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A computer aided classification of certain groups of prime power order

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A classification of two-generator 3-groups of second maximal class and low order is presented. All such groups with orders up to 3^8 are described, and in some cases with orders up to 3^{10}. The classification is based on computer aided computations. A description of the computations and their results are presented, together with an indication of their significance.

1. Introduction

The groups considered are two-generator groups $\underline{\underline{P}}$ of order 3^n and class $n - 2$. If $\underline{\underline{P}}/\underline{\underline{P}}' \cong C_3 \times C_3$ we consider $6 \le n \le 10$, and if $\underline{\underline{P}}/\underline{\underline{P}}' \cong C_9 \times C_3$ we consider $5 \le n \le 8$.

A considerable amount of work, most of it still unpublished, is being done on p-groups of large class, that is groups of order p^n and class $n - r$, for fixed r and varying n. In particular, the suggestion that such groups should have solubility length bounded in terms of p and r alone is being investigated. For $r = 1$, the groups in question are well understood and the suggestion is a theorem, originally due to Alperin [*1*] and more explicitly due to Shepherd [*9*]. When $r = 2$ and $p = 2$, this is also true and the groups in question have been classified by James [*4*]. One result of the computation discussed here is a proof, in [*5*], that the groups occurring when $r = 2$, $p = 3$ have derived length at most 4.

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The analysis of two-generator 3-groups of second maximal class goes along the following lines*. Let $\underline{\underline{P}}$ be such a group of order 3^n. Put $\underline{\underline{M}}_i = C_{\underline{\underline{P}}}(\gamma_i/\gamma_{i+2})$, where $\gamma_j = \gamma_j(\underline{\underline{P}})$, the jth term of the lower central series of $\underline{\underline{P}}$, for all i such that $[\gamma_i : \gamma_{i+2}] = 9$. Then $\underline{\underline{M}}_i$ is a maximal subgroup of $\underline{\underline{P}}$, and the crucial result that $\bigcup_i \underline{\underline{M}}_i \neq \underline{\underline{P}}$ (that is, that $\{\underline{\underline{M}}_i\}$ consists of at most 3 of the 4 maximal subgroups of $\underline{\underline{P}}$) is proved. Consider $a \notin \bigcup_i \underline{\underline{M}}_i$ and let $\underline{\underline{Q}} = \langle a, \gamma_2(\underline{\underline{P}})\rangle$. There are two possibilities.

(i) For all such a, $a^3 \in \zeta_1(\underline{\underline{P}})$. In this case $\underline{\underline{P}}$ is said to be of *maximal type*. Then $\gamma_i(\underline{\underline{P}}) = \gamma_{i-1}(\underline{\underline{Q}})$, and $\gamma_i(\underline{\underline{P}})^3 = \gamma_{i+2}(\underline{\underline{P}})$, for $i \geq 4$ ($i \geq 3$ if $\underline{\underline{P}}$ is a CF group, as defined in §4).

(ii) There is such an a with $a^3 \notin \gamma_2(\underline{\underline{P}})$. In this case $\underline{\underline{Q}}$ is a CF group of second maximal class and positive degree of commutativity, $[\underline{\underline{P}} : \underline{\underline{Q}}] = 3$, and $\underline{\underline{P}}$ is said to be of *non-maximal type*. Again $\gamma_i(\underline{\underline{P}}) = \gamma_{i-1}(\underline{\underline{Q}})$ for $i \geq 4$ ($i \geq 3$ if $\underline{\underline{P}}$ is a CF group), but $\gamma_i(\underline{\underline{P}})^3 = \gamma_{i+6}(\underline{\underline{P}})$ with the possible exception of a few small values of i.

It is possible to determine quite accurately the structure of CF groups of second maximal class and positive degree of commutativity. Thus the structure of 3-groups of second maximal class is quite well understood. The proof of the above results was much facilitated by the computer calculations presented here, which show that various cases which would otherwise need to be considered do not in fact arise.

The algorithm used is very similar to that described by Newman in [8]. Many ideas used in adapting the algorithm to this special problem were taken from Maung (a student of Leedham-Green) who computed 5-groups of maximal class, up to order 5^{16}.

In this paper no proofs are given, and many of the observations made are based on work the publication of which is still in preparation. A

* [Added 25 August 1977]. See, however, the Corrigendum, pp. 317-319.

complete list of groups under consideration is given. In addition precise descriptions of all the groups are included in a microfiche supplement, which is attached to page 320.

2. The algorithm

If $\underline{\underline{P}}$ is a finite p-group, and $1 \to \underline{\underline{R}} \to \underline{\underline{F}} \to \underline{\underline{P}} \to 1$ is a presentation of the group $\underline{\underline{P}}$ in which the rank of $\underline{\underline{F}}$ is minimal, then $\underline{\underline{F}}/[\underline{\underline{R}}, \underline{\underline{F}}]\underline{\underline{R}}^p$ and $\underline{\underline{R}}/[\underline{\underline{R}}, \underline{\underline{F}}]\underline{\underline{R}}^p$ are independent of the choice of presentation. We call them the *p-covering group* and *p-multiplicator* of $\underline{\underline{P}}$, respectively. The heart of the computation is a *nilpotent quotient algorithm*, which given a 'suitable' presentation of $\underline{\underline{P}}$ produces a 'suitable' presentation of the p-covering group of $\underline{\underline{P}}$. Here 'suitable' means 'in terms of a composition series defined by $a_1, a_2, \ldots$ '. Details of the nilpotent quotient algorithm appear elsewhere [7]. Let us define an *extension algorithm* as one that solves the following problem.

