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RESEARCH ARTICLE

Curves of maximal moduli on K3 surfaces

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Abstract
We prove that if X is a complex projective K3 surface and $g > 0$, then there exist infinitely many families of curves of geometric genus g on X with maximal, i.e., g-dimensional, variation in moduli. In particular, every K3 surface contains a curve of geometric genus 1 which moves in a nonisotrivial family. This implies a conjecture of Huybrechts on constant cycle curves and gives an algebro-geometric proof of a theorem of Kobayashi that a K3 surface has no global symmetric differential forms.

Contents

1. Introduction

Building on the work of many people [MM83, Che99, BT00, BHT11, LL11], it was recently proved in [CGL19] that, for any integer $g \geq 0$ and any complex projective K3 surface X, there is an infinite sequence of integral curves $C_n \subset X$ of geometric genus $g \geq 0$ such that for any ample divisor H

$$\lim_{n\to\infty} HC_n = \infty.$$

The aim of this paper is to strengthen and give a new proof of this result for curves of genus $g > 0$, assuming only the case $g = 0$, and then to derive a number of applications to the geometry of K3 surfaces. In particular, we prove the following.

Theorem A. *Let X be a K3 surface over an algebraically closed field of characteristic zero and $g > 0$ an integer. There exists a sequence of integral curves $C_n \subset X$ of geometric genus g such that*

$$\lim_{n\to\infty} C_n^2 = \infty,$$

and the normalisation morphism of each C_n deforms in a family of morphisms from smooth genus g curves to X with maximal variation in moduli.

More precisely, for each such $C_n \subset X$ there exists a diagram

$$\begin{array}{ccc} \mathcal{C}_n & \xrightarrow{F_n} & X \\ \downarrow{\scriptstyle f_n} & & \\ T_n & \xrightarrow{\phi_n} & \mathcal{M}_g, \end{array}$$

where f_n is a smooth family of curves over an irreducible variety T_n so that there exists a point $t \in T_n$ so that $F_{n,t} : \mathcal{C}_{n,t} \to X$ is the normalisation morphism of C_n composed with the inclusion, and $\dim T_n = \dim \phi(T_n) = g$, where ϕ_n is the moduli map to the moduli space of curves.

We give first an idea of the proof of this theorem. As mentioned, its proof relies on the existence of infinitely many rational curves on a K3 surface and not on the full statement of [CGL19, Theorem A] so provides a new proof and a strengthening of the higher genus case of loc. cit., both in that $C_n^2 \to \infty$ implies $HC_n \to \infty$ by the Hodge index theorem but also that the curves produced vary in moduli.

The second key ingredient in proving the above theorem is the logarithmic Bogomolov–Miyaoka–Yau inequality, which allows us, using local analysis of Orevkov–Zaidenberg, which we expand on in Section 3, to control the singularities of rational curves in X as their self-intersection increases. In particular, we show first in Proposition 3.4 that if

$$C^2 > 4690$$

for C is a rational curve on a K3 surface, then C must have a locally reducible singularity (i.e., one with at least two branches). As it is not known whether such a rational curve always exists on a K3 surface, we also show in Proposition 3.5 that if C_1, C_2 are two rational curves so that C_1C_2 is large enough with respect to C_1^2, C_2^2, then they must meet in at least two distinct points (e.g., if $C^2 \leq 4690$ for all rational curves in the K3, then $C_1C_2 > 1,299,546$ suffices). As a consequence, a partial normalisation of such a C or of such a union $C_1 \cup C_2$ may now be deformed in $\overline{\mathcal{M}}_1(X,\beta)$ to produce a genus one curve which necessarily deforms with maximal moduli. The argument then proceeds by induction on the genus.

By results of Mukai, the general curve of genus g is contained in a K3 surface if and only if $2 \leq g \leq 9$ or $g = 11$. Our result above, however, says that, for any fixed K3 surface X and any $g \geq 0$, there exist g-dimensional subvarieties of $\mathcal{M}_g$ whose general member parametrises a curve which admits a morphism to X birational onto its image. In the opposite direction, it is worth noting that it is expected yet not known that a very general K3 surface cannot be dominated by the product of two curves, which would imply that curves of constant moduli should not exist on most K3 surfaces.

As far as applications are concerned, although the existence of rational curves is satisfying to know, they do not provide much to work with. It turns out that the existence of one single genus 1 curve produced by Theorem 1 has numerous applications, so we begin by stating it as a separate corollary.

Corollary. *A K3 surface in characteristic zero contains a nonisotrivial family of integral curves of geometric genus 1.*

It is well-known that any K3 surface contains a family of genus 1 curves, so what is new in the above is the variation in moduli. As an application, combined with a result of Voisin [Huy14, Theorem 11.1] (where the existence of curves produced by the corollary is implicitly asked), the above immediately implies a conjecture of Huybrechts [Huy14, Conjecture 2.3].

Corollary. *There are infinitely many constant cycle curves of bounded order on every complex K3 surface X, and their union is dense in the strong topology.*

In a different direction, even though $\mathrm{H}^0(X, \Omega_X^1) = 0$ is easy to see for a complex K3 surface X via Hodge theory, Kobayashi [Kob80, Corollary 8] also proved that a simply connected Calabi–Yau

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manifold has no symmetric differentials or, in other words, that

$$\mathrm{H}^0(X, \mathrm{Sym}^n\, \Omega^1_X) = 0 \text{ for any } n > 0.$$

His proof is also analytic in nature and relies on the resolution of the Calabi conjecture by Yau. We give an algebraic proof of this fact for K3 surfaces, using only the existence of one nonisotrivial family of genus 1 curves, which follows from the corollary above.

Theorem B (Kobayashi). *The cotangent bundle of a complex K3 surface is not* $\mathbb{Q}$*-effective.*

Based on his generalised Zariski decomposition, Nakayama in [Nak04] proved that this implies that the divisor $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$ is not even pseudoeffective (see Theorem 5.6 for a proof).

Even though we do not provide a proof of Kobayashi's theorem or Theorem A in positive characteristic, we state as many results as possible in that direction, and in the final Section 5, we prove a conditional vanishing of global 1-forms (known by theorems of Rudakov–Shafarevich or Nygaard) and stability of the cotangent bundle (which holds if X is not uniruled but is known to fail otherwise).

Notation. Throughout this paper a *K3 surface* will always be a smooth projective simply connected surface with trivial canonical divisor over an algebraically closed field.

2. Deformations and singular curves

Let A be an effective divisor on a complex K3 surface. We consider the moduli map

$$V_{A,g} \longrightarrow \overline{\mathscr{M}}_g\ ,$$

where $\overline{\mathscr{M}}_g$ is the moduli space of stable curves of genus g and $V_{A,g}$ is the Severi variety parametrising integral curves in $|A|$ of geometric genus g. It is expected that this map is generically finite over its image for 'most' divisors $A \in \mathrm{Pic}\,(X)$, and we call such variation in moduli *maximal* (see Definition 2.3 for a more rigorous definition). The problem of existence of curves moving with maximal moduli has been studied by various authors for generic complex K3 surfaces (cf. [FKPS08, Kem15, CFGK17]).

Definition 2.1. Let $\mathbf{k}$ be an algebraically closed field and C an integral curve over $\mathbf{k}$. We say that a point $p \in C$ is a locally reducible singularity of C if the formal completion $\widehat{\mathcal{O}}_{C,p}$ of the stalk of C at p is not an integral domain. Equivalently, $\nu^{-1}(p)$ consists of at least two distinct points under the normalisation $\nu : C^\nu \to C$ of C. Otherwise, we say that C is locally irreducible at p. The number of local branches of C at p is the number of points in $\nu^{-1}(p)$.

The following is standard and is the main reason we are interested in such singularities.

Lemma 2.2. *Let $p \in C$ be a locally reducible singularity of an integral curve. Then the normalisation $\nu : C^\nu \to C$ factors through a curve C' which has one node and is smooth otherwise.*

Proof. Choose a sufficiently ample line bundle L on C. For $q_1 \neq q_2 \in \nu^{-1}(p)$, consider the subspace

$$V = \nu^*\mathrm{H}^0(L) + \mathrm{H}^0(\nu^*L \otimes \mathcal{O}_{C^\nu}(-q_1 - q_2)) \subset \mathrm{H}^0(\nu^*L).$$

Then $s_1(q_1) = s_2(q_2)$ for all $s_1, s_2 \in V$. Let $f : C^\nu \to G \subset \mathbb{P}V^*$ be the morphism given by the linear series V. Clearly, ν factors through f. For L sufficiently ample, G has a node $q = f(q_1) = f(q_2)$ over p as the only singularity. □

For $C \subset X$ a curve on a K3 surface, we denote by

$$\overline{\mathscr{M}}_g(X, \mathcal{O}_X(C))$$

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the Kontsevich moduli space of stable maps of arithmetic genus g to X with image of class $\mathcal{O}(C)$. For $f : D \to X$ such a morphism, we denote by $[f]$ the induced point in moduli.

