

THE (CO)HOMOLOGY OF GROUPS GIVEN BY PRESENTATIONS IN WHICH EACH DEFINING RELATOR INVOLVES AT MOST TWO TYPES OF GENERATORS

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Abstract

Our set-up will consist of the following: (i) a graph with vertex set V and edge set E ; (ii) for each vertex $v \in V$ a non-trivial group G_v given by a presentation $\langle x_v; r_v \rangle$; (iii) for each edge $e = \{u, v\} \in E$ a group G_e given by a presentation $\langle x_u, x_v; r_e \rangle$ where r_e consists of the elements of $r_u \cup r_v$ together with some further words on $x_u \cup x_v$. We let $G = \langle x; r \rangle$ where $x = \bigcup_{v \in V} x_v$, $r = \bigcup_{e \in E} r_e$. Our aim is to try to describe the structure of G in terms of the groups G_v ($v \in V$), G_e ($e \in E$). Under suitable conditions the natural homomorphisms $G_v \rightarrow G$ ($v \in V$), $G_e \rightarrow G$ ($e \in E$) are injective; and there is a short exact sequence

$$0 \rightarrow \bigoplus_{v \in V} (\mathbb{Z}G \otimes_{G_v} IG_v)^{|\text{Star}(v)|-1} \rightarrow \bigoplus_{e \in E} (\mathbb{Z}G \otimes_{G_e} IG_e) \rightarrow IG \rightarrow 0$$

(where, for any group H , IH is the augmentation ideal). Some (co)homological consequences of these results are derived.

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1. Introduction

In [8], [9], [10] we considered groups given by presentations where the generating set is partitioned into non-empty disjoint subsets, and each defining relator involves symbols from exactly two of these subsets. Such a group G can be expressed in the form

$$\langle F_v (v \in V); \mathbf{t} \rangle,$$

where each F_v is a non-trivial free group, and \mathbf{t} is a set of cyclically reduced elements of $*_{v \in V} F_v$, with each element of \mathbf{t} involving symbols from exactly two factors. Associated with G we have a graph Γ with vertex set V and edge set

$$E = \{\{u, v\} : \text{some element of } \mathbf{t} \text{ involves symbols from } F_u \text{ and } F_v\}.$$

For $e = \{u, v\} \in E$ we have a group $G_e = \langle F_u, F_v; \mathbf{t}_e \rangle$, where \mathbf{t}_e consists of all elements of \mathbf{t} involving symbols from F_u and F_v .

$$\underbrace{F_u \quad G_e \quad F_v}_{\text{homomorphic image}} \quad (G_e \text{ a homomorphic image of } F_u * F_v).$$

The idea is then to try to describe the structure of G in terms of the groups G_e .

In this paper we replace the free groups F_v by arbitrary non-trivial groups G_v . Our set-up will thus consist of the following.

(a) A graph Γ with vertex set V and edge set E . We assume for convenience that no vertex of Γ is isolated.

(b) For each vertex $v \in V$ a group G_v given by a presentation $\langle \mathbf{x}_v; \mathbf{r}_v \rangle$, where the elements of \mathbf{r}_v are cyclically reduced words on \mathbf{x}_v .

(c) For each edge $e = \{u, v\} \in E$, a group G_e given by a presentation $\langle \mathbf{x}_u, \mathbf{x}_v; \mathbf{r}_e \rangle$ where \mathbf{r}_e consists of the elements of $\mathbf{r}_u \cup \mathbf{r}_v$, together with some cyclically reduced words each involving at least one \mathbf{x}_u -symbol and at least one \mathbf{x}_v -symbol.

We let

$$G = \langle \mathbf{x}; \mathbf{r} \rangle$$

where

$$\mathbf{x} = \bigcup_{v \in V} \mathbf{x}_v, \quad \mathbf{r} = \bigcup_{e \in E} \mathbf{r}_e.$$

We can assume without loss of generality that \mathbf{r} is *slender*, that is, if $R \in \mathbf{r}$ then no cyclic permutation of $R^{\pm 1}$ except for R itself belongs to \mathbf{r} .

EXAMPLE 1. If each G_v is cyclic of order 2 and each G_e is finite dihedral, then G is a *Coxeter group*.

This example in fact motivated the work for this paper. In [11], R. Stöhr and I obtained a short exact sequence connecting the augmentation ideal of certain Coxeter groups (*aspherical Coxeter groups*) with the augmentation ideals of the cyclic and dihedral groups G_v ($v \in V$), G_e ($e \in E$), and we used this to obtain results about (co)homology. The present paper arose from the realisation that the computations in [11] can be extended to the general set-up described above.

