LINEAR COMBINATIONS OF BERNSTEIN POLYNOMIALS

P. L. BUTZER

1. Introduction. If $f(x)$ is defined on $[0, 1]$, then its corresponding Bernstein polynomial

$$(1) \qquad B_n(x) = B_n^f(x) = \sum_{\nu=0}^{n} f(\nu n^{-1})\, p_{\nu,n}(x), \qquad p_{\nu,n}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu},$$

approaches $f(x)$ uniformly on $[0, 1]$, if $f(x)$ is continuous on $[0, 1]$. If $f(x)$ is bounded on $[0, 1]$, then at every point x where the second derivative $f''(x)$ exists (Voronowskaja [**7**], see also [**5**])

$$\lim_{n\to\infty} n[B_n^f(x) - f(x)] = \frac{x(1-x)}{2} f''(x),$$

hence if $f''(x)$ is not zero on $[0, 1]$, the order of approximation to $f(x)$ by the $B_n(x)$ is exactly $O(n^{-1})$. It follows that the existence of derivatives of higher order of $f(x)$ cannot improve this order of approximation.

In this paper we shall introduce certain linear combinations of Bernstein polynomials which, under definite conditions, approximate $f(x)$ more closely than the Bernstein polynomials.

Polynomials approaching $f(x)$ more closely than the Bernstein polynomials, but of a different type from those considered here, were also considered by Bernstein [**1**] namely,

$$Q_n^f(x) = \sum_{\nu=0}^{n} \left[f(\nu n^{-1}) - \frac{x(1-x)}{2n} f''(\nu n^{-1}) \right] p_{\nu,n}(x).$$

Then if $|f(x)| \leqslant M$ and if $f^{(4)}(x)$ exists at the point x, it can be shown that

$$\lim_{n\to\infty} n^2[Q_n^f(x) - f(x)] = \frac{x(1-x)(1-2x)}{6} f^{(3)}(x) - \frac{[x(1-x)]^2}{8} f^{(4)}(x).$$

We remark that the combinations we consider do not contain the values of the derivatives of $f(x)$.

2. Preliminary results. We shall here recall some known facts, for their proofs one may consult [**5**, §§1.5–1.6]. With Bernstein [**1**] we define

Received April 6, 1952; presented to the American Mathematical Society at the Summer Meeting, September 1952. This paper is based on a part of the author's thesis prepared under the supervision of Professor G. G. Lorentz, and accepted for a Ph.D. degree at the University of Toronto in November 1951. The author wishes to thank Professor Lorentz and Professor W. J. Webber for their helpful suggestions in connection with this paper, and the National Research Council of Canada for the supporting grant.

https://doi.org/10.4153/CJM-1953-063-7 Published online by Cambridge University Press

(2) $$S_{n,r}(x) = \sum_{\nu=0}^{n} (\nu n^{-1} - x)^r p_{\nu,n}(x) \qquad (n = 1, 2, \ldots ; r = 0, 1, 2, \ldots)$$

and for $n^r S_{n,r}(x)$ we shall often write $T_{n,r}(x)$. If $f(x)$ is defined on [0, 1] with $|f(x)| \leqslant M$ then at points where $f^{(2k)}(x)$ exists [**1**],

(3) $$B_n^f(x) = f(x) + \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} S_{n,r}(x) + \frac{\epsilon_n}{n^k}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

The recursion formula

$$T_{n,r+1}(x) = x(1 - x)[T'_{n,r}(x) + nrT_{n,r-1}(x)]$$

is known, and by induction we obtain, putting $x(1 - x) = X$,

$$T_{n,0} = 1, \quad T_{n,1} = 0, \quad T_{n,2} = nX, \quad T_{n,3} = n(1 - 2x)X,$$

(4) $$T_{n,4} = 3n^2X^2 + n(X - 6X^2), \quad T_{n,5} = (1 - 2x)[10n^2X^2 + n(X - 12X^2)],$$

$$T_{n,6} = 15n^3X^3 + 5n^2X^2(5 - 26X) + nX[1 - 30X + 120X^2],$$

$$\ldots \qquad \ldots$$

In general, for r fixed, every $0 \leqslant x \leqslant 1$, $T_{n,r}(x)$ can be written as a polynomial in n,

(5) $$T_{n,r}(x) = \phi_{r,r'}(x)\, n^{r'} + \phi_{r,r'-1}(x)\, n^{r'-1} + \phi_{r,r'-2}(x)\, n^{r'-2} + \ldots + \phi_{r,1}(x) n$$

of degree

$$r' \equiv [\tfrac{1}{2}r] = \begin{cases} \tfrac{1}{2}r & \text{for even } r, \\ \tfrac{1}{2}(r - 1) & \text{for odd } r, \end{cases}$$

where the $\phi_{r,r'-i}(x)$ are polynomials in x, independent of n.