One is given a group $\underline{\underline{P}}$ of order p^n and class c, with soluble automorphism group and $\gamma_c(\underline{\underline{P}})$ of order p, together with the action of aut $\underline{\underline{P}}$ on $\underline{\underline{P}}$ in a suitable form. (For each composition factor of out $\underline{\underline{P}}$, the automorphism class group, an element β of aut $\underline{\underline{P}}$, corresponding to a generator of this composition factor, and the action of β on a minimal generating set of $\underline{\underline{P}}$ are given.) The problem is to obtain one representative $\underline{\underline{Q}}$ of each isomorphism class of groups, with $\gamma_{c+1}(\underline{\underline{Q}})$ of order p and $\underline{\underline{Q}}/\gamma_{c+1}(\underline{\underline{Q}}) \cong \underline{\underline{P}}$, and also the action of aut $\underline{\underline{Q}}$ on $\underline{\underline{Q}}$ in the above form. (For each such extension, aut $\underline{\underline{Q}}$ will be soluble.)

The given groups $\underline{\underline{P}}$ are on two generators a_1, a_2 and have $\gamma_c(\underline{\underline{P}}) = \langle a_n \rangle$. Lifting these to elements $b_1, \ldots, b_n$ in a p-covering group $\underline{\underline{P}}^*$ of $\underline{\underline{P}}$, $\gamma_{c+1}(\underline{\underline{P}}^*)$ is an elementary abelian group, spanned by $\{[b_n, b_1], [b_n, b_2]\}$. The rank of $\gamma_{c+1}(\underline{\underline{P}}^*)$ is of some importance in the program.

The extension algorithm is based on the following simple principle. The isomorphism classes mentioned above correspond to the orbits under out $\underline{\underline{P}}$ of those maximal subgroups $\underline{\underline{V}}$ of the p-multiplicator of $\underline{\underline{P}}$, which do not contain $\gamma_{c+1}(\underline{\underline{P}}^*)$. Here the group $\underline{\underline{Q}} = \underline{\underline{P}}^*/\underline{\underline{V}}$ corresponds to the orbit containing $\underline{\underline{V}}$. The group of outer automorphisms of $\underline{\underline{Q}}$ is an extension of a central outer automorphism, of order p, by the stabilizer of $\underline{\underline{V}}$ in out $\underline{\underline{P}}$. This central outer automorphism acts trivially on the p-multiplicator of $\underline{\underline{Q}}$, and only has an effect when the algorithm is applied yet again. This is taken into account in the algorithm.

It follows that the rank of $\gamma_{c+1}(\underline{\underline{P}}^*)$ is zero if and only if the set of groups $\underline{\underline{Q}}$ is empty.

3. An example

Let

$$\underline{\underline{P}} = \Big\langle a_1, a_2, a_3, a_4, a_5, a_6;\ a_1^3 = e,\ a_2^3 = a_4^2,\ a_3^3 = a_6^2,\ a_4^3 = a_5^3 = a_6^3 = e,$$
$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_6,$$
$$\text{all other simple commutators are trivial}\Big\rangle .$$

$\underline{\underline{P}}$ is a presentation for the group O order 3^6, class 4, which appears in the tables. A composition series for aut $\underline{\underline{P}}$ modulo inner and central automorphisms is induced by the automorphisms

$$\beta_1 : \begin{matrix} a_1 \to a_1a_4 \\ a_2 \to a_2 \end{matrix}, \quad \beta_2 : \begin{matrix} a_1 \to a_1 \\ a_2 \to a_2a_4 \end{matrix}, \quad \beta_3 : \begin{matrix} a_1 \to a_1a_5 \\ a_2 \to a_2 \end{matrix}, \quad \beta_4 : \begin{matrix} a_1 \to a_1 \\ a_2 \to a_2^2 \end{matrix} .$$

These were obtained from the previous stage of the algorithm.

The nilpotent quotient algorithm yields the p-covering group $\underline{\underline{P}}^*$:

$$\underline{\underline{P}}^* = \Big\langle b_1, b_2, \ldots; b_{10};\ b_1^3 = b_9,\ b_2^3 = b_4^2b_{10},\ b_3^3 = b_6^2b_7b_8,\ b_4^3 = b_7^2,\ b_5^3 = b_8^2,$$
$$b_6^3 = b_7^3 = \ldots = b_{10}^3 = e,\ [b_2, b_1] = b_3,\ [b_3, b_1] = b_4,\ [b_3, b_2] = b_5,$$
$$[b_4, b_1] = b_6,\ [b_4, b_3] = b_8^2,\ [b_5, b_1] = b_8^2,\ [b_5, b_2] = b_7^2,$$
$$[b_6, b_1] = b_7,\ [b_6, b_2] = b_8,\ \text{all other simple commutators are trivial}\Big\rangle,$$

and hence the p-multiplicator of $\underline{\underline{P}}$ is

$$\Big\langle b_7, b_8, b_9, b_{10};\ b_7^3 = b_8^3 = b_9^3 = b_{10}^3 = e,\ \text{abelian}\Big\rangle .$$