We will need to make two assumptions, which will be explained shortly. Later (see Theorem 3) we will give some fairly general sufficient conditions which ensure that our two assumptions hold.

ASSUMPTION 1. The natural maps $G_v \rightarrow G$ ($v \in V$), $G_e \rightarrow G$ ($e \in E$) are injective (and so we regard the G_v and G_e as subgroups of G).

To state the second assumption we need some definitions and notation (which will be used throughout the rest of the paper).

Write $G = F/N$, where F is the free group on x and N is the normal closure of r in F . We let M denote the relation module for the given presentation of G . Thus M is the left $\mathbb{Z}G$ -module with underlying abelian group $N^{ab} = N/N'$, and G -action

$$WN \cdot UN' = WUW^{-1}N' \quad (W \in F, U \in N).$$

Similarly for $v \in V, e \in E$ we have relation modules M_v, M_e (which are, respectively, left $\mathbb{Z}G_v$ -, $\mathbb{Z}G_e$ -modules) corresponding to the presentations of G_v, G_e . We also have the submodules $\overline{M}_v, \overline{M}_e$ of M generated by $\{RN' : R \in r_v\}, \{RN' : R \in r_e\}$ respectively.

For $e \in E$, let P_e be the free left $\mathbb{Z}G$ -module with basis $\{t_R^e : R \in r_e\}$, and let K_e be the kernel of the epimorphism

$$P_e \rightarrow \overline{M}_e, t_R^e \mapsto RN' \quad (R \in r_e).$$

Let P be the free left $\mathbb{Z}G$ -module with basis $\{t_R : R \in r\}$, and let K be the kernel of the epimorphism

$$P \rightarrow M, t_R \mapsto RN' \quad (R \in r).$$

Now we have an epimorphism

$$\alpha : \bigoplus_{e \in E} P_e \rightarrow P$$

given by

$$t_R^e \mapsto t_R \quad (e \in E, R \in r_e).$$

This obviously carries $\bigoplus_{e \in E} K_e$ into K .

ASSUMPTION 2. α carries $\bigoplus_{e \in E} K_e$ onto K .

We will have more to say about this assumption at the end of this introduction. The assumption is intimately connected with the structure of identity sequences over the presentation of G .

In the statements of Theorems 1 and 2, and the Corollaries, below, it will be supposed that Assumptions 1 and 2 hold.

THEOREM 1. *There is a short exact sequence*

$$0 \rightarrow \bigoplus_{v \in V} (\mathbb{Z}G \otimes_{G_v} M_v)^{|S(v)|-1} \rightarrow \bigoplus_{e \in E} \mathbb{Z}G \otimes_{G_e} M_e \rightarrow M \rightarrow 0.$$

(Here $S(v)$ is the adjacency set of v , that is, the set of edges of Γ incident with v .)

COROLLARY 1.1. *There is a short exact sequence*

$$0 \rightarrow \bigoplus_{v \in V} (\mathbb{Z}G \otimes_{G_v} IG_v)^{|S(v)|-1} \rightarrow \bigoplus_{e \in E} \mathbb{Z}G \otimes_{G_e} IG_e \rightarrow IG \rightarrow 0.$$

(Here, for any group H , IH is the augmentation ideal.)

COROLLARY 1.2. *Let A be any right $\mathbb{Z}G$ -module, and let B be any left $\mathbb{Z}G$ -module.*

(i) *There is a long exact sequence*

$$\begin{aligned} \dots \rightarrow H_{n+1}(G, A) &\rightarrow \bigoplus_{v \in V} H_n(G_v, A)^{|S(v)|-1} \rightarrow \bigoplus_{e \in E} H_n(G_e, A) \\ &\rightarrow H_n(G, A) \rightarrow \dots \end{aligned}$$

terminating in

$$\dots \rightarrow H_2(G, A) \rightarrow \bigoplus_{v \in V} (A \otimes_{G_v} IG_v)^{|S(v)|-1} \rightarrow \bigoplus_{e \in E} A \otimes_{G_e} IG_e \rightarrow A \otimes_G IG \rightarrow 0.$$

(ii) *There is a long exact sequence*

$$\begin{aligned} \dots \rightarrow H^n(G, B) &\rightarrow \prod_{e \in E} H^n(G_e, B) \rightarrow \prod_{v \in V} H^n(G_v, B)^{|S(v)|-1} \\ &\rightarrow H^{n+1}(G, B) \rightarrow \dots \end{aligned}$$

starting with

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathbb{Z}G}(IG, B) &\rightarrow \prod_{e \in E} \text{Hom}_{\mathbb{Z}G_e}(IG_e, B) \\ &\rightarrow \prod_{v \in V} (\text{Hom}_{\mathbb{Z}G_v}(IG_v, B))^{|S(v)|-1} \rightarrow H^2(G, B) \rightarrow \dots \end{aligned}$$

REMARK. In the above Corollary, $()^{|S(v)|-1}$ is to be interpreted as direct sum in(i), and direct product in (ii).