Moreover, for every r, one can show [**5**] there exists a constant A_r (depending only on r) such that for every $0 \leqslant x \leqslant 1$,

(6) $$0 \leqslant T_{n,2r}(x) \leqslant A_r n^r.$$

Calling $p = 2r/\beta$ for a given $\beta > 0$, we have also

(7) $$\sum_{\nu=0}^{n} | \nu - nx |^\beta p_{\nu,n}(x) \leqslant A_r^{1/p} n^{\beta/2}.$$

If $\delta = n^{-\alpha}$, $0 < \alpha < \frac{1}{2}$, it is known that [**5**] for every $l > 0$, there is a constant C where $C = C(\alpha, l)$ such that

(8) $$\sum_{|\nu n^{-1} - x| \geqslant \delta} p_{\nu,n}(x) \leqslant Cn^{-l}.$$

3. The linear combination. If $f(x)$ is defined on [0, 1], we define the polynomials

$$\mathfrak{L}_n^{[0]} = [\mathfrak{L}_n^f(x)]^{[0]} = B_n^f(x)$$

(9) $$(2^k - 1)\, \mathfrak{L}_n^{(2k]} = 2^k \mathfrak{L}_{2n}^{[2k-2]} - \mathfrak{L}_n^{[2k-2]}, \qquad k = 1, 2, \ldots.$$

One can rewrite the relation (9) as

$$(10) \quad \mathfrak{L}_n^{[2k]}(x) = \alpha_k B_{2^k n}(x) + \alpha_{k-1} B_{2^{k-1} n}(x) + \alpha_{k-2} B_{2^{k-2} n}(x) + \ldots + \alpha_0 B_n(x)$$

where by induction, explicit values can be found for the constants α_i, $\alpha_i = \alpha_i(k)$. Note that

$$(11) \qquad \alpha_k + \alpha_{k-1} + \alpha_{k-2} + \ldots + \alpha_0 = 1.$$

The polynomial (10) is the linear combination of the ordinary Bernstein polynomials we consider in this paper.

For $r = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$, we also define

$$(12) \qquad \begin{aligned} \mathfrak{S}_{n,r}^{[0]} &= \mathfrak{S}_{n,r}^{[0]}(x) = S_{n,r}(x) \\ (2^k - 1)\,\mathfrak{S}_{n,r}^{[2k]} &= 2^k \mathfrak{S}_{2n,r}^{[2k-2]} - \mathfrak{S}_{n,r}^{[2k-2]}, \qquad k = 1, 2, \ldots. \end{aligned}$$

Corresponding to the relation (3), for the linear combination (10) we have the following result:

LEMMA 1. *If $f^{(2k+2s)}(x)$ exists at the point x, then*

$$(13) \qquad \mathfrak{L}_n^{[2k]}(x) = f(x) + \sum_{r=1}^{2(k+s)} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^{k+s}}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Proof. We prove this lemma by induction. So we suppose (13) holds and if $f^{(2k+2s+2)}(x)$ exists we show that (13) holds with $2k$ replaced by $2(k+1)$. We have

$$\mathfrak{L}_n^{[2k]}(x) = f(x) + \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^{k+s+1}}.$$

By the relations (9) and (12) we deduce that

$$\begin{aligned} (2^{k+1} - 1)[\mathfrak{L}_n^{[2k+2]}(x) - f(x)] &= 2^{k+1}[\mathfrak{L}_{2n}^{[2k]} - f] - [\mathfrak{L}_n^{[2k]} - f] \\ &= 2^{k+1} \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{2n,r}^{[2k]}(x) - \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^{k+s+1}} \\ &= (2^{k+1} - 1) \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k+2]}(x) + \frac{\epsilon_n}{n^{k+s+1}}. \end{aligned}$$

This establishes the lemma.

We shall now prove the approximation theorem for our linear combination.