The maximal subgroups of the p-multiplicator are given by

$$\langle b_8, b_9, b_{10}\rangle , \langle b_7b_8^{\alpha}, b_9, b_{10}\rangle , \langle b_7b_9^{\alpha}, b_8b_9^{\beta}, b_{10}\rangle ,$$
$$\langle b_7b_{10}^{\alpha}, b_8b_{10}^{\beta}, b_9b_{10}^{\gamma}\rangle , \quad \alpha, \beta, \gamma \in \{0, 1, 2\} .$$

There are 40 of these. Eliminate those which contain $\gamma_5(\underline{\underline{P}}^*) = \langle b_7, b_8\rangle$, namely, $\langle b_7, b_8, b_{10}\rangle$, $\langle b_7, b_8, b_9\rangle$, $\langle b_7, b_8, b_9b_{10}\rangle$, $\langle b_7, b_8, b_9b_{10}^2\rangle$. The remaining maximal subgroups of the p-multiplicator are as follows:

S_1 $\langle b_8, b_9, b_{10}\rangle$,	S_{19} $\langle b_7b_{10}, b_8b_{10}^2, b_9\rangle$,
S_2 $\langle b_7, b_9, b_{10}\rangle$,	S_{20} $\langle b_7b_{10}^2, b_8b_{10}^2, b_9\rangle$,
S_3 $\langle b_7b_8, b_9, b_{10}\rangle$,	S_{21} $\langle b_7b_{10}, b_8, b_9b_{10}\rangle$,
S_4 $\langle b_7b_8^2, b_9, b_{10}\rangle$,	S_{22} $\langle b_7b_{10}^2, b_8, b_9b_{10}\rangle$,
S_5 $\langle b_7b_9, b_8, b_{10}\rangle$,	S_{23} $\langle b_7, b_8b_{10}, b_9b_{10}\rangle$,
S_6 $\langle b_7b_9^2, b_8, b_{10}\rangle$,	S_{24} $\langle b_7b_{10}, b_8b_{10}, b_9b_{10}\rangle$,
S_7 $\langle b_7, b_8b_9, b_{10}\rangle$,	S_{25} $\langle b_7b_{10}^2, b_8b_{10}, b_9b_{10}\rangle$,
S_8 $\langle b_7b_9, b_8b_9, b_{10}\rangle$,	S_{26} $\langle b_7, b_8b_{10}^2, b_9b_{10}\rangle$,
S_9 $\langle b_7b_9^2, b_8b_9, b_{10}\rangle$,	S_{27} $\langle b_7b_{10}, b_8b_{10}^2, b_9b_{10}\rangle$,
S_{10} $\langle b_7, b_8b_9^2, b_{10}\rangle$,	S_{28} $\langle b_7b_{10}^2, b_8b_{10}^2, b_9b_{10}\rangle$,
S_{11} $\langle b_7b_9, b_8b_9^2, b_{10}\rangle$,	S_{29} $\langle b_7b_{10}, b_8, b_9b_{10}^2\rangle$,
S_{12} $\langle b_7b_9^2, b_8b_9^2, b_{10}\rangle$,	S_{30} $\langle b_7b_{10}^2, b_8, b_9b_{10}^2\rangle$,
S_{13} $\langle b_7b_{10}, b_8, b_9\rangle$,	S_{31} $\langle b_7, b_8b_{10}, b_9b_{10}^2\rangle$,
S_{14} $\langle b_7b_{10}^2, b_8, b_9\rangle$,	S_{32} $\langle b_7b_{10}, b_8b_{10}, b_9b_{10}^2\rangle$,
S_{15} $\langle b_7, b_8b_{10}, b_9\rangle$,	S_{33} $\langle b_7b_{10}^2, b_8b_{10}, b_9b_{10}^2\rangle$,
S_{16} $\langle b_7b_{10}, b_8b_{10}, b_9\rangle$,	S_{34} $\langle b_7, b_8b_{10}^2, b_9b_{10}^2\rangle$,
S_{17} $\langle b_7b_{10}^2, b_8b_{10}, b_9\rangle$,	S_{35} $\langle b_7b_{10}, b_8b_{10}^2, b_9b_{10}^2\rangle$,
S_{18} $\langle b_7, b_8b_{10}^2, b_9\rangle$,	S_{36} $\langle b_7b_{10}^2, b_8b_{10}^2, b_9b_{10}^2\rangle$.

The automorphisms $\beta_1, \ldots, \beta_4$ are extended to automorphisms $\beta_1', \ldots, \beta_4'$ of $\underline{\underline{P}}^*$, the definition of β_i' being obtained from the definition of β_i

by replacing a_j by b_j throughout. The action of these automorphisms on the p-multiplicator is as follows:

$$\begin{array}{llll}
\beta_1' : b_7 \mapsto b_7, & \beta_2' : b_7 \mapsto b_7, & \beta_3' : b_7 \mapsto b_7, & \beta_4' : b_7 \mapsto b_7^2, \\
\quad b_8 \mapsto b_8, & \quad b_8 \mapsto b_8, & \quad b_8 \mapsto b_8, & \quad b_8 \mapsto b_8 \\
\quad b_9 \mapsto b_9, & \quad b_9 \mapsto b_9, & \quad b_9 \mapsto b_8^2 b_9, & \quad b_9 \mapsto b_9 \\
\quad b_{10} \mapsto b_8 b_{10}, & \quad b_{10} \mapsto b_{10}, & \quad b_{10} \mapsto b_{10}, & \quad b_{10} \mapsto b_{10}^2 .
\end{array}$$

