1, m + 2, \ldots, n$ are regulated via a closed-loop integral control scheme so that the frequency across the microgrid network remains close to some nominal value. Let

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$z_k$ denote an $(n-m)$-dimensional vector whose entries correspond to the internal states at instant $k$ of said closed-loop integral control scheme. Let $\theta_k$ denote an $n$-dimensional vector whose entries correspond to the bus voltage phase angles at time instant $k$. Then, the microgrid closed-loop dynamics can be written as follows:

$$\begin{aligned} \theta_{k+1} &= h_1(\theta_k, z_k, \xi_k), \\ z_{k+1} &= h_2(z_k, \xi_k), \end{aligned} \tag{1.26}$$

where the functions $h_1 \colon \mathbb{R}^n \times \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^n$ and $h_2 \colon \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$ encapsulate the microgrid network topology and transmission line parameters, and the control gains of the frequency regulation scheme. Then, given either a probabilistic model or a set-theoretic model describing the values that $\xi_k,\ k \geq 0$, can take, the problem is to characterize the values that $\theta_k,\ k \geq 0$, and $z_k,\ k \geq 0$, can take. This information can then be used to characterize the system frequency. We explore such settings in detail in Chapters 5 and 8. In addition, in Chapter 6, we formulate the continuous-time counterpart of the model in (1.26) and study the system performance when the measurements utilized by the frequency regulation scheme are corrupted.

### 1.4.3 Reliability Analysis of Static Systems

Consider a static system comprising $r$ components indexed by the elements in $\mathcal{C} = \{c_1, c_2, \ldots, c_r\}$. Assume that each component $c_i$ can operate in one of two modes: nominal and off-nominal (failed). Further assume that initially all the components are operating in their nominal mode, and as time evolves, transitions from the nominal to the failed mode may occur, and once a transition occurs, the component remains in its failed mode. Such static systems are called nonrepairable and their input-to-state behavior can be described by

$$x = h_i(w), \tag{1.27}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ is constant, and $h_i \colon \mathbb{R}^m \to \mathbb{R}^n,\ i \in \mathcal{Q} = \{1, 2, \ldots, N\}$, is determined by (i) the components that are operating in their nominal mode, and (ii) the components that have transitioned from operating in their nominal to their failed mode, and (iii) the chronological order in which these transitions occurred.

Assume that the time it takes for each component $c_i$ to transition from its nominal mode to its failed mode is random and can be described by a continuous random variable, $T_{c_i}$, with continuously differentiable cumulative distribution function, $F_{T_{c_i}}(\cdot)$. Further, assume that these random variables are pairwise independent. Then, transitions among the elements in $\mathcal{Q}$ are governed by a continuous-time Markov chain, $Q = \{Q(t) \colon t \geq 0\},\ Q(t) \in \mathcal{Q}$, whose transition

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rates can be obtained from the individual component failure rates, denoted by $\lambda_{c_i}(t),\ t \geq 0$, and defined as follows:

$$\lambda_{c_i}(t) = \frac{f_{T_{c_i}}(t)}{1 - F_{T_{c_i}}(t)}, \tag{1.28}$$

where $f_{T_{c_i}}(t) = \frac{dF_{T_{c_i}}(t)}{dt}$.

Let $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, with $\mathcal{R}_i = (x_i^m, x_i^M)$, where $x_i^m, x_i^M \in \mathbb{R}$ respectively denote the minimum and maximum values that the $i^{th}$ entry of $x$ can take so as to guarantee that the system is performing its intended function. Now, because $w$ is assumed to be constant, the value that $x$ takes for each $i$ is also constant. Then, given $i \in \mathcal{Q}$, the system is said to be operational if $x = h_i(w) \in \mathcal{R}$, and nonoperational otherwise. Let $\pi_i(t)$ denote the probability that the aforementioned continuous-time Markov chain is in state $i \in \mathcal{Q}$. Then, the reliability of the system at time $t$, which we denote by $R(t)$, can be defined as the probability that the non-repairable system is operating at time $t$, and can be computed as follows:

$$R(t) = \sum_{i \in \mathcal{Q}_0} \pi_i(t), \tag{1.29}$$

where $\mathcal{Q}_0 = \{i \in \mathcal{Q}\colon x = h_i(w) \in \mathcal{R}\}$.