THEOREM 1. *If $f(x)$ is defined on $[0, 1]$ with $|f(x)| \leqslant M$ and if $f^{(2k)}(x)$ exists at the point x, then*

$$(14) \qquad |\mathfrak{L}_n^{[2k-2]}(x) - f(x)| = O(n^{-k}),$$

and moreover,

$$(15) \qquad |\mathfrak{L}_n^{[2k]}(x) - f(x)| = o(n^{-k}) \text{ as } n \to \infty, \qquad k = 1, 2, \ldots.$$

https://doi.org/10.4153/CJM-1953-063-7 Published online by Cambridge University Press

Proof. By Lemma 1 we have

$$\mathfrak{L}_n^{[2k]} - f = \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^k}, \qquad \epsilon_n \to 0 \text{ as } n \to \infty,$$

and if we can show that

$$\sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k]}(x) = O(n^{-k-1}) \tag{16}$$

then the relation (15) will hold. For this purpose we need

LEMMA 2. *With* $\mathfrak{S}_{n,r}^{[2k]}$ *defined by* (12),

$$\mathfrak{S}_{n,r}^{[2k]}(x) = 0 \qquad \textit{for } 1 \leqslant r \leqslant k+1, \tag{17}$$

$$\mathfrak{S}_{n,r}^{[2k]}(x) = O(n^{-k-1}) \qquad \textit{for } r = 1, 2, 3, \ldots. \tag{18}$$

To prove this lemma, we note that by (5)

$$\mathfrak{S}_{n,r}^{[0]}(x) = \phi_{r,r'}(x)\, n^{-(r-r')} + \phi_{r,r'-1}(x)\, n^{-(r-r'+1)} + \ldots + \phi_{r,1}(x)\, n^{-(r-1)}. \tag{19}$$

Applying to n^{-s} the difference operator which connects $\mathfrak{S}^{[2k]}$ with $\mathfrak{S}^{[2k-2]}$ in (12), we obtain

$$2^k (2n)^{-s} - n^{-s} = (2^{k-s} - 1)\, n^{-s}.$$

This is of form an^{-s}, and $a = 0$ if $k = s$. Operating on the right-hand side of (19) with difference operators for $s = 1, 2, 3, \ldots, k$ and omitting vanishing terms we therefore have

$$\mathfrak{S}_{n,r}^{[2k]}(x) = \phi_{k+1}(x)\, n^{-(k+1)} + \ldots + \phi_{r-1}(x)\, n^{-(r-1)}$$

where the $\phi_i(x)$ are polynomials in x independent of n. This proves (18). If $k + 1 > r - 1$, all terms vanish, and we obtain (17). The lemma is complete.

The relation (15) now follows. By Lemma 1, we find

$$\mathfrak{L}_n^{[2k-2]}(x) - f(x) + \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k-2]}(x) + \frac{\epsilon_n}{n^k}$$

and on account of (18), we deduce (14). This establishes the theorem.

In the particular case of the previous theorem for $k = 3$, i.e., if $f^{(6)}(x)$ exists at x, we have by the relations (4) and (13)

$$\lim_{n\to\infty} n^3 [\mathfrak{L}_n^{[4]}(x) - f(x)] \equiv \lim_{n\to\infty} n^3 \left[\frac{8}{3} B_{4n}^f(x) - 2B_{2n}^f(x) + \frac{1}{3} B_n^f(x) - f(x)\right]$$

$$= \frac{1}{8}(X - 6X^2)\frac{f^{(4)}(x)}{4!} + \frac{5}{4}(1 - 2x)\, X^2 \frac{f^{(5)}(x)}{5!} + \frac{15}{8} X^3 \frac{f^{(6)}(x)}{6!},$$

and also

$$\lim_{n\to\infty} n^3 [\mathfrak{L}_n^{[6]}(x) - f(x)]$$

$$\equiv \lim_{n\to\infty} n^3 \left[\frac{64}{21} B_{8n}(x) - \frac{56}{21} B_{4n}(x) + \frac{14}{21} B_{2n}(x) - \frac{1}{21} B_n(x) - f(x)\right] = 0.$$

https://doi.org/10.4153/CJM-1953-063-7 Published online by Cambridge University Press

4. Further theorems on the order of approximation. If $f(x)$ is defined and continuous on [0, 1], then

$$\omega(\delta) = \omega^f(\delta) \equiv \max_{|h|\leqslant\delta} |f(x+h) - f(x)|, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant x+h \leqslant 1$$

is called the modulus of continuity of the function $f(x)$.