These automorphisms correspond to the following permutations of the maximal subgroups:

β_1' : $(S_2S_{15}S_{18})(S_3S_{17}S_{19})(S_4S_{16}S_{20})(S_7S_{31}S_{26})(S_8S_{32}S_{28})(S_9S_{33}S_{27})$
$(S_{10}S_{23}S_{34})(S_{11}S_{25}S_{35})(S_{12}S_{24}S_{36})$;

β_3' : $(S_2S_{10}S_7)(S_3S_{11}S_9)(S_4S_{12}S_8)(S_{15}S_{23}S_{31})(S_{16}S_{24}S_{32})(S_{17}S_{25}S_{33})$
$(S_{18}S_{34}S_{26})(S_{19}S_{35}S_{27})(S_{20}S_{36}S_{28})$;

β_4' : $(S_3S_4)(S_5S_6)(S_8S_9)(S_{11}S_{12})(S_{15}S_{18})(S_{16}S_{19})(S_{17}S_{20})(S_{21}S_{29})(S_{22}S_{30})$
$(S_{23}S_{34})(S_{24}S_{35})(S_{25}S_{36})(S_{26}S_{31})(S_{27}S_{32})(S_{28}S_{33})$.

Under these permutations the maximal subgroups form the following equivalence classes:

S_1 ;

$S_2, S_7, S_{10}, S_{15}, S_{18}, S_{23}, S_{26}, S_{31}, S_{34}$;

$S_3, S_4, S_8, S_9, S_{11}, S_{12}, S_{16}, S_{17}, S_{19}, S_{20}, S_{24}, S_{25}, S_{27}, S_{28}, S_{32}, S_{33}, S_{35}, S_{36}$;

S_5, S_6 ;

S_{13} ;

S_{14} ;

S_{21}, S_{29} ;

S_{22}, S_{30} .

Thus given $\underline{\underline{P}}$ and its automorphisms the extension algorithm yields a batch of eight new groups,

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$\underline{\underline{P}}^*/S_1$, $\underline{\underline{P}}^*/S_2$, $\underline{\underline{P}}^*/S_3$, $\underline{\underline{P}}^*/S_5$, $\underline{\underline{P}}^*/S_{13}$, $\underline{\underline{P}}^*/S_{14}$, $\underline{\underline{P}}^*/S_{21}$, $\underline{\underline{P}}^*/S_{22}$.

These groups are denoted by *O#1*, *O#2*, *O#3*, *O#4*, *O#5*, *O#6*, *O#7*, *O#8* in the microfiche supplement.

All these groups have order 3^7 and class 5 . $\beta'_1, \beta'_2, \beta'_3, \beta'_4$ all stabilize S_1 and so induce automorphisms of $\underline{\underline{P}}^*/S_1$. The automorphisms

$$\beta_5 : \begin{matrix} a_1 \to a_1a_6 \\ a_2 \to a_2 \end{matrix} , \quad \beta_6 : \begin{matrix} a_1 \to a_1 \\ a_2 \to a_2a_6 \end{matrix}$$

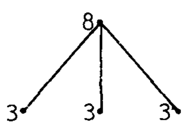
are central and β_5 is an inner automorphism of $\underline{\underline{P}}$, so $\beta'_1, \beta'_2, \beta'_3, \beta'_4$, β'_6 correspond to generators of a composition series for $\mathrm{aut}\left(\underline{\underline{P}}^*/S_1\right)$ modulo inner and central automorphisms.

The remaining seven groups are dealt with similarly. Thus, for example, β'_2, β'_6 correspond to generators for $\mathrm{aut}\left(\underline{\underline{P}}^*/S_3\right)$ modulo inner and central automorphisms.

4. Organisation of results

By applying the extension algorithm to a group $\underline{\underline{P}}$ we obtain a set of "immediate descendants" $\underline{\underline{Q}}$ which we call a *batch*. We also say that the groups $\underline{\underline{Q}}$ *arise* from $\underline{\underline{P}}$, and $\underline{\underline{P}}$ *gives rise* to the groups $\underline{\underline{Q}}$.

After repeated applications of the extension algorithm the information obtained is conveniently displayed as a labelled tree. Each node in the tree represents a batch of groups. For example



indicates that we are starting with a batch of eight groups. Of these eight, five give rise to no groups at all but the remaining three each give rise to a batch of three groups. The order of the groups at each level appears on the left-hand side of the page. Above each tree appearing in the tables is a letter. This indicates a specific group which gives rise to the first node of the tree. The computer produces the groups in each batch in a specific order. While this order is of no theoretical

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significance we keep the groups in the tables in the same order to facilitate correlations being made between the computer output and the tables.