Definition 2.3. Let $f : C \to X$ be a stable map of arithmetic genus g to a K3 surface over an algebraically closed field. We say that f deforms

1. *in the expected dimension* if $\dim M = g$ for every irreducible component $[f] \in M \subset \overline{\mathscr{M}}_g(X, \mathcal{O}(C))$ and
2. *with maximal moduli* if the induced moduli map $\phi_M : M \to \overline{\mathscr{M}}_g$ satisfies $\dim(\operatorname{im}\phi_M) \geq g$ for at least one irreducible component $[f] \in M \subset \overline{\mathscr{M}}_g(X, \mathcal{O}(C))$.

We say that an integral curve $C \subset X$ satisfies one of the above properties if its normalisation morphism $\nu : C^\nu \to C \subset X$ composed with the embedding into X does so.

Remark 2.4. From [CGL19, Theorem 2.11], for any $C \subset X$ integral with normalisation morphism contained in some irreducible component $[\nu : C^\nu \to X] \in M \subset \overline{\mathscr{M}}_g(X, \mathcal{O}(C))$, we have $\dim M \geq g$. Moreover, in characteristic zero any such C deforms in the expected dimension (from Proposition 2.5 below), but it is not necessarily the case that C deforms with maximal moduli, as seen, for example, by the existence of isotrivial elliptic fibrations. In positive characteristic, the situation is more complicated, as on a uniruled K3 there exist genus 0 curves which deform too much. Nodal rational curves on a K3 surface are always rigid though, and on a nonuniruled K3 surface, every curve of geometric genus 1 deforms in the expected dimension (see [CGL19, Proposition 2.9]). We do not know any examples of curves that do not deform in the expected dimension on a nonuniruled K3 surface.

The following is basically the Arbarello–Cornalba lemma (see [AC81, Lemma 1.4] or [ACG11, §XXI.9] for a more thorough reference) in the case of K3 surfaces.

Proposition 2.5. *Let X be a K3 surface over an algebraically closed field of characteristic zero, and $C \subset X$ an integral curve of geometric genus $g \geq 1$. Then if $[\nu] \in M \subset \overline{\mathscr{M}}_g(X, \mathcal{O}(C))$ is an irreducible component containing the normalisation $\nu : C^\nu \to C$, we have:*

1. *A general element $[f : D \to X] \in M$ corresponds to an unramified morphism.*
2. $\dim M = g$.
3. *If $D' \subset X$ an integral curve and $[f : D \to X] \in M$ general, then the support of $f^*\mathcal{O}_X(D')$ consists of $D'f(D)$ distinct points.*

Proof. The first claim is an application of the usual Arbarello–Cornalba lemma in the case of K3 surfaces (see, e.g., [DS17]), whereas the second and third follow essentially from the first (see [CGL19, §2] and the proof of [CGL19, Lemma 6.3]). □

Remark 2.6. In positive characteristic, it is not the case that (1) in the above is true (e.g., in a quasi-elliptic fibration the general fibre has ramified normalisation as it is a cusp), but we expect it to be true in most cases (see Question 4.6). It is, however, true that (1) implies (2) and (3).

We recall the following argument, essentially due to Bogomolov–Mumford, cf. [Huy16, §13.2.1].

Proposition 2.7. *Let X be a K3 surface over an algebraically closed field and $C \subset X$ be an integral curve of geometric genus g. Assume further that C*

1. *deforms in the expected dimension,*
2. *deforms with maximal moduli and*
3. *has a locally reducible singularity at a point p.*

Then C deforms to an integral curve D of geometric genus $g+1$ which deforms in the expected dimension and with maximal moduli.

Proof. As the singularity at p is locally reducible, from Lemma 2.2 we may take $f : \widetilde{C} \to X$ to be a partial normalisation of C which has one node over the point p and is smooth otherwise. In particular,

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$[f] \in \overline{\mathcal{M}}_{g+1}(X, \mathcal{O}(C))$. Let M be an irreducible component of $\overline{\mathcal{M}}_{g+1}(X, \mathcal{O}(C))$ containing $[f]$. From [CGL19, Theorem 2.11], $\dim M \geq g + 1$. Consider now the moduli map

$$\phi : \overline{\mathcal{M}}_{g+1}(X, \mathcal{O}(C)) \longrightarrow \overline{\mathcal{M}}_{g+1}.$$

Let D_M be an irreducible component of $M \cap \phi^{-1}(\partial\overline{\mathcal{M}}_{g+1})$ containing $[f]$, where $\partial\overline{\mathcal{M}}_{g+1} = \overline{\mathcal{M}}_{g+1} - \mathcal{M}_{g+1}$ is the boundary divisor of $\overline{\mathcal{M}}_{g+1}$.

For a general point, $[h] \in D_M$, $h : \Gamma \to X$ is a stable map such that Γ is an integral curve of geometric genus g with a node and $h(\Gamma)$ and C lie on the same component of $V_{C,g}$. Since C deforms in the expected dimension, $\dim D_M \leq g$, and hence, $D_M \subsetneq M$. On the other hand, since $\partial\overline{\mathcal{M}}_{g+1}$ is a $\mathbb{Q}$-Cartier divisor, D_M has codimension one in M. We must have

$$g + 1 \leq \dim M = \dim D_M + 1 \leq g + 1,$$

and hence, $\dim M = g + 1$. This proves that for a general point $h : \Gamma \to X$ of M, $D = h(\Gamma)$ is an integral curve of geometric genus $g + 1$ that deforms in the expected dimension.

Since C deforms with maximal moduli, there exists an irreducible component D of $\phi^{-1}(\partial\overline{\mathcal{M}}_{g+1})$ containing $[f]$ such that $\dim \phi(D) = g$. Let M' be an irreducible component of $\overline{\mathcal{M}}_{g+1}(X, \mathcal{O}(C))$ containing D. Since $\phi(M')$ is not contained in $\partial\overline{\mathcal{M}}_{g+1}$, we conclude

$$g + 1 = \dim M' \geq \dim \phi(M') \geq \dim \phi(D) + 1 = g + 1,$$

and hence, $\dim \phi(M') = g + 1$. Therefore, for a general point $h : \Gamma \to X$ of M', $D = h(\Gamma)$ is an integral curve of geometric genus $g + 1$ that deforms with maximal moduli. □

Although we will not be using it in this paper, we include the following immediate corollary, which is well-known to experts, as an application.

Corollary 2.8. *Let X be a K3 surface over an algebraically closed field and $R \subset X$ be a nodal rational curve of arithmetic genus $g \geq 1$. For any $1 \leq d \leq g$, R deforms to a nodal integral curve C of geometric genus d which deforms in the expected dimension and with maximal moduli.*

Proof. The result follows by induction, Proposition 2.7 and the fact that a general deformation of a nodal curve will be nodal and as such has unramified normalisation morphism, hence deforms in the expected dimension from [CGL19, Proposition 2.9]. □

One similarly obtains the following.

Proposition 2.9. *Let X be a K3 surface over an algebraically closed field and $C_1, C_2 \subset X$ be two integral curves of geometric genus g_1, g_2, respectively. Assume further that*

1. *C_i deforms in the expected dimension for $i = 1, 2$,*
2. *C_i deforms with maximal moduli for $i = 1, 2$,*
3. *$|C_1 \cap C_2|$ contains at least two distinct points.*

Then $C_1 \cup C_2$ deforms to an integral curve D of geometric genus $g_1 + g_2 + 1$ which deforms in the expected dimension and with maximal moduli.

3. Families of curves of maximal moduli

There are two main ingredients in the proof of Theorem A

- the existence of infinitely many rational curves on every complex K3 surface [CGL19],
- the logarithmic Bogomolov–Miyaoka–Yau (BMY) inequality [Miy84].