COROLLARY 1.3. *Suppose there exists $n \geq 1$ such that G_v has cohomological dimension less than or equal to n for all $v \in V$. Then any finite subgroup of G is contained in a conjugate of some subgroup G_e ($e \in E$).*

A proof of Theorem 1 will be given in Section 2.

The proof of Corollary 1.1 is similar to arguments used in the proof of [11, Theorem 4]. (Details are left to the reader.) The proof of Corollary

1.2 is obtained by considering the long exact Tor and Ext sequences (in the second and first variables respectively) derived from the short exact sequence in Corollary 1.1, and then using dimension shifting arguments and Shapiro’s Lemma. The computations are similar to those in [11, Section 4], and are left to the reader. To prove Corollary 1.3, observe that under the stated conditions we have

$$H^{n+2}(G, B) \cong \prod_{e \in E} H^{n+2}(G_e, B)$$

for all left $\mathbb{Z}G$ -modules B . Now use Serre’s Theorem [6].

Before stating the next theorem, we recall the definition of a *combinatorially aspherical group* (for further information see [2] or [3]).

Let a group H be given by a presentation $\langle y; s \rangle$, so that H is (isomorphic to) Φ/Σ , where Φ is the free group on y and Σ is the normal closure of s in Φ . For $S \in s$ write $S = \dot{S}^{p(S)}$, where \dot{S} is not a proper power and $p(S)$ is a positive integer. Let X be the free left $\mathbb{Z}H$ -module with basis $\{b_S : S \in s\}$. We say that H (or more precisely the given presentation of H) is *combinatorially aspherical* (CA) if the kernel of the epimorphism

$$b_S \mapsto S\Sigma' \quad (S \in s)$$

from X onto the relation module Σ/Σ' , is generated by the elements $(1 - \dot{S}\Sigma)b_S$ ($S \in s$).

THEOREM 2. *If each G_e is CA then G is CA.*

This theorem will be proved in Section 3.

We now give conditions which guarantee that Assumptions 1 and 2 hold. Let $e = \{u, v\}$ be an edge of Γ . We will say that G_e (or more precisely the given presentation of G_e) has *property- W_k* if no non-trivial element of $G_u * G_v$ of free product length less than or equal to $2k$ lies in the kernel of the natural epimorphism $G_u * G_v \rightarrow G_e$.

EXAMPLE 2 (E. Fennessey). For $e = \{u, v\}$, let D_e denote the cartesian subgroup of $G_u * G_v$ (that is, D_e is the kernel of the natural epimorphism $G_u * G_v \rightarrow G_u \times G_v$). Let $\hat{r}_e = r_e - (r_u \cup r_v)$, where we regard the elements of \hat{r}_e as belonging to $G_u * G_v$. Then G_e has property- W_1 if $\hat{r}_e \subseteq D_e$, and G_e has property- W_2 if $\hat{r}_e \subseteq D_e^{p(e)} D_e'$ for some prime $p(e)$.

EXAMPLE 3 (see M. Edjvet [4, Example 4.1]). For $v \in V$, let x_v consist of a single element (which we take to be v itself). Let φ be a mapping from $V \cup E$ into $\{2, 3, 4, \dots\}$, and let

$$G_e = \langle u, v; u^{\varphi(u)}, v^{\varphi(v)}, (uv)^{\varphi(e)} \rangle \quad (e = \{u, v\} \in E).$$

Then G_e always has property- W_1 , and has property- W_2 if $\varphi(e) > 2$.

THEOREM 3. *Assumptions 1 and 2 hold if either of the following conditions is satisfied.*

- (I) *Each G_e has property- W_2 .*
- (II) *Γ is triangle-free and each G_e has property- W_1 .*

REMARKS. (1) A proof of the theorem will be given in Section 4. Another proof (based on extending results on van Kampen and spherical diagrams over free groups obtained in [8], [9], [10], to analogous results for diagrams over free products) has been given by E. Fennessey [5].

(2) Note that Theorem 3 applies to many Coxeter groups (take $\varphi(v) = 2$ for all $v \in V$ in Example 3). We remark that, for Coxeter groups, Assumption 1 always holds [12, p. 245]. To guarantee that Assumption 2 holds we can get by with a weaker condition than (I)/(II), namely the *asphericity condition* [11] referred to earlier.