A p-group $\underline{\underline{P}}$ is said to be a CF *group* if $\gamma_i(\underline{\underline{P}})/\gamma_{i+1}(\underline{\underline{P}})$ is of order at most p for i greater than or equal to two. If $\underline{\underline{P}}$ is any p-group with $\underline{\underline{P}}/\gamma_{p+1}(\underline{\underline{P}})$ of maximal class, then $\underline{\underline{P}}$ is of maximal class [2, Theorem 3.9]. Thus a two-generator 3-group $\underline{\underline{P}}$ of second maximal class is either a CF group or satisfies $\underline{\underline{P}}/\gamma_2(\underline{\underline{P}}) \cong \gamma_3(\underline{\underline{P}})/\gamma_4(\underline{\underline{P}}) \cong C_3 \times C_3$,

$$\gamma_2(\underline{\underline{P}})/\gamma_3(\underline{\underline{P}}) \cong \gamma_4(\underline{\underline{P}})/\gamma_5(\underline{\underline{P}}) \cong \gamma_5(\underline{\underline{P}})/\gamma_6(\underline{\underline{P}}) \cong \ldots \cong \gamma_{n-2}(\underline{\underline{P}})/\gamma_{n-1}(\underline{\underline{P}}) \cong C_3 ,$$

where $\underline{\underline{P}}$ is of order 3^n , provided n is greater than or equal to six.

The non CF groups are investigated first. There are twenty four of these groups of order 3^6 and class 4 . They are denoted $A, B, \ldots, X$, and were obtained, together with their automorphism groups, by hand. The calculations were checked by machine, and against other calculations. They and the groups to which they give rise, up to order 3^{10} , are dealt with in Tables 1 to 5.

The eight CF groups $\underline{\underline{P}}$ of order 3^5 and class 3 , with $\underline{\underline{P}}/\underline{\underline{P}}' \cong C_3 \times C_9$ were also obtained, together with their automorphisms, both by machine and by hand. They are denoted $A, B, \ldots, H$, and they and the groups to which they give rise, up to order 3^8 , are dealt with in Tables 6 and 7.

5. Reliability of results

The nilpotent quotient algorithm has been very thoroughly tested on much larger groups than those considered here. The algorithm was highly automated to reduce the risk of human error, the results of one computation being read automatically into the next. Programming errors should show up very readily; in particular by yielding non-soluble automorphism groups, which cause the program to signal an error. Finally, all central extensions of 3-groups of maximal class have been obtained by Conlon in [3], and his calculations agree with ours. It is to be hoped that further work will verify other parts of these calculations.

6. Non CF groups

TABLE 1

Order	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
3^6	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
3^7	6*	16*	5	3	0	0	0	6*	6	0	4*	1	1	4*	8*	1	8*	1	6	1	8*	1	1*	1
3^8	0	27	0	0				45	54		0	0	0	0	9	0	15	0	6	0	15	0	0	0 .

Table 1 gives, under each group $A, B, \ldots, X$, the number of groups of order 3^7 that arise from this group by one application of the extension algorithm, and the number of groups of order 3^8 that arise by applying the extension algorithm once to the groups of order 3^7.

A star in the 3^7 row indicates that for the corresponding group $\underline{\underline{P}}$ of order 3^6, $\gamma_5(\underline{\underline{P}}^*)$ has rank 2. The reasons for placing the star on this row rather than on the row above are firstly to be consistent with Table 4, where the more natural convention would be impracticable, and secondly because the rank of $\gamma_5(\underline{\underline{P}}^*)$ is not known, until calculating the groups of order 3^7. There are five groups, order 3^5, class 3, which give rise to the groups $A, B, \ldots, X$. These five groups fall into one isoclinism class. Each of them may be generated by elements $a_1, \ldots, a_5$, subject to the relations $[a_2, a_1] = a_3$, $[a_3, a_1] = a_4$, $[a_3, a_2] = a_5$, class 3, $a_3^3 = a_4^3 = a_5^3 = e$, $a_1^3 = u$, $a_2^3 = v$, where u and v are given as follows:

A to J	K to N	O to S
$u = v = e$,	$u = e, v = a_5^2$,	$u = e, v = a_4^2$,

T, U, V	W, X
$u = a_5^2, v = a_4^2$,	$u = a_5^2, v = a_4$.

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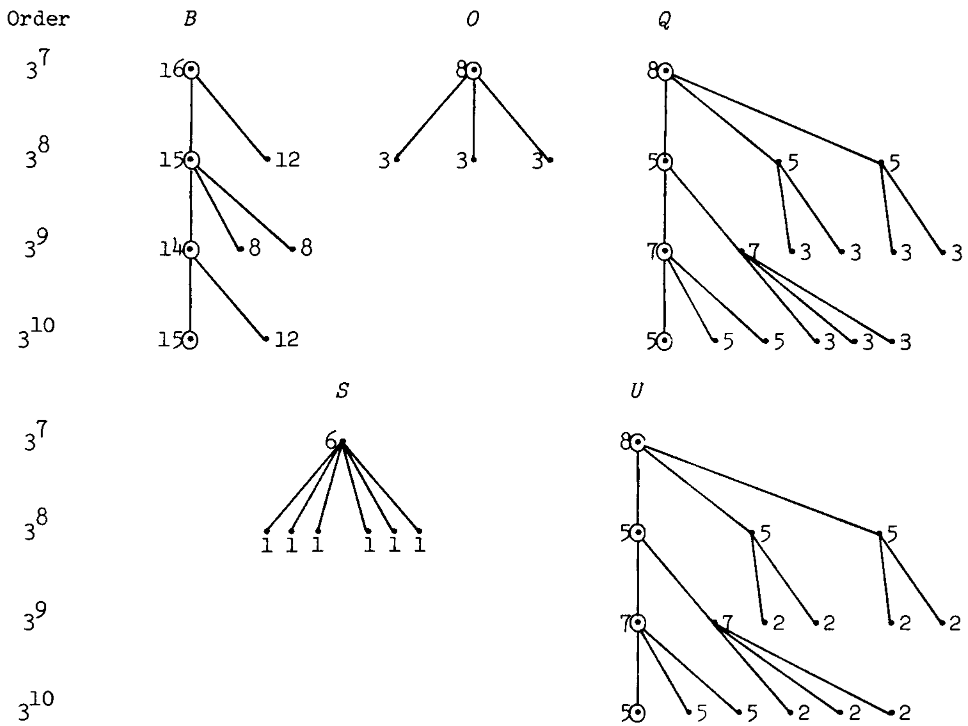
TABLE 2