The ideas above can be generalized in several directions. First, it is possible to take into account input uncertainty by modeling $w$ as a random vector with known first and second moments (or known pdf). In addition, it is possible to take into account events that may cause several components to simultaneously transition from operating in their nominal mode to their failed mode; such events are referred to as common cause failures. Finally, one can extend the framework to analyze static systems comprising components that can be repaired. All these ideas are explored in detail in Chapter 4.

## 1.5 Book Road Map

Chapter 2 starts by reviewing important concepts from probability theory and stochastic processes. Subsequent chapters on probabilistic input and structural uncertainty make heavy use of random vectors and vector-valued stochastic process, so the reader should be familiar with the material included on these concepts. Next, the chapter provides a review of set-theoretic notions. The material on sets in Euclidean space included in this part is key to understanding the set-theoretic approach to input uncertainty modeling. The chapter concludes with a review of several fundamental concepts from the theory of discrete- and continuous-time linear dynamical systems.

Chapter 3 covers the analysis of static systems under probabilistic input uncertainty. The first part of the chapter is devoted to analyzing both linear and nonlinear static systems when the first and second moments of the input vector are known, and provides techniques for characterizing the first and second moments of the state vector. For the linear case, the techniques provide the exact moment characterization, whereas for the nonlinear case, the characterization, which is based on a linearization of the system model, is approximate. The second part of the chapter provides techniques for the analysis of both linear and nonlinear static systems when the pdf of the input vector is known. The techniques included provide exact characterizations of the state pdf for both linear and nonlinear systems. In both cases, the inversion of the input-to-state mapping is required, which in the linear case involves the computation of the inverse of a matrix; however, for the nonlinear, it involves obtaining an analytical expression for the input-to-state mapping, which might be difficult in general. Thus, for the nonlinear case, we also provide a technique based on linearization that yields an approximation of the state pdf. The chapter concludes by utilizing the techniques developed to study the power flow problem under active power injection uncertainty.

Chapter 4 studies static systems under structural uncertainty. The first part of the chapter is devoted to the development of a model describing the system stochastic behavior. To this end, we assume that the system can only adopt a finite number of input-to-state mappings, and that transitions among these different mappings are random and governed by a Markov chain. We consider both discrete- and continuous-time settings and provide expressions governing the evolution of the probability distribution associated with the resulting Markov chains. The second part of the chapter tailors the techniques developed earlier to analyze multi-component systems subject to component failures and repairs. Techniques for constructing the system input-to-state model are extensively covered, as this is in general the most difficult part of the analysis when analyzing systems with a large number of components.

Chapter 5 provides techniques for analyzing discrete-time dynamical systems under probabilistic input uncertainty. Here, the relation between the input and the state is described by a discrete-time state-space model. The input vector is modeled as a vector-valued stochastic process with known first and second moments (or known pdf). The first part of the chapter is devoted to the analysis of linear systems and provides techniques for characterizing the first and second moments and the pdf of the state vector. The second part of the chapter is devoted to the analysis of nonlinear systems, where we use the techniques developed in Chapter 4 to exactly characterize the distribution of the state vector when the pdf of the input vector is given. In addition, we rely on linearization techniques to obtain expressions that approximately characterize the first and second moments and the pdf of the state vector. The third part of the chapter illustrates the application of the techniques developed to the analysis of inertia-less AC microgrids under random active power injections.

Chapter 6 is the continuous-time counterpart of Chapter 5, i.e., it studies continuous-time dynamical systems described by a continuous-time state-space model whose input is subject to probabilistic uncertainty. The first part of the

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chapter is devoted to the analysis of linear systems and provides techniques for computing the first and second moments of the state vector when the evolution of the input vector is governed by a "white noise" process with known mean and covariance functions. Then, by additionally imposing this "white noise" process to be Gaussian, we provide a partial differential equation whose solution yields the pdf of the state vector. The second part of the chapter extends these techniques to the analysis of nonlinear systems, with a special focus on the case when the "white noise" governing the evolution of the input vector is Gaussian. The third part of the chapter illustrates the application of the techniques developed to the analysis of inertia-less AC microgrids when the measurements utilized by the frequency control system are corrupted by additive disturbances.