(3) We can, in fact, formulate an asphericity condition for our general set-up, as follows. For $e = \{u, v\} \in E$, let m_e be the length of the shortest non-trivial element of $\text{Ker}(G_u * G_v \rightarrow G_e)$. (Note that either $m_e = 1$ (which means that one of the natural homomorphisms $G_u \rightarrow G_e, G_v \rightarrow G_e$ is not injective), or m_e is even.) The *asphericity condition* would require that:

- (i) $m_e \geq 4$ for all e (that is, each G_e has property- W_1);
- (ii) for any triangle Γ (with edges e_1, e_2, e_3 , say)

$$\frac{1}{m_{e_1}} + \frac{1}{m_{e_2}} + \frac{1}{m_{e_3}} \leq \frac{1}{2}.$$

It seems plausible that an analogue of Theorem 3 could be obtained under the weaker asphericity condition.

To complete this introduction we now discuss Assumption 2 in more detail. We first briefly review some basic ideas concerning identity sequences (for further information, see [2], especially Sections 2, 3).

Let \mathbf{w} be the set of words on \mathbf{x} (reduced or not). For $\mathbf{s} \subseteq \mathbf{r}$, let $\mathbf{s}^{\mathbf{w}}$ denote the subset of \mathbf{w} consisting of all words of the form

$$WS^{\varepsilon}W^{-1} \quad (W \in \mathbf{w}, S \in \mathbf{s}, \varepsilon = \pm 1).$$

If $C = WR^{\varepsilon}W^{-1}$ is an element of $\mathbf{r}^{\mathbf{w}}$ then we define $\text{eval}(C)$ to be the element $\varepsilon W N t_R$ of the free module P . If $\sigma = (C_1, C_2, \dots, C_n)$ is a finite sequence of elements of $\mathbf{r}^{\mathbf{w}}$ then we define $\text{eval}(\sigma)$ to be $\sum_{i=1}^n \text{eval}(C_i)$. If $V \in \mathbf{w}$ then σ^V is the sequence $(VC_1V^{-1}, VC_2V^{-1}, \dots, VC_nV^{-1})$.

A finite sequence (C_1, C_2, \dots, C_n) of elements $C_i = W_i R_i^{\varepsilon_i} W_i^{-1}$ of elements of $\mathbf{r}^{\mathbf{w}}$ is called an *identity sequence* if $C_1 C_2 \dots C_n$ is freely equal to 1. We define operations on such sequences as follows.

- (1) Replace W_i by a word freely equal to it for $i = 1, \dots, n$.

(2) Delete two successive terms C_j, C_{j+1} if C_{j+1} is identically equal to C_j^{-1} .

(3) Replace two successive terms C_j, C_{j+1} by either $C_{j+1}, C_{j+1}^{-1}C_jC_{j+1}$ or $C_jC_{j+1}C_j^{-1}, C_j$. Two identity sequences are said to be *Peiffer equivalent* if one can be obtained from the other by a finite number of operations (1), (2), (2)⁻¹ (the operation inverse to (2)), and (3). The Peiffer equivalence class of a sequence σ will be denoted by $\langle \sigma \rangle$. The set of Peiffer equivalence classes is a left $\mathbb{Z}G$ -module with addition defined by

$$\langle \sigma \rangle + \langle \tau \rangle = \langle \sigma\tau \rangle$$

and G -action defined by

$$WN \cdot \langle \sigma \rangle = \langle \sigma^W \rangle.$$

This $\mathbb{Z}G$ -module is isomorphic to K under the mapping

$$\langle \sigma \rangle \mapsto \text{eval}(\sigma).$$

Consider the following assumption.

ASSUMPTION 2'. Every identity sequence σ is Peiffer equivalent to a product $\sigma_1\sigma_2 \dots \sigma_k$ of identity sequences σ_i ($1 \leq i \leq k$), where for $i = 1, \dots, k$ there is an edge e_i of Γ such that all terms of σ_i belong to $\mathfrak{r}_{e_i}^W$.

It is easy to see that Assumption 2' implies Assumption 2. Moreover, one can show that if Assumption 1 holds then Assumption 2 implies Assumption 2' (see the second remark at the end of Section 3). Thus *Assumptions 1 and 2 are equivalent to Assumptions 1 and 2'*.

2. Proof of Theorem 1

We will need the following results.