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Let us first review the basics of the latter. For the applications that we have in mind, we start with a reduced but possibly reducible curve D on a smooth projective surface X over $\mathbb{C}$. Take now a log resolution

$$(\widehat{X}, \widehat{D}) \longrightarrow (X, D),$$

i.e., a birational projective morphism $f : \widehat{X} \to X$ such that the total transform $\widehat{D} = f^{-1}(D) = \sum_{i=1}^{n} \Gamma_i$ of D has simple normal crossings, with irreducible components Γ_i and $X \backslash D \cong \widehat{X} \backslash \widehat{D}$. We usually choose $(\widehat{X}, \widehat{D})$ to be the minimal resolution of (X, D).

Now, for such a pair $(\widehat{X}, \widehat{D})$ of a smooth projective surface and a SNC divisor, the log BMY inequality says that if $K_{\widehat{X}} + \widehat{D}$ is $\mathbb{Q}$-effective, then

$$(K_{\widehat{X}} + \widehat{D})^2 \leq 3c_2(\Omega^1_{\widehat{X}}(\log \widehat{D})). \tag{3.1}$$

We recall that $\Omega^1_{\widehat{X}}(\log \widehat{D})$ is the locally free sheaf which sits in the following short exact sequence

$$0 \longrightarrow \Omega^1_{\widehat{X}} \longrightarrow \Omega^1_{\widehat{X}}(\log \widehat{D}) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\Gamma_i} \longrightarrow 0,$$

and we refer, for example, to [EV92, §2] for further details.

Remark 3.1. Note that there is a version of the log BMY inequality over fields of positive characteristic, proven recently by Langer [Lan16]. The conclusion is essentially the same inequality; however, one requires that the pair $(\widehat{X}, \widehat{D})$ lifts in a compatible way to $W_2(k)$.

Over the complex numbers, we have

$$c_2(\Omega^1_{\widehat{X}}(\log \widehat{D})) = e(\widehat{X} \backslash \widehat{D}) = e(X \backslash D) = e(X) - e(D), \tag{3.2}$$

where $e(\bullet)$ is the topological Euler characteristic.

For the applications we have in mind, X will be a K3 surface, and hence, $K_{\widehat{X}} + \widehat{D}$ will always be effective.

Although $c_2(\Omega^1_{\widehat{X}}(\log \widehat{D}))$ can be computed topologically by equation (3.2) over $\mathbb{C}$, we want to give a purely algebraic formula for it in terms of $c_2(X)$, $p_a(D)$ and the invariants of the singularities of D (we refer to [dJP00, §5] for the basics of curve singularities). As the proof of this works in arbitrary characteristic, we state it in this generality.

Lemma 3.2. *Let X be a smooth projective surface over an algebraically closed field and D be a reduced curve on X. Let $(\widehat{X}, \widehat{D})$ be the minimal log resolution of (X, D). Then*

$$c_2(\Omega^1_{\widehat{X}}(\log \widehat{D})) = c_2(X) + (K_X + D)D - \sum_{p \in D} (2\delta_p - \gamma_p + 1), \tag{3.3}$$

where δ_p and γ_p are the δ-invariant and the number of local branches of D at p, respectively.

Proof. Let $\widehat{D} = \sum_{i=1}^{n} \Gamma_i$, where Γ_i are the irreducible components of $\widehat{D}$. From the exact sequences

$$0 \longrightarrow \Omega^1_{\widehat{X}}(\log \sum_{i=1}^{m-1} \Gamma_i) \longrightarrow \Omega^1_{\widehat{X}}(\log \sum_{i=1}^{m} \Gamma_i) \longrightarrow \mathcal{O}_{\Gamma_m} \longrightarrow 0$$

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for $m = 1, \ldots, n$, we obtain

$$\begin{aligned}
\mathrm{ch}(\Omega^1_{\widehat{X}}(\log \widehat{D})) &= \mathrm{ch}(\Omega^1_{\widehat{X}}) + \sum_{m=1}^{n} \mathrm{ch}(\mathcal{O}_{\Gamma_m}) \\
&= \mathrm{ch}(\Omega^1_{\widehat{X}}) + \sum_{m=1}^{n} (\mathrm{ch}(\mathcal{O}_{\widehat{X}}) - \mathrm{ch}(\mathcal{O}_{\widehat{X}}(-\Gamma_m))) \\
&= K_{\widehat{X}} + \widehat{D} + \frac{1}{2}(K^2_{\widehat{X}} - 2c_2(\widehat{X}) - \sum_{m=1}^{n} \Gamma_m^2),
\end{aligned}$$

where $\mathrm{ch}(\bullet)$ is the Chern character. It follows that

$$\begin{aligned}
c_2(\Omega^1_{\widehat{X}}(\log \widehat{D})) &= c_2(\widehat{X}) + \frac{1}{2}(K_{\widehat{X}} + \widehat{D})^2 - \frac{1}{2}K^2_{\widehat{X}} + \frac{1}{2}\sum_{m=1}^{n} \Gamma_m^2 \\
&= c_2(\widehat{X}) + (K_{\widehat{X}} + \widehat{D})\widehat{D} - \sum_{1 \le i < j \le n} \Gamma_i \Gamma_j.
\end{aligned}$$

Note that further blowing up $\widehat{X}$ at a singularity of $\widehat{D}$ does not change $c_2(\Omega^1_{\widehat{X}}(\log \widehat{D}))$. The minimal log resolution of (X, D) does not blow up all singularities of D in case that D is reducible: If D has an ordinary double point at p where two components of D meet transversely, we do not need to blow up X at p. On the other hand, we can choose to blow up X at such p since it does not change $c_2(\Omega^1_{\widehat{X}}(\log \widehat{D}))$. This has the advantage of streamlining our argument. Hence, we choose a log resolution $(\widehat{X}, \widehat{D})$ of (X, D) which is minimal with the properties that $\widehat{D}$ has simple normal crossings and the proper transforms of the components of D are disjoint from each other.

Let us write

$$\widehat{D} = \sum_{i=1}^{n} \Gamma_i = \Delta + \sum_{p \in D_s} E_p,$$

where Δ is the proper transform of D under $\pi : \widehat{X} \to X$ and $E_p = \pi^{-1}(p)$ for $p \in D_s$, where D_s is the set of singularities of D. Clearly, E_p is a tree of smooth rational curves for all $p \in D_s$. Then the above equality takes the form

$$\begin{aligned}
c_2(\Omega^1_{\widehat{X}}(\log \widehat{D})) = c_2(\widehat{X}) + (K_{\widehat{X}} + \Delta)\Delta + \sum_{p \in D_s} (K_{\widehat{X}} + E_p)E_p + \sum_{p \in D_s} \Delta E_p \\
- \sum_{p \in D_s} \sum_{\substack{1 \le i < j \le n \\ \Gamma_i \cup \Gamma_j \subset E_p}} \Gamma_i \Gamma_j.
\end{aligned}$$

Since Δ is the normalisation of D,

$$\begin{aligned}
(K_{\widehat{X}} + \Delta)\Delta = 2p_a(\Delta) - 2 &= 2p_a(D) - 2 - 2\sum_{p \in D} \delta_p \\
&= (K_X + D)D - 2\sum_{p \in D} \delta_p.
\end{aligned}$$

For every $p \in D_s$, $p_a(E_p) = 0$, and hence,

$$\sum_{p \in D} (K_{\widehat{X}} + E_p)E_p = -2 \sum_{p \in D_s} 1.$$

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It is also clear that ΔE_p equals the number of local branches of D at $p \in D_s$. Therefore,

$$\sum_{p \in D_s} \Delta E_p = \sum_{p \in D_s} \gamma_p.$$

Since E_p is a tree of smooth rational curves,

$$\sum_{\substack{1 \le i < j \le n \\ \Gamma_i \cup \Gamma_j \subset E_p}} \Gamma_i \Gamma_j = |E_p| - 1$$

for $p \in D_s$, where $|E_p|$ is the number of irreducible components of E_p. Finally,

$$c_2(\widehat{X}) = c_2(X) + \sum_{p \in D_s} |E_p|.$$

Combining all the above, we obtain equation (3.3). □

For convenience, we write

$$\mu_p = 2\delta_p - \gamma_p + 1.$$

Over the complex numbers, μ_p agrees with the Milnor number of D at p (see [Mil68, Theorem 10.5]). However, this can fail in positive characteristic, so we will call μ_p the *pseudo-Milnor number* of D at p.

We now work towards constructing a lower bound for $(K_{\widehat{X}} + \widehat{D})^2$ in terms of $(K_X + D)^2$ and the local contribution of the singularities of D. The following lemma is basically due to Orevkov–Zaidenberg [OZ95, §4], but we give here a simple proof that works in all characteristics.