THEOREM 2. *If $f^{(2k)}(x)$ exists and is continuous on* [0, 1] *having a modulus of continuity $\omega_{2k}(\delta)$, then*

$$|\mathfrak{L}_n^{[2k]}(x) - f(x)| \leqslant \max\left\{\frac{C}{n^k}\,\omega_{2k}\,(n^{-\frac{1}{2}}), \frac{C'}{n^{k+1}}\right\}$$

where $C = C(k)$ and $C' = C'(k; f)$.

Proof. Since $f^{(2k)}(x)$ exists and is continuous, for x_1, x_2 there is an η, $x_1 < \eta < x_2$ such that

$$f(x_2) - f(x_1) = \sum_{i=1}^{2k} (x_2 - x_1)^i \frac{f^{(i)}(x_1)}{i!} + \frac{(x_2 - x_1)^{2k}}{(2k)!}[f^{(2k)}(\eta) - f^{(2k)}(x_1)].$$

By (11) we have

$$\begin{aligned}
\mathfrak{L}_n^{[2k]} - f &= \sum_{j=0}^{k}\left\{\alpha_j[B_{2^jn}(x) - f(x)]\right\}\\
&= \sum_{j=0}^{k}\left\{\alpha_j\sum_{\nu=0}^{2^jn}[f(2^{-j}\nu n^{-1}) - f(x)]\,p_{\nu,2^jn}(x)\right\}\\
&= \sum_{j=0}^{k}\left\{\alpha_j\sum_{\nu=0}^{2^jn}\left[\sum_{i=1}^{2k}(2^{-j}\nu n^{-1} - x)^i\frac{f^{(i)}(x)}{i!}\right]p_{\nu,2^jn}(x)\right\}\\
&\qquad + \sum_{j=0}^{k}\left\{\alpha_j\sum_{\nu=0}^{2^jn}\left[\frac{(2^{-j}\nu n^{-1} - x)^{2k}}{(2k)!}(f^{(2k)}(\xi_j) - f^{(2k)}(x))\right]p_{\nu,2^jn}(x)\right\}\\
&= \Sigma_1 + \Sigma_2,
\end{aligned}$$

where $\xi_j = \xi_j(\nu)$, $x < \xi_j < 2^{-j}\nu/n$, $0 \leqslant j \leqslant k$. Now

$$\begin{aligned}
\sum_{\nu=0}^{2^jn}\sum_{i=1}^{2k}(2^{-j}\nu n^{-1} - x)^i\frac{f^{(i)}(x)}{i!}\,p_{\nu,2^jn}(x) &= \sum_{i=1}^{2k}\sum_{\nu=0}^{2^jn}(2^{-j}\nu n^{-1} - x)^i\,p_{\nu,2^jn}(x)\frac{f^{(i)}(x)}{i!}\\
&= \sum_{i=1}^{2k}\mathfrak{S}_{2^jn,i}^{[0]}(x)\frac{f^{(i)}(x)}{i!}.
\end{aligned}$$

Therefore

$$\Sigma_1 = \sum_{i=1}^{2k}\sum_{j=0}^{k}\alpha_j\mathfrak{S}_{2^jn,i}^{[0]}(x)\frac{f^{(i)}(x)}{i!} = \sum_{i=1}^{2k}\mathfrak{S}_{n,i}^{[2k]}(x)\,\frac{f^{(i)}(x)}{i!},$$

and by Lemma 2 we obtain

$$\Sigma_1 \leqslant C'n^{-k-1}.$$

We now evaluate Σ_2. Since for a modulus of continuity, $\omega(\lambda\delta) \leqslant (1+\lambda)\omega(\delta)$ for any $\lambda > 0$ we have

https://doi.org/10.4153/CJM-1953-063-7 Published online by Cambridge University Press

$$\sum_{\nu=0}^{2^j n} \frac{(2^{-j}\nu n^{-1} - x)^{2k}}{(2k)!} \mid f^{(2k)}(\xi_j) - f^{(2k)}(x) \mid p_{\nu,2^j n}(x)$$

$$\leqslant \frac{\omega_{2k}(\delta)}{(2k)!}\left\{ \sum_{\nu=0}^{2^j n} (2^{-j}\nu n^{-1} - x)^{2k} p_{\nu,2^j n}(x) + \frac{1}{\delta}\sum_{\nu=0}^{2^j n} (2^{-j}\nu n^{-1} - x)^{2k} \mid \xi_j - x \mid p_{\nu,2^j n}(x)\right\}$$