In Table 2 a circle around a node indicates that some of the groups in that batch have centre of order 9 .

As can be seen from Table 1, only seven of the groups of order 3^6 give rise to groups of order 3^8 . In Table 2 groups arising from five of these, namely B, O, Q, S , and U are given while H and I are dealt with in Table 4 and Table 5. O and S give rise to no groups of order 3^9 ; the trees for B, Q , and U continue indefinitely.

Groups arising from B, O, Q, S , and U (not necessarily of order less than or equal to 3^{10}) are distinguished by the fact that $a_6^3 \equiv a_8^{\pm 1} \bmod \gamma_7$. From this it follows that γ_4 is of class at most two. Moreover, it can be shown that each of these groups has a subgroup of maximal class and index 9 . These groups are of maximal type. It can be shown that the centralizers of the quotients

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$\gamma_4/\gamma_6, \gamma_5/\gamma_7, \ldots, \gamma_{n-3}/\gamma_{n-1} = \gamma_{n-3}$ for a group of maximal type and order 3^n are all equal, and that the rank of $\gamma_{c+1}(\underline{P}^*)$ is always one. The structure of the upper central series of the groups can be read off from Tables 2 and 3. In a non CF group of order 3^n and class $n - 2$, either $\zeta_{n-i}/\zeta_{n-i-1} \cong C_3 \times C_3$ for one value of i, $n-1 \geq i \geq 4$, or $\zeta_1 \cong C_9$. In the former case, say that the group is of type E_i, in the latter that it is of type C (these groups are cyclic-by-maximal class).

For $i = n - 1$, so that $\zeta_1 \cong C_3 \times C_3$ and the group is centre-by-maximal class, every group of order 3^{n+1} arising from this is either of type C or E_{n-1} or E_n. For $i < n-1$, every group of order 3^{n+1} arising from a group of type E_i is of type E_i. The groups of type C do not give rise to new groups. Thus the upper central structure is determined once the groups of type C and those of order 3^n and type E_{n-1} have been located.

For $n = 7$, the results are given in Table 3. The types of the groups are listed in the order in which they are constructed.

The circled nodes in the graphs B, O, Q, and U in Table 2 correspond to sets of groups of order 3^n, say, some of which are of type E_{n-1} or C. Each circled node corresponding to groups of order 3^n, $n > 7$, contains exactly one group of type C. For graphs Q and U, every other group in a circled node is of type E_{n-1}. For graph B, of the groups in a circled node corresponding to groups of order 3^n, ten are of type E_{n-2} and four of type E_{n-1} for $n = 8$ or 10; seven are of type E_{n-2} and six are of type E_{n-1} for $n = 7$ or 9. In graph B, a group in a circled node giving rise to groups in an uncircled node is of type E_{n-2}; a group in a circled node giving rise to a circled node is (necessarily) of type E_{n-1}.

The groups of maximal type and order less than or equal to 3^{10} are

TABLE 3

Central structure of groups of order 3^7

A	six groups, type E_4
B	E_5, E_6, C, C, E_5, C, E_6, E_5, E_6, E_5, E_6, E_5, E_6, E_5, E_6, E_5
C	five groups, type E_4
D	E_5, E_5, C
H	six groups, type E_4
I	six groups, type E_4
K	four groups, type E_4
L	one group, type E_4
M	one group, type E_4
N	four groups, type E_4
O	E_5, C, E_5, E_5, E_5, E_5, E_5, E_5
P	one group, type E_5
Q	E_6, C, E_6, E_6, E_6, E_6, E_6, C
R	one group, type C
S	six groups, type E_5
T	one group, type C
U	E_6, C, E_6, E_6, E_6, E_6, E_6, C
V	one group, type C
W	one group, type E_4
X	one group, type E_4 .

all centre-by-metabelian. Thus being metabelian is a necessary, but not sufficient, condition for a group to give rise to new groups. However since the nodes of order 3^{10} that give rise to new groups are the left hand node in graph B and the three left most nodes in graphs Q and U, if no group at a node gives rise to new groups then no group at that node is metabelian.