Lemma 3.3. *Let X be a smooth projective surface over an algebraically closed field and D be a reduced curve on X. Let $(\widehat{X}, \widehat{D})$ be the minimal log resolution of (X, D). Then*

$$(K_{\widehat{X}} + \widehat{D})^2 \ge (K_X + D)^2 - \sum_{p \in D} \left(1 - \frac{1}{m_p}\right)\mu_p, \tag{3.4}$$

where m_p and μ_p are the multiplicity and pseudo-Milnor number of D at p, respectively.

Proof. As in the proof of Lemma 3.2, further blowing up $\widehat{X}$ at a singularity of $\widehat{D}$ does not change $(K_{\widehat{X}} + \widehat{D})^2$. So we choose a log resolution $(\widehat{X}, \widehat{D})$ of (X, D) which is minimal with the properties that $\widehat{D}$ has simple normal crossings and the proper transforms of the components of D are disjoint from each other.

The proof of Lemma 3.2 already gives

$$(K_{\widehat{X}} + \widehat{D})\widehat{D} = (K_X + D)D - \sum_{p \in D_s} \mu_p + \sum_{p \in D_s} (\gamma_p - 1). \tag{3.5}$$

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From now on, we denote $K_{\widehat{X}/X} = K_{\widehat{X}} - \pi^* K_X$. Then, equation (3.5) and the fact that $K_{\widehat{X}/X}.\pi^* F = 0$ for any divisor F on X yield

$$\begin{aligned}(K_{\widehat{X}} + \widehat{D})^2 - (K_X + D)^2 &= -\sum_{p\in D_s} \mu_p + \sum_{p\in D_s} (\gamma_p - 1) + (K_{\widehat{X}}^2 - K_X^2)\\ &\quad + \sum_{p\in D_s} K_{\widehat{X}} E_p + (K_{\widehat{X}}\Delta - K_X D)\\ &= -\sum_{p\in D_s} \mu_p + \sum_{p\in D_s} (\gamma_p - 1) + K_{\widehat{X}/X}^2\\ &\quad + \sum_{p\in D_s} K_{\widehat{X}} E_p + K_{\widehat{X}}(\Delta - \pi^* D).\end{aligned}$$

Thus, equation (3.4) holds as long as we can prove

$$(\gamma_p - 1) + K_{\widehat{X}} E_p + (K_{\widehat{X}/X}^2)_p + (K_{\widehat{X}}(\Delta - \pi^* D))_p \geq \frac{\mu_p}{m_p} \tag{3.6}$$

for all $p \in D_s$. The problem is local, so we work in a formal neighbourhood of a point $p \in D_s$ in X. For simplicity, we drop the subscript p in all notation so that $m = m_p$, $\mu = \mu_p$, $\gamma = \gamma_p$ and $E = E_p$.

We can factor $\pi : \widehat{X} \to X$ into a sequence of blowups:

$$\widehat{X} = X_a \xrightarrow{\pi_{a,a-1}} X_{a-1} \xrightarrow{\pi_{a-1,a-2}} \dots \xrightarrow{\quad} X_1 \xrightarrow{\pi_{1,0}} X_0 = X,$$

where each $\pi_{i,i-1} : X_i \to X_{i-1}$ is the blowup of X_{i-1} at one point for $i = 1, 2, \dots, a$. Let $\pi_{i,j} = \pi_{j+1,j} \circ \pi_{j+2,j+1} \circ \dots \circ \pi_{i,i-1}$ be the birational map $X_i \to X_j$ for $0 \leq j < i \leq a$, and let F_i be the exceptional divisor of $\pi_{i,i-1} : X_i \to X_{i-1}$ for $i = 1, 2, \dots, a$. Then

$$\begin{aligned}K_{\widehat{X}/X} &= \pi_{a,1}^* F_1 + \pi_{a,2}^* F_2 + \dots + \pi_{a,a-1}^* F_{a-1} + F_a\\ \Delta &= \pi^* D - m_1 \pi_{a,1}^* F_1 - m_2 \pi_{a,2}^* F_2 - \dots - m_{a-1}\pi_{a,a-1}^* F_{a-1} - m_a F_a\end{aligned}$$

for some $m_i \in \mathbb{Z}_+$ satisfying that

$$m = m_1 = \max_{1\leq i\leq a} m_i.$$

It follows (see, e.g., [dJP00, Theorem 5.4.13]) that

$$\begin{aligned}\mu + \gamma - 1 = 2\delta &= \sum_{i=1}^{a} m_i(m_i - 1)\\ K_{\widehat{X}/X}^2 + K_{\widehat{X}}(\Delta - \pi^* D) &= \sum_{i=1}^{a} (m_i - 1).\end{aligned}$$

Therefore, equation (3.6) holds provided that we can prove

$$(\gamma - 1) + K_{\widehat{X}} E \geq 0. \tag{3.7}$$

We recall that $E = \pi^{-1}(p)$ is a tree of smooth rational curves. Thus, from the adjunction formula,

$$K_{\widehat{X}} E = (K_{\widehat{X}} + E)E - E^2 = -2 - E^2 \geq -1,$$

where $E^2 \leq -1$ because the components of E have negative definite intersection matrix. So we have equation (3.7) if $\gamma \geq 2$. Otherwise, $\gamma = 1$, i.e., D has a locally irreducible or unibranch singularity at p. We claim that $K_{\widehat{X}}E \geq 0$ in this case.

Let $E_i = \pi_{i,0}^{-1}(p)$ for $i = 1, 2, \ldots, a$. Then $E_1 = F_1$ and $K_{X_1}E_1 = -1$. If $\pi_{i,i-1} : X_i \to X_{i-1}$ is the blowup of X_{i-1} at a smooth point of E_{i-1}, then

$$E_i = \pi_{i,i-1}^* E_{i-1} \text{ and } K_{X_i}E_i = K_{X_{i-1}}E_{i-1}.$$

Otherwise, if $\pi_{i,i-1} : X_i \to X_{i-1}$ is the blowup of X_{i-1} at a singular point of E_{i-1}, then

$$E_i = \pi_{i,i-1}^* E_{i-1} - F_i \text{ and } K_{X_i}E_i = K_{X_{i-1}}E_{i-1} + 1.$$

In conclusion, we have

$$K_{X_1}E_1 = -1 \text{ and } K_{X_i}E_i = \begin{cases} K_{X_{i-1}}E_{i-1} & \text{if } \pi_{i,i-1}(F_i) \notin (E_{i-1})_{\mathrm{sing}} \\ K_{X_{i-1}}E_{i-1} + 1 & \text{if } \pi_{i,i-1}(F_i) \in (E_{i-1})_{\mathrm{sing}} \end{cases}$$

for $2 \leq i \leq a$. Therefore, $K_{\widehat{X}}E = K_{X_a}E_a \geq 0$ as long as one of $\pi_{i,i-1}$ is the blowup of X_{i-1} at a singular point of E_{i-1}. For a locally irreducible singularity $p \in D_s$, it is easy to see that $\pi_{a,a-1} : X_a \to X_{a-1}$ blows up X_{a-1} at a singular point of E_{a-1}. Consequently, $K_{\widehat{X}}E \geq 0$ when $\gamma = 1$. This proves equation (3.7) and hence equation (3.6), giving equation (3.4). □

Combining equations (3.1), (3.3) and (3.4), we obtain

$$(K_X + D)^2 - \sum_{p \in D} \left(1 - \frac{1}{m_p}\right)\mu_p \leq 3\left(c_2(X) + (K_X + D)D - \sum_{p \in D} \mu_p\right). \tag{3.8}$$

We are now in a position to put all the above together for K3 surfaces in the characteristic zero case, where the BMY inequality holds.

Proposition 3.4. *Let $D \subset X$ be an integral curve of geometric genus g in a K3 surface over an algebraically closed field of characteristic zero. If*

$$D^2 > 4690 + 550g + 16g^2,$$

then D has at least one locally reducible singularity.