Chapter 7 is the set-theoretic counterpart of Chapter 3, i.e., it covers the analysis of static systems under set-theoretic input uncertainty. In the first part of the chapter, we assume that the input belongs to an ellipsoid and analyze both linear and nonlinear systems. For the linear case, we provide techniques to exactly characterize the set containing all possible values that the state can take. For the nonlinear case, we again resort to linearization to approximately characterize the set containing all possible values that the state can take. The second part of the chapter considers linear and nonlinear systems when the input is known to belong to a zonotope. For the linear case, we are able to compute the exact set containing all possible values the state can take, whereas for the nonlinear case, we settle for an approximation thereof obtained via linearization. The techniques developed are utilized to analyze the power flow problem under uncertain active power injections.

Chapter 8 is the set-theoretic counterpart of Chapter 5, i.e., it analyzes linear and nonlinear discrete-time systems described by a discrete-time state-space model whose inputs are uncertain but known to belong to an ellipsoid. For the linear case, even if the input set is an ellipsoid, the set containing all possible values that the state can take is not an ellipsoid in general; however, it can be outer bounded by an ellipsoid. We develop techniques for recursively computing a family of such outer-bounding ellipsoids. Within this family, we then show how to choose ellipsoids that are optimal in some sense, e.g., they have minimum volume. For the nonlinear case, we will again resort to linearization techniques to approximately characterize the set containing all possible values that the state can take. Application of the techniques is illustrated using the same inertia-less AC microgrid model used in Chapter 5.

Chapter 9 is the continuous-time counterpart of Chapter 8, i.e., it covers the analysis of linear and nonlinear continuous-time dynamical systems described by a continuous-time state-space model whose input belongs to an ellipsoid. Similar to the linear discrete-time case, the set containing all possible values that the state can take is not an ellipsoid in general; however, it can be outer bounded by a family of ellipsoids whose evolution is governed by a differential equation that can be derived from the system state-space model. As in the discrete-time case, it is possible to choose ellipsoids within this family that are optimal in some sense. The nonlinear case is again handled using linearization. The techniques developed in the chapter are used to analyze the performance of a buck DC-DC power converter. In addition, we show how the techniques can be used to assess the effect of variability associated with renewable-based electricity generation on bulk power system dynamics, with a focus on time-scales involving electromechanical phenomena.

There are several ways in which the material in subsequent chapters can be studied depending on the end goal of the reader. For example, a reader familiar with the foundations on which the book builds – probability and stochastic processes, set theory, and linear dynamical system theory – can skip Chapter 2, although its reading is recommended so one becomes familiar with the notation adopted. A reader interested exclusively in the theory can skip the final section in each subsequent chapter as these provide detailed case studies drawn from electric power applications that showcase the potential of the theoretical tools for tackling real-world problems. A reader interested exclusively in probabilistic uncertainty analysis tools can focus on Chapters 3 to 6, whereas a reader interested exclusively in set-theoretic uncertainty analysis tools can focus on Chapters 7 to 9. A reader interested in the analysis of static systems can focus on Chapters 3, 4, and 7. A reader interested in the analysis of discrete-time dynamical systems can focus on Chapters 5 and 8, whereas a reader interested in continuous-time dynamical systems can focus on Chapters 6 and 9.

## 1.6 Notes and References

Formal definitions of a dynamical system, including the notions of input and state and the state-space formalism, can be found in [1, 2]. Early references on the use of state-space models to describe the behavior of dynamical systems include [3, 4]. The topic is now standard and covered in detail in many modern control theory textbooks (e.g., [5, 6, 7, 8, 9]). Several examples on how to describe particular dynamical systems, e.g., an inverted pendulum and a nonlinear circuit, are included in [7]. The use of probabilistic and set-theoretic uncertainty models in the context of static and dynamical systems is extensively covered in [10] and many of the results discussed in later chapters can be traced back to this seminal book.

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