In Tables 4 and 5 a circle round a node indicates that for the group

TABLE 4

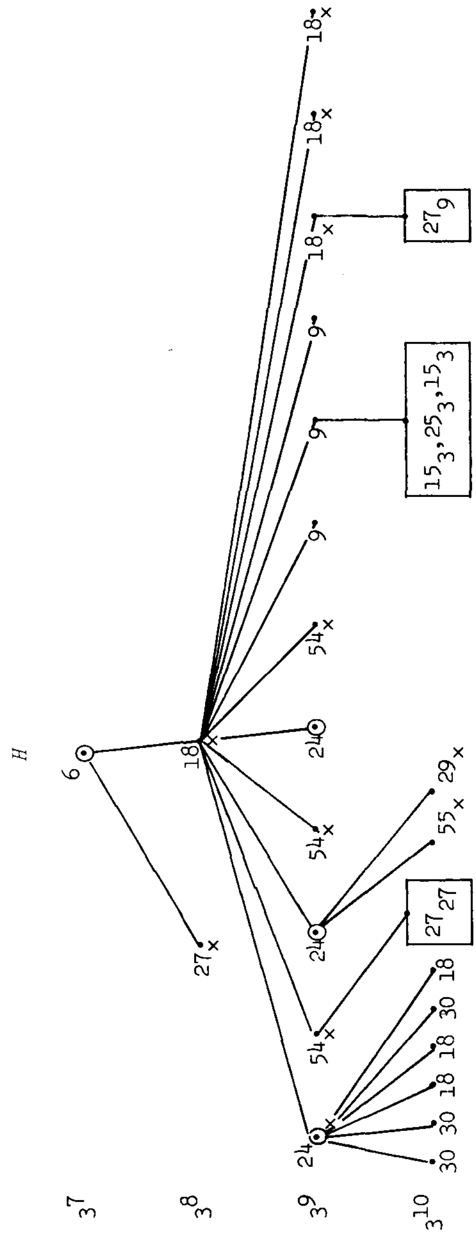
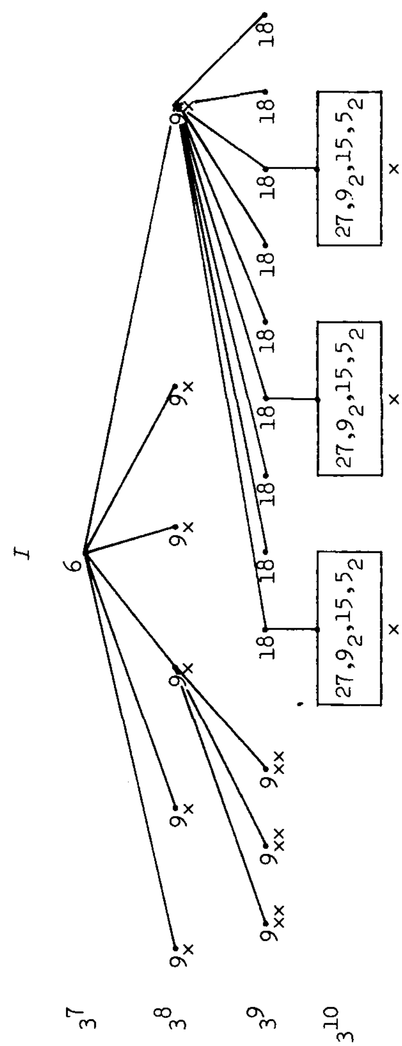


TABLE 5



I
3^7
3^8
3^9
3^{10}
6
9x
9x
9x
9x
9x
9x
9xx
9xx
9xx
18
18
18
18
18
18
18
18
18
18
$27,9_2,15,5_2$
x
$27,9_2,15,5_2$
x
$27,9_2,15,5_2$
x

$\underline{\underline{P}}$ which gave rise to the batch, $\gamma_{n+1}(\underline{\underline{P}}^*)$ has rank 2. $\boxed{a_b}$ indicates a sequence of b batches with a groups in each.

The groups in Tables 4 and 5 have the property that $\gamma_i^3 \subseteq \gamma_{i+6}$ for $i \geq 4$, and can certainly not have a subgroup of maximal class and 'small' index; however they can be shown to contain a CF subgroup of index 3 with abelianization $C_3 \times C_9$. These groups are of non-maximal type. It is known that such groups are 'wilder' and 'more numerous' than groups of maximal type, so the large number of groups in graphs H and I is to be expected.

The groups of non-maximal type and order greater than 3^7 have γ_5 as their second derived group, in contradistinction to the groups of maximal type, which are centre-by-metabelian.

The following information has been found important in examining the structure of the groups of non-maximal type. They can be generated by elements s and t such that

$$[t, s, s, s] \in \gamma_5 , \quad [t, s, t, t] \in \gamma_5 , \quad [t, s, s, t, t] \in \gamma_6 .$$

This information is used for analysing the maximal subgroups $\underline{\underline{M}}_i = C_{\underline{\underline{P}}}(\gamma_i/\gamma_{i+2})$, $4 \leq i \leq n-3$, of the group $\underline{\underline{P}}$. To calculate the maximal subgroups it is sufficient to consider the case $i = n-3$, smaller values of i being dealt with by quotients of $\underline{\underline{P}}$.