Proof. Suppose that D only has locally irreducible singularities. Then

$$(K_X + D)D - \sum_{p \in D} \mu_p = (K_X + D)D - 2\sum_{p \in D} \delta_p = 2g - 2. \tag{3.9}$$

By equation (3.8) and $c_2(X) = 24$, we have

$$D^2 - \sum_{p \in D} \left(1 - \frac{1}{m_p}\right)\mu_p \leq 66 + 6g. \tag{3.10}$$

Combining equations (3.9) and (3.10), we have

$$\sum_{p \in D} \frac{\mu_p}{m_p} \leq 68 + 4g. \tag{3.11}$$

On the other hand,

$$\mu_p \geq m_p(m_p - 1) \tag{3.12}$$

for all $p \in D$. Putting equations 3.9–3.12 together gives

$$68 + 4g \geq \sum_{p\in D} \frac{\mu_p}{m_p} \geq \sum_{p\in D}\left(\sqrt{\mu_p + \frac{1}{4}} - \frac{1}{2}\right) \geq \sqrt{D^2 + \frac{9}{4} - 2g} - \frac{1}{2},$$

where we note that we have used that the function $f(x) = \sqrt{x + \frac{1}{4}} - \frac{1}{2}$ vanishes at 0 and has everywhere negative second derivative, hence is concave and $\sum f(x_i) \geq f(\sum x_i)$ for positive real x_i. It follows that $D^2 \leq 4690 + 550g + 16g^2$. Therefore, D has at least one locally reducible singularity if $D^2 > 4690 + 550g + 16g^2$. □

Proposition 3.5. *Let $D_1, D_2 \subset X$ be two distinct integral curves in a K3 surface X over an algebraically closed field of characteristic 0. If*

$$2D_1D_2 > \left(\sqrt{4D_1^2 + 9} + \sqrt{4D_2^2 + 9} + 2\right)(37 + D_1^2 + D_2^2) + 1,$$

then D_1 and D_2 meet at (at least) two distinct points.

Proof. Suppose that D_1 and D_2 meet at a unique point q. Applying equation (3.8) to $(X, D = D_1 + D_2)$, we have

$$D^2 - \sum_{p\in D}\left(1 - \frac{1}{m_{D,p}}\right)\mu_{D,p} \leq 72 + 3(D^2 - \sum_{p\in D}\mu_{D,p}), \tag{3.13}$$

where we use $\mu_{C,p}$ and $m_{C,p}$ to denote the pseudo-Milnor number and multiplicity of a reduced curve C at p, respectively.

Note the following simple facts for $i = 1, 2$, $p \in D$ and $D_1 \cap D_2 = \{q\}$ as above

$$\begin{aligned} \mu_{D,p} &= \mu_{D_1,p} + \mu_{D_2,p} + 2(D_1.D_2)_p - 1 \\ &= \mu_{D_1,p} + \mu_{D_2,p} - 1 \text{ if } p \neq q \\ \mu_{D,q} &= \mu_{D_1,q} + \mu_{D_2,q} + 2D_1D_2 - 1 \\ m_{D,p} &= m_{D_1,p} + m_{D_2,p} \leq M := \sqrt{D_1^2 + \frac{9}{4}} + \sqrt{D_2^2 + \frac{9}{4}} + 1. \end{aligned} \tag{3.14}$$

Combining equations (3.13) and (3.14), we obtain

$$\begin{aligned} 75 - 3\sum_{i=1}^{2}\sum_{p\in D_i}\mu_{D_i,p} &= 72 + 3(D^2 - \sum_{p\in D}\mu_p) - 3(D_1^2 + D_2^2) \\ &\geq D^2 - \sum_{p\in D}\left(1 - \frac{1}{m_{D,p}}\right)\mu_{D,p} - 3(D_1^2 + D_2^2) \\ &\geq 2(D_1D_2 - D_1^2 - D_2^2) - \sum_{p\in D}\left(1 - \frac{1}{M}\right)\mu_{D,p} \\ &= \frac{2}{M}D_1D_2 - 2(D_1^2 + D_2^2) + \frac{M-1}{M}\left(1 - \sum_{i=1}^{2}\sum_{p\in D_i}\mu_{D_i,p}\right). \end{aligned}$$

Hence,

$$75 \geq \frac{2}{M} D_1 D_2 + \frac{M-1}{M} - 2(D_1^2 + D_2^2),$$

and the proposition follows. □

The lower bounds in the above propositions are almost certainly not optimal. Better results can be achieved with improvement to equation (3.4) (cf. [Moe15]).

We are now ready to prove Theorem A.

Proof of Theorem A. Let us first prove it for $g = 1$.

By [CGL19, Theorem A], there are infinitely many integral rational curves C_n on X. Suppose that C_n^2 is unbounded. Then C_n has a locally reducible singularity by Proposition 3.4 for C_n^2 sufficiently large. Such C_n can be deformed to a nonisotrivial family of curves of geometric genus 1 by Proposition 2.7.

Suppose that $C_n^2 \leq c$ for all n. We claim that

$$\overline{\lim_{\min(m,n)\to\infty}} C_m C_n = \infty. \tag{3.15}$$

Fixing $N \in \mathbb{Z}_+$, since $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) \leq 20$, $C_N, C_{N+1}, \ldots, C_{N+20}$ are linearly dependent in $\operatorname{Pic}(X)_{\mathbb{Q}}$. Suppose that

$$a_0 C_N + a_1 C_{N+1} + \ldots + a_{20} C_{N+20} = 0 \tag{3.16}$$

in $\operatorname{Pic}(X)$ for some integers a_i, not all zero. Since C_i are effective, a_i cannot be all positive or negative. Let us rewrite equation (3.16) as

$$F = \sum_{a_i > 0} a_i C_{N+i} = -\sum_{a_j < 0} a_j C_{N+j}.$$

Since $C_N, C_{N+1}, \ldots, C_{N+20}$ are distinct integral curves, it is easy to see that F is nef. This implies that there are only finitely many integral rational curves R such that $FR = 0$, since if $F^2 = 0$, then F can only be zero on the (up to 24) singular fibres of the elliptic fibration induced by F, and if $F^2 > 0$, then from the Hodge index theorem the orthogonal space $F^\perp$ in the effective cone is negative definite and spanned by finitely many -2-curves. Hence, there exists $m \geq N$ such that $FC_m \geq 1$. Then $C_m + 2F$ is nef and big, and hence,

$$\lim_{n\to\infty} (C_m + 2F) C_n = \infty.$$

Thus, there exists $C \in \{C_N, C_{N+1}, \ldots, C_{N+20}, C_m\}$ such that CC_n is unbounded. This proves equation (3.15).

By Proposition 3.5, C_m and C_n meet at (at least) two distinct points for $C_m C_n$ sufficiently large since $C_m^2 \leq c$ and $C_n^2 \leq c$. There are infinitely many such pairs C_m and C_n by equation (3.15), and

$$\overline{\lim_{\min(m,n)\to\infty}} (C_m + C_n)^2 = \infty.$$

Such $C_m \cup C_n$ can be deformed to a nonisotrivial family of curves of geometric genus 1 by Proposition 2.9, which as pointed out above will have unbounded self-intersection. This proves the theorem for $g = 1$. The remaining cases follow from Propositions 2.7 and 3.4 by induction. □

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4. An algebraic proof of Kobayashi's theorem

We say that a vector bundle E on a quasi-projective variety X is $\mathbb{Q}$-effective if

$$\mathrm{H}^0(X, \mathrm{Sym}^m E) \neq 0$$

for some positive integer m, where $\mathrm{Sym}^m E$ is the m-th symmetric product of E. We call E pseudoeffective if, for every $n \in \mathbb{Z}_+$, there exists $m \in \mathbb{Z}_+$ such that

$$\mathrm{H}^0(X, \mathrm{Sym}^{mn} E \otimes \mathcal{O}_X(mA)) \neq 0,$$

where A is a fixed ample divisor on X. Alternatively, let

$$Y = \mathbb{P}(E^\vee) = \mathrm{Proj}\ (\mathrm{Sym}^\bullet E) = \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{Sym}^m E$$

be the projectivisation of $E^\vee$, and let $\mathcal{O}_Y(1)$ be the tautological bundle of Y over X. By the Leray spectral sequence, the $\mathbb{Q}$-effectivity (resp. pseudoeffectivity) of E coincides with that of $\mathcal{O}_Y(1)$.

Let now X be a K3 surface, and let $Y = \mathrm{Proj}\ (S^\bullet \Omega^1_X)$ with $L = \mathcal{O}_Y(1)$ being the tautological bundle of $\pi : Y \to X$. The following follows easily from Hodge theory over the complex numbers, whereas in positive characteristic is a theorem of Rudakov–Shafarevich [RS76] (see also Nygaard [Nyg79]).

Theorem 4.1. *Let X be a K3 surface over an algebraically closed field. Then* $\mathrm{H}^0(X, \Omega^1_X) = 0$.