LEMMA 1. *The submodules \overline{M}_v ($v \in V$) generate their direct sum in M .*

PROOF. Let Q be the free left $\mathbb{Z}G$ -module with basis $\{b_x : x \in \mathfrak{x}\}$. There is a standard short exact sequence

$$(2.1) \quad 0 \rightarrow M \xrightarrow{\mu} Q \rightarrow IG \rightarrow 0$$

where the embedding μ is given by

$$WN' \mapsto \sum_{x \in \mathfrak{x}} \pi \frac{\partial W}{\partial x} b_x \quad (W \in F),$$

with $\partial/\partial x$ Fox differentiation, and π the natural projection $\mathbb{Z}F \rightarrow \mathbb{Z}G$ [1, p. 43]. The image of \overline{M}_v under μ lies in the submodule of Q generated by $\{b_x : x \in \mathbf{x}_v\}$.

LEMMA 2. *For $e \in E$, $v \in V$ we have $\overline{M}_e \cong \mathbb{Z}G \otimes_{G_e} M_e$, $\overline{M}_v \cong \mathbb{Z}G \otimes_{G_v} M_v$.*

PROOF. Let $e = \{u, v\} \in E$, and let Q_e be the free left $\mathbb{Z}G_e$ -module with basis $\{b_x^e : x \in \mathbf{x}_u \cup \mathbf{x}_v\}$. Then we have a short exact sequence

$$0 \rightarrow M_e \xrightarrow{\mu_e} Q_e \rightarrow IG_e \rightarrow 0,$$

analogous to (2.1). Tensoring by the free right $\mathbb{Z}G_e$ -module $\mathbb{Z}G$ we get the exact sequence

$$0 \rightarrow \mathbb{Z}G \otimes_{G_e} M_e \xrightarrow{1 \otimes \mu_e} \mathbb{Z}G \otimes_{G_e} Q_e \rightarrow \mathbb{Z}G \otimes_{G_e} IG_e \rightarrow 0.$$

Now $\mathbb{Z}G \otimes_{G_e} Q_e$ is isomorphic to the submodule of Q generated by $\{b_x : x \in \mathbf{x}_u \cup \mathbf{x}_v\}$, under the mapping $1 \otimes b_x^e \mapsto b_x$. The image of $(1 \otimes \mu_e)(\mathbb{Z}G \otimes_{G_e} M_e)$ under this isomorphism is precisely $\mu(\overline{M}_e)$. Thus $\overline{M}_e \cong \mathbb{Z}G \otimes_{G_e} M_e$, as required.

The other result is similar.

Assumption 2 implies that the kernel of the induced homomorphism

$$\alpha^* : \frac{\bigoplus P_e}{\bigoplus K_e} \rightarrow \frac{P}{K}$$

consists of the cosets $\text{mod } \bigoplus K_e$ of the elements of $\text{Ker } \alpha$. We note that $\text{Ker } \alpha$ is (freely) generated by the elements

$$t_R^e - t_R^{e(v)} \quad (v \in V, e \in S'(v), R \in \mathbf{r}_v).$$

(Here $e(v)$ is some fixed element of $S(v)$, and $S'(v) = S(v) - \{e(v)\}$.)

Now $\bigoplus P_e / \bigoplus K_e \cong \bigoplus \overline{M}_e$. Let $f = \{u, v\}$ be an edge of Γ and let \overline{M}_u^f be the module \overline{M}_u regarded as a submodule of the factor \overline{M}_f of $\bigoplus \overline{M}_e$, and similarly for v . Then $\overline{M}_u^f \cap \overline{M}_v^f = 0$, by Lemma 1, so we get the submodule

$$\begin{aligned} L &= \bigoplus_{\{u,v\} \in E} (\overline{M}_u^{\{u,v\}} \oplus \overline{M}_v^{\{u,v\}}) \\ &= \bigoplus_{v \in V} \bigoplus_{e \in S(v)} \overline{M}_v^e \end{aligned}$$

of $\bigoplus \overline{M}_e$. The submodule of L corresponding to $\text{Ker } \alpha^*$ is generated by the elements

$$m^e - m^{e(v)} \quad (v \in V, \quad e \in S'(v), \quad m \in \overline{M}_v).$$

(Here m^e is the element m of \overline{M}_v thought of as an element of the factor \overline{M}_e of $\bigoplus \overline{M}_e$, and similarly for $m^{e(v)}$.) Now there is an automorphism of L defined by

$$\begin{aligned} m^e &\mapsto m^e - m^{e(v)} && (v \in V, \quad e \in S'(v), \quad m \in \overline{M}_v), \\ m^{e(v)} &\mapsto m^{e(v)} && (v \in V, \quad m \in \overline{M}_v). \end{aligned}$$

The image of $\bigoplus_{v \in V} \bigoplus_{e \in S'(v)} \overline{M}_v^e$ under this automorphism is $\text{Ker } \alpha^*$. Consequently we have a short exact sequence

$$0 \rightarrow \bigoplus_{v \in V} \bigoplus_{e \in S'(v)} \overline{M}_v^e \rightarrow \bigoplus_{e \in E} \overline{M}_e \rightarrow M \rightarrow 0.$$

Now Theorem 1 follows using Lemma 2.