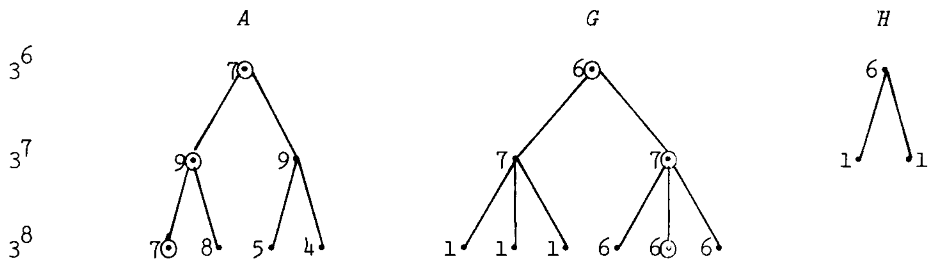
Let $\underline{\underline{C}}_i = \underline{\underline{M}}_i/\gamma_2$, so that $\underline{\underline{C}}_i = \langle s\rangle, \langle t\rangle, \langle st\rangle$, or $\langle s^2t\rangle \pmod{\gamma_2}$. For $n = 7$, $i = 4$, $\underline{\underline{C}}_i = \langle t\rangle$ by choice of t.

If a batch of groups $\underline{\underline{Q}}$ arise from $\underline{\underline{P}}$ such that the rank of $\gamma_{c+1}(\underline{\underline{P}}^*)$ is two, then clearly all four possible values of $\underline{\underline{C}}_{n-3}$ will occur in that batch; otherwise only one value of $\underline{\underline{C}}_{n-3}$ can occur in the batch. For groups of order less than or equal to 3^{10} this only occurs with groups arising from H, and any two groups in the same such batch which give rise to new groups have the same value for $\underline{\underline{C}}_{n-3}$; in Table 5 we treat such a batch as if every group in the batch had this value of $\underline{\underline{C}}_{n-3}$, and the corresponding node is circled.

A cross is placed under a node whenever $\underline{\underline{C}}_{n-3} \neq \underline{\underline{C}}_{n-4}$ for the groups in that batch. Now $\underline{\underline{C}}_i = \langle s \rangle$ or $\langle t \rangle$ for all groups in the tables and all values of i, with the above mentioned exceptions, and the exception of the groups in three batches in Table 5, marked with a double cross, which have $\underline{\underline{C}}_{n-3} = \langle st \rangle$.

7. CF groups

TABLE 6



In Table 6 a circle round a node indicates that all groups at that node have centre of order 9. All groups at other nodes have centre of order 3.

Of the eight CF groups $A, B, \ldots, H$ of order 3^5 only four give rise to groups of order 3^6. Here we deal with A, G, and H, while E is dealt with in Table 7. The graphs A and G continue indefinitely, whereas H terminates. Groups arising from A, G, H, not necessarily of order less than or equal to 3^8, are distinguished by the fact that $\left[\gamma_2 : \gamma_2^3\right] \leq 9$, from which it follows that γ_2 is of class at most two. These groups are of maximal type. All CF groups of maximal type have positive degree of commutativity; that is to say, the centralizers of the quotients $\gamma_2/\gamma_4, \gamma_3/\gamma_5, \ldots$ are all the same maximal subgroup, $\underline{\underline{M}}$ say. For the groups arising from A, $\underline{\underline{M}}/\gamma_2(\underline{\underline{P}}) \cong C_9$, and for groups arising from G and H, $\underline{\underline{M}}/\gamma_2(\underline{\underline{P}}) \cong C_3 \times C_3$.

The structure of the upper central series of the groups in Table 6 resembles that of the groups in Table 2. A group is centre-by-maximal

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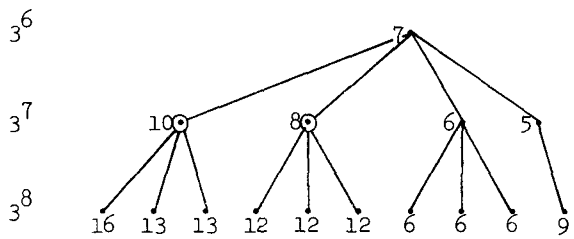
class if and only if it appears in a circled node. The circled nodes in graph A corresponding to orders 3^6 and 3^8 have two groups with centre C_9, and the node corresponding to order 3^7 has four groups with centre C_9. The circled nodes in graph G have three groups with centre C_9. The groups $\underline{\underline{P}}$ appearing in this table are all metabelian and have the rank of $\gamma_{c+1}(\underline{\underline{P}}^*)$ equal to 1.

TABLE 7

E



In Table 7 a circle round a node indicates that the groups at that node arise from a group $\underline{\underline{P}}$ for which $\gamma_{c+1}(\underline{\underline{P}}^*)$ has rank two.

Here we deal with the CF groups arising from E. This graph will continue indefinitely, and the corresponding groups are of non-maximal type. Typically, for such a group $\gamma_i^3 = \gamma_{i+6}$, at least for i large enough. A precise statement, with proof, [6], and examples, will appear elsewhere. The wreath product $C_3 \text{ wr } C_9$ has $\gamma_2^3 = \gamma_{10} = \langle e \rangle \neq \gamma_9$ and is an example of a group of non-maximal type.

M.F. Newman has constructed, in an unpublished note, three infinite chains of CF groups $\underline{\underline{P}}$ of non-maximal type distinguished by their prime power structure, with $\underline{\underline{P}}/\gamma_2(\underline{\underline{P}}) \cong C_3 \times C_9$.

All groups in Table 7, except for some at circled nodes, have positive degree of commutativity, with a_1 in every two-step centralizer. The groups that appear in the table are all centre-by-metabelian; however it can be shown that CF groups of non-maximal type with arbitrarily large second derived groups exist.

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