See Proposition 5.1 for a simple, conditional algebraic proof of the above. In what follows, we will give an algebraic proof of Kobayashi's theorem (i.e., Theorem B of the introduction) by reducing it to the above. The proof in fact works in arbitrary characteristic under the following, minimal assumption.

Hypothesis 4.2. There exists an unramified morphism $f : E \to X$ from a smooth genus 1 curve which deforms in the expected dimension and with maximal moduli.

In characteristic zero, Theorem A (in combination with Proposition 2.5) produces infinitely many such curves, whereas in positive characteristic we are not able to produce such a curve, although in remarks after the proof we will give various cases in which such a curve does exist.

Theorem 4.3. *Let X be a K3 surface over an algebraically closed field. If we assume Hypothesis (4.2), then*

$$\mathrm{H}^0(X, \mathrm{Sym}^m \Omega^1_X) = 0 \textit{ for } m \geq 1.$$

Proof. We maintain the notation for Y, L from the beginning of this section. Suppose for a contradiction that L is $\mathbb{Q}$-effective. Let m be the smallest positive integer such that mL is effective, and let $G \in |mL|$. We write

$$G = \sum b_i D_i,$$

where $D_i \in |a_i L + \pi^* F_i|$ are the irreducible components of G for some $a_i \in \mathbb{N}$ and some divisors $F_i \in \mathrm{Pic}\ (X)$, and $b_i \in \mathbb{Z}_+$ is the multiplicity of D_i in G. Since $mL = \sum a_i b_i L + \sum b_i \pi^* F_i$, we obtain that

$$\sum b_i F_i = 0 \text{ in } \mathrm{Pic}\ (X). \tag{4.1}$$

Let $C \subset X$ be an integral curve of geometric genus 1 as given by Hypothesis 4.2. From the assumption, there exists an irreducible curve $B \subset |C|$ with C as member and such that every curve $\Gamma \in B$ is of geometric genus 1.

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When $a_i = 0$, F_i is necessarily effective and $CF_i \geq 0$. Note also that there exists at least one i such that $CF_i \leq 0$ and $a_i > 0$ since otherwise, $CF_i > 0$ for all $a_i > 0$ and so $\sum CF_i > 0$, contradicting equation (4.1).

From now on, we denote by $a = a_i$, $D = D_i$ and $F = F_i$ so that $a_i > 0$ and $CF_i \leq 0$.

From the assumption, the general deformation of the normalisation of C is an immersion. We henceforth replace C by a general member of B and let $\nu : E = C^\nu \to X$ be its normalisation, i.e., we have that $\nu^*\Omega^1_X \to \Omega^1_E$ is a surjection. As the kernel is torsion-free on a smooth curve, it is a line bundle, and by taking determinants we see that it must be isomorphic to $(\Omega^1_E)^\vee \cong \mathcal{O}_E$. This leads to the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}_\nu^\vee \arrow[r] \arrow[d, equal] & \nu^*\Omega^1_X \arrow[r] & \Omega^1_E \arrow[r] \arrow[d, equal] & 0, \\ & \mathcal{O}_E & & \mathcal{O}_E & \end{tikzcd} \tag{4.2}$$

where $\mathcal{N}_\nu$ is the normal bundle of ν. From our assumption and the following lemma, the above sequence does not split.

Lemma 4.4. *Sequence (4.2) splits if and only if B parametrises an isotrivial family of elliptic curves.*

Proof. If $f : \mathcal{C} \to B$ the family with B a smooth projective curve and E the generic fibre of f, then a section $\Omega^1_E \to \nu^*\Omega^1_X$ also induces a splitting of

$$0 \longrightarrow f^*\Omega^1_B|_U \longrightarrow \Omega^1_{\mathcal{C}}|_U \longrightarrow \Omega^1_f|_U \longrightarrow 0$$

on some open subset $U \subset B$. Dualising this sequence and pushing forward to U, we get a split sequence whose first coboundary map in cohomology is the Kodaira–Spencer map. Hence, this map is necessarily zero so the family over U is isotrivial. □

Since $aL + F$ is effective, $\mathrm{H}^0(S^a\Omega^1_X \otimes \mathcal{O}_X(F)) \neq 0$ and as C is a general member of a covering family of curves on X, we see that

$$\mathrm{H}^0(E, S^a\nu^*\Omega^1_X \otimes \mathcal{O}_E(\nu^*F)) \neq 0$$

as otherwise a global section of $S^a\Omega^1_X \otimes \mathcal{O}_X(F)$ would vanish everywhere. By equation (4.2), $S^a\nu^*\Omega^1_X \otimes \mathcal{O}_E(\nu^*F)$ has a filtration

$$0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{a+1} := S^a\nu^*\Omega^1_X \otimes \mathcal{O}_E(\nu^*F)$$

with graded pieces all isomorphic to $\mathcal{O}_E(\nu^*F)$. If the global section $\mathcal{O}_E \to E_{a+1}$ from above vanishes when mapped to $E_{a+1}/E_a = \mathcal{O}_E(\nu^*F)$, then it must induce a nonzero global section of E_a. By induction, one of the quotients E_i/E_{i-1} must have a nonzero global section, and hence, $\mathrm{H}^0(\mathcal{O}_E(\nu^*F)) \neq 0$. On the other hand, $CF \leq 0$ and $\deg \nu^*F \leq 0$. So we necessarily have $\mathcal{O}_E(\nu^*F) = \mathcal{O}_E$.

This proves that for all i satisfying $a_i > 0$ and $CF_i \leq 0$ we have $\mathcal{O}_E(\nu^*F_i) = \mathcal{O}_E$, and hence, $CF_i = 0$. For the remaining i, we clearly have $CF_i \geq 0$. Therefore, we conclude that $CF_i = 0$ for all i from equation (4.1). In summary, we have

- if $a_i > 0$, $\mathcal{O}_E(\nu^*F_i) = \mathcal{O}_E$,
- if $a_i = 0$, F_i is effective and $CF_i = 0$.

As exact sequence (4.2) does not split,

$$h^0(E, S^n\nu^*\Omega^1_X) = 1 \tag{4.3}$$

for all $n \in \mathbb{Z}_+$.

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Work now again with a fixed i so that $a_i > 0$ as above, keeping the notation D, F, a. Since D is reduced, $Y_p = \pi^{-1}(p)$ meets D transversely for $p \in X$ general, and as C is a general member of a covering family of curves on X, also $Y_p = \pi^{-1}(p)$ meets D transversely for $p \in C$ general. Let now $R = E \times_X Y \cong \mathrm{Proj}\,(S^\bullet(\nu^*\Omega_X^1))$ with diagram

$$\begin{tikzcd} R \arrow[r, "\rho"] \arrow[d] & Y \arrow[d, "\pi"] \\ E \arrow[r, "\nu"] & X. \end{tikzcd}$$

Since Y_p and D meet transversely for $p \in C$ general, R_q and ρ^*D meet transversely for $q \in E$ general, where R_q is the fibre of R over q.

Note that ρ^*D is a section of $a\rho^*L$. From equation (4.3), $h^0(R, n\rho^*L) = 1$ for all $n \geq 0$, and so we must have $\rho^*D = a\Gamma$, where Γ is the unique section of ρ^*L. Then we must have $a = 1$ because R_q and ρ^*D meet transversely for $q \in E$ general.

Hence, we have concluded that $a_i = 0$ or 1 for all i. If there are two distinct components D_i and D_j of G such that $a_i = a_j = 1$, then $\rho^*D_i = \rho^*D_j = \Gamma$. Therefore,

$$D_i \cap \pi^{-1}(C) = D_j \cap \pi^{-1}(C)$$

for $C \in B$ general, and hence, $D_i = D_j$. Consequently, G has only one component D_i with $a_i = 1$, and so we have $\mathrm{H}^0(\Omega_X^1 \otimes \mathcal{O}_X(F)) \neq 0$ for some $F \in \mathrm{Pic}\,(X)$ such that $-F = \sum F_i$ is effective. As $\mathrm{H}^0(\Omega_X^1 \otimes \mathcal{O}_X(F)) \subset \mathrm{H}^0(\Omega_X^1)$, we obtain a contradiction from the case $m = 1$, namely Theorem 4.1. $\square$

In conclusion, we have proved that Ω_X^1 is not $\mathbb{Q}$-effective if Hypothesis (4.2) holds. This of course is a consequence of Theorem A in characteristic zero, but in the following remark, we outline various cases where this is true in characteristic zero under far weaker assumptions than the existence of infinitely many rational curves on X.