3. Proof of Theorem 2

For $e = \{u, v\}$ an edge of Γ , write $G_e = F_e/N_e$, where F_e is the free group on $x_u \cup x_v$, and N_e is the normal closure of r_e in F_e . The relation module M_e is then N_e/N'_e . Let A_e be the free left $\mathbb{Z}G$ -module with basis $\{a_R^e : R \in r_e\}$. By assumption the kernel B_e of the epimorphism

$$A_e \rightarrow M_e, \quad a_R^e \mapsto RN'_e \quad (R \in r_e)$$

is generated by the elements $(1 - \dot{R}N_e)a_R^e$ ($R \in r_e$). Tensoring the exact sequence

$$0 \rightarrow B_e \rightarrow A_e \rightarrow M_e \rightarrow 0$$

by the free right $\mathbb{Z}G_e$ -module $\mathbb{Z}G$ gives the exact sequence

$$0 \rightarrow \mathbb{Z}G \otimes_{G_e} B_e \rightarrow \mathbb{Z}G \otimes_{G_e} A_e \rightarrow \mathbb{Z}G \otimes_{G_e} M_e \rightarrow 0.$$

Now we have isomorphisms

$$\begin{aligned} \mathbb{Z}G \otimes_{G_e} A_e &\xrightarrow{1 \otimes a_R^e \mapsto t_R^e} P_e, \\ \mathbb{Z}G \otimes_{G_e} M_e &\xrightarrow{1 \otimes RN'_e \mapsto RN'} \overline{M}_e \end{aligned}$$

(the latter by Lemma 2). Thus we obtain an isomorphism

$$(3.1) \quad \mathbb{Z}G \otimes_{G_e} B_e \rightarrow K_e$$

carrying the generating set

$$\{1 \otimes (1 - \dot{R}N_e)a_R^e : R \in \mathfrak{r}_e\}$$

of $\mathbb{Z}G \otimes_{G_e} B_e$ to

$$\{(1 - \dot{R}N)t_R^e : R \in \mathfrak{r}_e\}.$$

Since $\alpha : \bigoplus K_e \rightarrow K$ is onto, we deduce that K is generated by the elements

$$(1 - \dot{R}N)t_R \quad (R \in \mathfrak{r}_e, e \in E),$$

as required.

REMARKS. (1) We can obviously prove the following result by similar arguments.

Suppose Assumptions 1 and 2 hold. For $e \in E$, let X_e be a set of generators for the $\mathbb{Z}G$ -module B_e . Then $1 \otimes_{G_e} X_e = \{1 \otimes x : x \in X_e\}$ is a set of generators for the $\mathbb{Z}G$ -module $\mathbb{Z}G \otimes_{G_e} B_e$, and so the image of $\bigcup_{e \in E} 1 \otimes_{G_e} X_e$ under the composition

$$(3.2) \quad \bigoplus_{e \in E} \mathbb{Z}G \otimes_{G_e} B_e \xrightarrow{(3.1)} \bigoplus_{e \in E} K_e \xrightarrow{\alpha} K$$

is a generating set of K .

(2) In particular, for $e = \{u, v\} \in E$, let w_e be the set of words on $x_u \cup x_v$. Let Σ_e be the set of identity sequences of elements of $\mathfrak{r}_e^{w_e}$. If $\tau = (W_1 R_1^{e_1} W_1^{-1}, \dots, W_n R_n^{e_n} W_n^{-1})$ is an element of Σ_e , let $\text{eval}_e(\tau)$ denote the element $\varepsilon_1 W_1 N_e a_{R_1}^e + \dots + \varepsilon_n W_n N_e a_{R_n}^e$ of A_e . Then $B_e = \{\text{eval}_e(\tau) : \tau \in \Sigma_e\}$. The image of $\{1 \otimes \text{eval}_e(\tau) : \tau \in \Sigma_e, e \in E\}$ under (3.2) is $\{\text{eval}(\tau) : \tau \in \Sigma_e, e \in E\}$. From this we deduce that if σ is an identity sequence of elements of \mathfrak{r}^w then

$$\text{eval}(\sigma) = V_1 N \text{eval}(\tau_1) + \dots + V_m N \text{eval}(\tau_m)$$

for certain elements $V_i N$ of G and elements τ_i of $\bigcup_{e \in E} \Sigma_e$. Thus

$$\langle \sigma \rangle = \langle \tau_1^{V_1} \dots \tau_m^{V_m} \rangle.$$

4. Proof of Theorem 3

For $e = \{u, v\} \in E$, we will write $\hat{\mathfrak{r}}_e$ for the set $\mathfrak{r}_e - (\mathfrak{r}_u \cup \mathfrak{r}_v)$, and we will denote the union of all the $\hat{\mathfrak{r}}_e$'s by $\hat{\mathfrak{r}}$. The free product $G_u * G_v$ will be denoted by \tilde{G}_e .

We will be interested in *diagrams* over the presentation $\langle \mathbf{x}; \mathfrak{r} \rangle$ of G . (For the general theory of diagrams over presentations, see [7, Chapter V].) Our

main concern will be with what we will call r -discs and r -spheres. An r -disc (respectively r -sphere) is a finitely tessellated disc (respectively, sphere) whose oriented edges are labelled by elements of $x \cup x^{-1}$, and where each region has a boundary path labelled by an element of r .

Let M be an r -disc or r -sphere. We let $E(M)$ denote the set of regions of M which are labelled by elements of \hat{r} . If $\Delta \in E(M)$, and if the label on Δ belongs to \hat{r}_e then we define $e(\Delta)$ to be e . For any path α in M , we let $tp(\alpha)$ denote the set of all $v \in V$ for which some element of x_v occurs in the label of α .

Suppose $E(M) \neq \emptyset$. Choose some $\Delta \in E(M)$, and consider the collection of all regions Δ' of M for which the following holds: there exist regions $\Delta = \Delta_0, \Delta_1, \dots, \Delta_n = \Delta'$ with $tp(\partial\Delta_i) \subseteq e(\Delta)$ ($i = 1, \dots, n$), and where Δ_i, Δ_{i+1} have an edge in common for $i = 0, \dots, n - 1$. We call this collection, and also the subdiagram F of M which it determines, a *federation*. We define $e(F)$ to be $e(\Delta)$.

Let F_1 be a federation of M . If $F_1 \neq M$, then construct a federation F_2 of $M - F_1$. (The reader at this point may object, since we have only defined the concept of federation for r -discs and r -spheres. However, it is clear that we can define the concept of federation for subdiagrams of r -discs.) If $F_2 \neq M - F_1$, then choose a federation F_3 of $M - (F_1 \cup F_2)$, and so on. Eventually, we will get a collection F_1, \dots, F_n of subdiagrams of M which cover M , with F_{i+1} a federation of $M - \bigcup_{j=1}^i F_j$. We call $\{F_i\}_{i=1}^n$ a *federal subdivision* of M .

LEMMA 3. *Let D be an r -disc having a federal subdivision $\{F_i\}_{i=1}^n$ with the following properties: (i) each F_i is simply connected; (ii) the label on ∂F_i defines a non-trivial element of the kernel of the natural epimorphism $\tilde{G}_{e(F_i)} \rightarrow G_{e(F_i)}$ ($i = 1, \dots, n$). Then there exists some F_j such that $F_j \cap \partial D$ contains at least one edge, and $e(F_j) \subseteq tp(\partial D)$.*

This is a special case of the proof of the main lemma of Edjvet's paper [4]. (We remark that, strictly speaking, Edjvet's result only applies when each x_v is a singleton. However, the extension to the more general situation where the x_v 's may have more than one element is easy.)

The proof that Assumption 1 holds now follows from Lemma 3 using arguments similar to those in the proof of Theorem 1 of [6] (see also the proof of Theorem 4 of [6]).

LEMMA 4. *Let S be an r -sphere having a federal subdivision $\{F_i\}_{i=1}^n$ with $n > 1$. Then there is a simply connected federation F whose boundary label*

defines the identity of $\text{Ker}(\tilde{G}_{e(\mathbf{F})} \rightarrow G_{e(\mathbf{F})})$. Moreover, for any given vertex of \mathbf{S} , \mathbf{F} can be chosen so that that vertex does not lie in the interior of \mathbf{F} .

PROOF. By arguing as on p. 573 of [9], we can find some federation \mathbf{F}_p and a component \mathbf{C} of $\mathbf{S} - \mathbf{F}_p$ with all $\mathbf{F}_j \subseteq \mathbf{C}$ simply connected. Suppose that for each $\mathbf{F}_j \subseteq \mathbf{C}$ the boundary label of \mathbf{F}_j defines a *non-trivial* element of $\text{Ker}(\tilde{G}_{e(\mathbf{F}_j)} \rightarrow G_{e(\mathbf{F}_j)})$ (the boundary label must define an element of this kernel). Then by Lemma 1 (applied to \mathbf{C}), there exists $\mathbf{F}_q \subseteq \mathbf{C}$ with $\mathbf{F}_q \cap \partial\mathbf{C}$ containing at least one edge, and $e(\mathbf{F}_q) \subseteq tp(\partial\mathbf{C})$. Now $tp(\partial\mathbf{C}) \subseteq e(\mathbf{F}_p)$, so $e(\mathbf{F}_q) = e(\mathbf{F}_p)$. Since \mathbf{F}_q and \mathbf{F}_p have an edge in common, this contradicts the fact that \mathbf{F}_p and \mathbf{F}_q are federations (they should have been incorporated into one larger federation!).