Remark 4.5.

1. Recall that from Propositions 2.7 and 2.9, the existence of either one rational curve $C \subset X$ with a locally reducible singularity or two distinct rational curves meeting in at least two distinct points guarantee the existence of a nonisotrivial family of genus 1 curves in X.
2. More generally, we can produce a nonisotrivial family of genus 1 curves on X if there are distinct rational curves $C_1, \ldots, C_n \subset X$ and points $p_i \neq q_i \in C_i^\nu$ on their normalisations such that for all $1 \leq i < n$

$$\nu(p_i) = \nu(q_{i+1}) \text{ and } \nu(p_n) = \nu(q_1),$$

where $\nu : \sqcup C_i^\nu \to X$ is the normalisation of $\cup C_i$. In this case, we can find a stable map $f : \Gamma \to X$ such that $\Gamma = \cup \Gamma_i$, $\Gamma_i \cong C_i^\nu$, $f(\Gamma_i) = C_i$,

$$|\Gamma_1 \cap \Gamma_2| = \ldots = |\Gamma_n \cap \Gamma_1| = 1 \text{ and } \Gamma_i \cap \Gamma_j = \emptyset \text{ otherwise.}$$

In positive characteristic, even though there exist rational curves which deform too much and without unramified deformations (e.g., a quasi-elliptic fibration on a supersingular K3 surface), a version of the Arbarello–Cornalba lemma (Proposition 2.5) eludes us for the time being. One could ask the following.

Question 4.6. Let $f : C \to X$ be a morphism from a smooth projective curve of genus $g \geq 1$ to a K3 surface over an algebraically closed field. If f deforms in the expected dimension, is a general deformation of f unramified?

Assuming the above and that all rational curves in Remark 4.5 are rigid, Propositions 2.7 and 2.9 imply that the cases listed in Remark 4.5 also provide a genus 1 curve satisfying the properties of Hypothesis 4.2, and hence, Kobayashi's theorem holds.

5. Global 1-forms and stability

As mentioned in the introduction and in the previous section (see Theorem 4.1), the proof that a K3 surface does not have any global 1-forms uses analytic techniques in characteristic zero (Hodge theory) and is rather nontrivial in positive characteristic. In this section, we gather some auxiliary results and questions, giving simple, conditional algebraic proofs of the fact that for a K3 surface X we have that $\mathrm{H}^0(X, \Omega_X^1) = 0$ and that Ω_X^1 is slope-stable (with respect to any ample divisor), using only the existence of special curves in X.

Proposition 5.1. *Let X be a smooth projective variety of dimension n over an algebraically closed field and $f : C \to X$ an unramified morphism from a smooth curve of genus $g > 1$ so that f deforms in a family which dominates X and varies with maximal moduli. Then* $\mathrm{H}^0(X, T_X) = 0$.

Proof. Taking cohomology of the sequence

$$0 \longrightarrow T_C \longrightarrow f^*T_X \longrightarrow N_f \longrightarrow 0,$$

the Kodaira–Spencer map $\mathrm{H}^0(C, N_f) \to \mathrm{H}^1(C, T_C)$ must be injective, as it is the induced differential to the moduli map and C deforms with maximal moduli. This implies that $\mathrm{H}^0(C, f^*T_X) = 0$, but as C deforms to cover X, we obtain the result. □

In the case of K3 surfaces, the existence of such curves in characteristic zero is guaranteed by Theorem A, but the current proof relies on the existence of infinitely many rational curves, whose proof in fact uses the vanishing of 1-forms in a number of ways. The assumptions of the above do hold unconditionally for K3 surfaces in the cases listed in Remark 4.5.

We move now to the question of stability of the (co)tangent bundle. We recall that for an ample divisor $A \in \mathrm{Pic}\,(X)$ on a projective variety X we say that a vector bundle E on X is *μ_A-(semi)stable* (often just μ) if

$$\mu_A(F) := \frac{\det(F)A^{\dim X-1}}{\mathrm{rk}(F)} \underset{(\leq)}{<} \frac{\det(E)A^{\dim X-1}}{\mathrm{rk(E)}}$$

for all torsion-free subsheaves $F \subsetneq E$. In fact, if F does not satisfy the above inequality, then we say that F *destabilises* E, and we may assume that F is a subvector bundle with torsion-free quotient. In particular, if, for a K3 surface X, $E = \Omega_X^1$ is not semistable, then there exists a destabilising line bundle $L \subset \Omega_X^1$, i.e., $LA \geq 0$.

The assumption we will be making to give a quick proof of stability of the tangent bundle will be the following.

Question 5.2. Let X be a K3 surface over an algebraically closed field. Is it true that for any ample divisor $D \in \mathrm{Pic}\,(X)_{\mathbb{Q}}$ there exist integral curves $E_1, \ldots, E_n \subset X$ of geometric genus 1 so that $D = \sum_{i=1}^n a_i E_i$ for $a_i \in \mathbb{Q}_{\geq 0}$?

Remark 5.3. We note that the above is known to be true in the following cases

1. The Picard rank of X is ≤ 2 [CGL19, Corollary 7.3, Theorem 8.4].
2. X contains no smooth rational curves: In many such cases, the effective cone is generated by smooth genus 1 curves even (see [Kov94]). For the rest (in particular the case where the cone is not polyhedral), one can use the fact that every nef divisor can be written as a sum of minimal nef divisors and that each such divisor is linearly equivalent to an integral curve of geometric genus 1 (see [CGL19, §3] for the definition and for this result).

We claim that the stability of Ω^1_X follows from a positive answer to Question 5.2 for K3 surfaces X. In fact, we can prove a more general statement. For that purpose, let us recall some basic facts about Harder–Narasimhan filtrations and the cone of curves.

Let E be a vector bundle on a smooth projective variety X. We use the notation $\mu_{A,\max}(E)$ to denote that the maximum of the slopes $\mu_A(F)$ for all subsheaves $F \subset E$ and some ample A, which we from now on suppress in the notation. This number is given by the Harder–Narasimhan filtration

$$E = E_0 \supsetneq E_1 \supsetneq \ldots \supsetneq E_m \supsetneq E_{m+1} = 0$$

of E, where $F_i = E_i/E_{i+1}$ are torsion-free and semistable sheaves satisfying

$$\mu(F_0) < \mu(F_1) < \ldots < \mu(F_m)$$

and $\mu_{\max}(E)$ is given by $\mu(F_m) = \mu(E_m)$. Using Harder–Narasimhan filtrations, we have

$$r\mu_{\max}(E) \geq \mu_{\max}(\wedge^r E)$$

for all $1 \leq r \leq \operatorname{rank}(E)$.

For a smooth projective variety X, we let $N_1(X)$ denote the group of 1-cycles modulo numerical equivalence and let $N_1(X)_{\mathbb{Q}}$ and $N_1(X)_{\mathbb{R}}$ denote $N_1(X) \otimes \mathbb{Q}$ and $N_1(X) \otimes \mathbb{R}$, respectively. For X over $\mathbb{C}$, we have

$$N_1(X)_{\mathbb{Q}} \cong H^{n-1,n-1}(X, \mathbb{Q}) = H^{n-1,n-1}(X) \cap H^{2n-2}(X, \mathbb{Q}).$$

For lack of a better term, we call the classes $A_1 A_2 \ldots A_{n-1} \in N_1(X)$ for ample $A_1, A_2, \ldots, A_{n-1} \in \operatorname{Pic}(X)$ *ample complete intersection classes*. We call the cone $\operatorname{Amp}_1(X)_{\mathbb{R}} \subset N_1(X)_{\mathbb{R}}$ generated by these classes the *cone of ample complete intersection curves*.

Theorem 5.4. *Let X be a smoooth projective variety of dimension n over an algebraically closed field of characteristic 0, and let $G \subset N_1(X)_{\mathbb{R}}$ be the set consisting of numerical classes ξ with the following property: There exists a sequence $f_m : C_m \to X$ of morphisms from smooth projective curves C_m to X such that*

- *$f_m(C_m)$ passes through a general point of X, i.e., the deformation of f_m dominates X for each m,*
- *the numerical classes $[(f_m)_* C_m]$ of $(f_m)_* C_m$ satisfy*

$$\lim_{m\to\infty} \frac{[(f_m)_* C_m]}{\deg (f_m)_* C_m} = \xi$$

- *and the conormal bundles*

$$M_{f_m} = \ker(f_m^* \Omega^1_X \to \Omega^1_{C_m})$$

of f_m satisfy

$$\overline{\lim_{m\to\infty}} \frac{n \max(\mu_{\max}(M_{f_m}), \deg K_{C_m}) - \deg f_m^* K_X}{n \deg (f_m)_* C_m} \leq 0,$$

where $\deg(f_m)_* C_m$ is the degree of $(f_m)_* C_m$ with respect to a fixed ample line bundle on X.