The last statement of the lemma follows as on the bottom of p. 104 of [10].

We will now show that Assumption 2' holds.

If σ is an identity sequence then we define the *rank* $\text{rk}(\sigma)$ of σ to be the ordered pair (n_1, n_2) , where n_1 is the number of terms belonging to $\hat{\mathbf{r}}^w$, and n_2 is the number of terms belonging to $\mathbf{r}^w - \hat{\mathbf{r}}^w$. The proof is by induction on rank (where ordered pairs are given the lexicographical ordering).

If $\text{rk}(\sigma) = (0, 0)$ there is nothing to prove.

Suppose $\text{rk}(\sigma) > (0, 0)$. We are going to use the theory of *spherical pictures*, which is a standard geometric tool for detailing with identity sequences (see [2], [10], for the general theory).

Construct a spherical picture \mathbf{P} , having a spray whose associated identity sequence is σ (see [10, p. 103]). Let \mathbf{P}' be a spherical subpicture of \mathbf{P} containing at least one disc of \mathbf{P} , and which is minimal with respect to this property.

Assume first that some disc of \mathbf{P}' is labelled by an element of $\hat{\mathbf{r}}$. Then dualising \mathbf{P}' to obtain an r -sphere, and arguing as in [10, pp. 103–105] (making use of Lemma 4 above), we can find a simple closed transverse path δ in \mathbf{P} with the following properties.

(4.1) δ does not intersect any disc of \mathbf{P} , and does not intersect the boundary of \mathbf{P} .

(4.2) There is at least one disc of \mathbf{P} inside δ .

(4.3) There is some $e = \{u, v\} \in E$ such that all discs inside δ are labelled by elements of \mathbf{r}_e .

(4.4) At least one disc inside δ is labelled by an element of $\hat{\mathbf{r}}_e$.

(4.5) The label U on δ defines the identity of $\tilde{G}_e = G_u * G_v$.

Let k, l be the number of discs inside and outside δ respectively. Then

we can construct a spray for \mathbf{P} where the first k paths are connected to the discs inside δ and the remaining l paths are connected to the discs outside δ , and where the identity sequence $\sigma' = (C_1, C_2, \dots, C_k, C_{k+1}, \dots, C_{k+l})$ associated with the spray is such that $C_1 C_2 \dots C_k$ is freely equal to a conjugate ZUZ^{-1} of U (see [10, pp. 105–106]). It is important to note that σ and σ' are Peiffer equivalent. For since σ and σ' both arise from sprays, $\sigma^{-1}\sigma'$ has the primary identity property as defined in [2, p. 177] (see [10], especially statement (1.3) on p. 102, in this regard). Hence [2, Proposition 16(ii)] applies.

Now since U defines the identity of $G_u * G_v$ there is a sequence (D_1, \dots, D_m) of elements of $(\mathbf{r}_u \cup \mathbf{r}_v)^{\mathbb{W}}$ such that $D_1 D_2 \dots D_m$ is freely equal to ZUZ^{-1} . Let

$$\begin{aligned}\sigma_1 &= (C_1, \dots, C_k, D_m^{-1}, \dots, D_1^{-1}) \\ \tau &= (D_1, \dots, D_m, C_{k+1}, \dots, C_{k+l}).\end{aligned}$$

Then $\sigma_1 \tau$ is Peiffer equivalent to σ' (and hence to σ). Also, σ_1 is an identity sequence all of whose terms lie in $\mathbf{r}_e^{\mathbb{W}}$, and τ is an identity sequence with $\text{rk}(\tau) < \text{rk}(\sigma)$. Now apply the inductive hypothesis to τ .

Next suppose that no disc of \mathbf{P}' is labelled by an element of $\hat{\mathbf{r}}$. Then all discs of \mathbf{P}' are labelled by elements of \mathbf{r}_v for some $v \in V$. By taking δ to be a suitable path encircling \mathbf{P}' , and by choosing an edge e incident with v , we have that (4.1)–(4.3) hold, and instead of (4.4) and (4.5) we have that the label U on δ is the empty word. It is now an easy matter to adapt the arguments of the previous two paragraphs to obtain the required result.

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