If $\operatorname{Amp}_1(X)_{\mathbb{R}}$ is asymptotically generated by G, i.e., $\operatorname{Amp}_1(X)_{\mathbb{R}}$ is contained in the closure of the cone generated by G, then Ω^1_X is μ-semistable for all ample divisors A on X. More precisely, if Ω^1_X contains a locally free subsheaf E of rank r such that $\mu(E) \geq \mu(\Omega^1_X)$, then $nc_1(E) - rK_X$ is numerically trivial.

In particular, if X is a complex K3 surface, A is an ample divisor on X and there is a positive answer to Question 5.2, then Ω^1_X is μ_A-stable.

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Proof. Suppose that there exists a locally free subsheaf $E \subset \Omega^1_X$ of rank r such that $\mu(E) \geq \mu(\Omega^1_X)$. Then $L = \wedge^r E$ is a subsheaf of Ω^r_X, and hence, $H^0(\Omega^r_X(-L)) \neq 0$.

Let $\xi \in G$ and $f_m : C_m \longrightarrow X$ be the sequence of morphisms associated to ξ. Since $f_m(C_m)$ passes through a general point of X, we see that

$$H^0(C_m, f_m^*\Omega^r_X(-L)) \neq 0.$$

Then we have

$$h^0(M^r_{f_m}(-f_m^*L)) + h^0(M^{r-1}_{f_m}(-f_m^*L) \otimes K_{C_m}) \geq h^0(f_m^*\Omega^r_X(-L)) > 0$$

by the left exact sequence

$$0 \longrightarrow M^r_{f_m} \longrightarrow f_m^*\Omega^r_X \longrightarrow M^{r-1}_{f_m} \otimes K_{C_m},$$

where $M^a_{f_m} = \wedge^a M_{f_m}$. On the other hand, we know that

$$H^0(V(-B)) = 0 \text{ if } \deg B > \mu_{max}(V)$$

for a vector bundle V and a divisor B on a smooth projective curve. It follows that

$$\begin{aligned} L.(f_m)_*C_m = \deg f_m^*L &\leq \max\left(\mu_{max}(M^r_{f_m}), \mu_{\max}(M^{r-1}_{f_m}) + \deg K_{C_m}\right) \\ &\leq \max\left(r\mu_{\max}(M_{f_m}), (r-1)\mu_{\max}(M_{f_m}) + \deg K_{C_m}\right) \\ &\leq r\max(\mu_{\max}(M_{f_m}), \deg K_{C_m}). \end{aligned}$$

Therefore,

$$\left(\frac{L}{r} - \frac{K_X}{n}\right)\frac{(f_m)_*C_m}{\deg(f_m)_*C_m} \leq \frac{n\max(\mu_{\max}(M_{f_m}), \deg K_{C_m}) - \deg f_m^*K_X}{n\deg(f_m)_*C_m}.$$

By our definition of G, we conclude that

$$\left(\frac{L}{r} - \frac{K_X}{n}\right)\xi \leq 0$$

for all $\xi \in G$. On the other hand, since $\mu(E) \geq \mu(\Omega^1_X)$,

$$\left(\frac{L}{r} - \frac{K_X}{n}\right)A^{n-1} \geq 0.$$

Fixing $\xi \in G$, since $\mathrm{Amp}_1(X)_{\mathbb{R}}$ is open in $N_1(X)_{\mathbb{R}}$,

$$A^{n-1} - t\xi \in \mathrm{Amp}_1(X)_{\mathbb{R}}$$

for some $t > 0$ sufficiently small. Since $\mathrm{Amp}_1(X)_{\mathbb{R}}$ is asymptotically generated by G,

$$A^{n-1} - t\xi = \sum_{m=1}^{\infty} t_m\xi_m$$

for some $t_m > 0$ and $\xi_m \in G$. Finally, from

$$(nL - rK_X)A^{n-1} \geq 0,\ (nL - rK_X)\xi \leq 0 \text{ and } (nL - rK_X)\xi_m \leq 0,$$

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we conclude that $(nL - rK_X)\xi = 0$. Therefore,

$$(nL - rK_X)\xi = 0$$

for all $\xi \in G$. This implies that $nL - rK_X$ is numerically trivial since G also generates $N_1(X)_{\mathbb{R}}$.

For a complex K3 surface X, it is easy to see that $E/\deg E \in G$ for every elliptic curve E on X. Since by hypothesis the elliptic curves generate the ample cone $\mathrm{Amp}(X)$ of X, $\mathrm{Amp}(X)$ is generated by G. If Ω^1_X is destabilised by a line bundle L, then L is numerically trivial. For K3 surfaces, this implies that $L = \mathcal{O}_X$ so that $H^0(\Omega^1_X(-L)) = H^0(\Omega^1_X) \neq 0$ which is a contradiction. □

Remark 5.5. In positive characteristic, Langer [Lan15, §4] proves that K3 surfaces not dominated by $\mathbb{P}^2$ have strongly semistable cotangent bundle and that K3 surfaces admitting a quasi-elliptic fibration (e.g., unirational K3 surfaces in characteristic 2) do not have semistable cotangent bundle. If semistable, then Ω^1_X must also be stable as $\mathrm{H}^0(X, \Omega^1_X) = 0$ is known for an arbitrary K3. We expect Question 5.2 to still have a positive answer here though. In fact if one could furthermore assume that all the genus 1 curves generating the nef cone admit normalisations which deform to unramified morphisms (something which does not occur for fibres of a quasi-elliptic fibrations), the above proof goes through.

We conclude this section by giving the proof of Nakayama's theorem in arbitrary characteristic. This proof is essentially the same as in [BDPP13, Theorem 7.8] (which draws from Nakayama's original proof from [Nak04]) with the necessary adjustments for positive characteristic in place.

Theorem 5.6 (Nakayama in characteristic $p \geq 0$). *Let X be a K3 surface over an algebraically closed field k. Assume further that Ω^1_X is μ-stable and that*

$$\mathrm{H}^0(X, \mathrm{Sym}^n\, \Omega^1_X) = 0 \textit{ for all } n > 0.$$

Then Ω^1_X is not pseudoeffective.

Proof. Since stability persists if we pass to a larger algebraically closed field, we may assume k is uncountable. Let $Y = \mathbb{P}(\Omega^1_X)$, and suppose for a contradiction that $L = \mathcal{O}_Y(1)$ is pseudoeffective. Then there is a Nakayama–Zariski decomposition of $L = E + N$, where E is an effective $\mathbb{R}$-divisor and N is nef in codimension 1 (due to [Nak04] in characteristic 0 and [Mus13, FL17] otherwise).

From [Lan10, Theorem 4.1] (or Flenner or Mehta–Ramanathan's theorem in characteristic zero), we may pick a very ample smooth curve C on X so that $\Omega^1_X|_C$ is strongly semistable (or just semistable in characteristic zero). Then on the ruled surface $R = \mathbb{P}(\Omega^1_X|_C)$ every pseudoeffective line bundle is nef (in fact for the projectivisation of a degree zero strongly semistable bundle on a curve, these cones agree). On the other hand, $L|_R$ is not ample, since $L^2|_R = c_1(\Omega^1_X) \cdot C = 0$. Hence, $L|_R$ is on the boundary of the nef cone of R. Write $E = aL + \pi^* E'$. As the Picard number of R is two and $E|_R$ is also $\mathbb{R}$-effective, it must be that $E'.C \geq 0$. If $E'.C > 0$, then E' is effective on X (as C can vary), so in particular, $E|_R$ is big and hence ample. This contradicts $L|_R = E|_R + N|_R$ being boundary on the nef cone though. In other words, $CE' = 0$ and as C can vary, $E' = 0$, forcing $E = aL$. Then $a = 0$ since from the assumption L has no effective multiple. It follows that $E = 0$ and L is nef in codimension 1. In particular, it fails to be nef on at most countably many curves C_i. Taking a hyperplane section H of Y, we see then that $L|_H$ is nef. In particular, $L^2 \cdot H \geq 0$. In terms of Chern classes, this means that

$$-c_2(T_X) \geq 0,$$

which contradicts $c_2(T_X) = 24$. □

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