$$\leqslant \frac{\omega_{2k}(\delta)}{(2k)!}\left\{ \sum_{\nu=0}^{2^j n} (2^{-j}\nu n^{-1} - x)^{2k} p_{\nu,2^j n}(x) + \frac{1}{\delta}\sum_{\nu=0}^{2^j n} \mid 2^{-j}\nu n^{-1} - x\mid^{2k+1} p_{\nu,2^j n}(x)\right\},$$

and by (6) and (7), this expression does not exceed

$$\frac{\omega_{2k}(\delta)}{(2k)!}\left\{\frac{A_k}{(2^j n)^k} + \frac{A'_k}{\delta(2^j n)^{k+\frac{1}{2}}}\right\}.$$

Hence

$$\mid \Sigma_2 \mid \leqslant \frac{\omega_{2k}(\delta)}{(2k)!}\sum_{j=0}^{k} \mid \alpha_j \mid \left\{\frac{A_k}{(2^j n)^k} + \frac{A'_k}{\delta(2^j n)^{k+\frac{1}{2}}}\right\}$$

and putting in particular $\delta = n^{-\frac{1}{2}}$ we have

$$\mid \Sigma_2 \mid \leqslant \frac{C}{n^k}\, \omega_{2k}\,(n^{-\frac{1}{2}})\,.$$

This proves Theorem 2.

COROLLARY. *If* $f^{(2k)}(x)$ *exists and belongs to* Lip α, $0 < \alpha \leqslant 1$, *that is if*

$$\mid f^{(2k)}(x+h) - f^{(2k)}(x) \mid \leqslant K \mid h \mid^{\alpha},$$

then

$$\mid \mathfrak{L}_n^{[2k]}(x) - f(x) \mid \leqslant M n^{-k-\frac{1}{2}\alpha}$$

where M *is a constant.*

In connection with Theorem 2, we remark that if $f(x)$ is continuous on [0, 1] having a modulus of continuity, $\omega(\delta)$, then for the ordinary $B_n(x)$, Popoviciu [**6**] has shown that

$$\mid B_n^f(x) - f(x) \mid \leqslant \tfrac{3}{2}\, \omega(n^{-\frac{1}{2}}).$$

Regarding the case $k = 1$ of the preceding corollary, compare [**2**].

5. Another property of the $\mathfrak{L}_n^{[2k]}(x)$. If $f(x)$ satisfies only a Lipschitz condition of order α, $0 < \alpha \leqslant 1$, then

$$\mid \mathfrak{L}_n^{[2k]}(x) - f(x) \mid \leqslant \sum_{j=0}^{k} \mid \alpha_j \mid \mid B_{2^j n}(x) - f(x) \mid = O(n^{-\frac{1}{2}\alpha}),$$

and we shall show that this order of approximation cannot in general be improved, that is, one can find functions $f(x) \in$ Lip α $(0 < \alpha \leqslant 1)$ such that

$$\mid \mathfrak{L}_n^{[2k]}(x) - f(x) \mid \geqslant C_1 n^{-\frac{1}{2}\alpha}$$

with some constant $C_1 > 0$. This shows that in general, $\mathfrak{L}_n^{[2k]}(x)$, $k \geqslant 1$ does not approximate $f(x)$ more closely than $B_n^f(x)$. We shall show this for the particular case $k = 1$, the general case $k \geqslant 1$ can be treated along similar lines. We have the following theorem.

THEOREM 3. *For every* $0 < \alpha \leqslant 1$, *there exist functions* $f(x) \in \text{Lip}\, \alpha$ *such that the order of approximation given by*

$$| \mathfrak{L}_n^{[2]}(x) - f(x) | = O(n^{-\frac{1}{2}\alpha})$$

cannot be improved.

Proof. We shall consider the function $f(x) = |x - x_0|^\alpha$ with fixed $0 < x_0 < 1$ (and where $0 < \alpha \leqslant 1$). This function satisfies the Lipschitz condition of order α, namely

$$| x + h - x_0 |^\alpha - | x - x_0 |^\alpha \leqslant | h |^\alpha.$$

Now for fixed $0 < x_0 < 1$ and $\gamma > \frac{1}{3}$, for all ν which satisfy

$$(20) \qquad | \nu\mu^{-1} - x_0 | \leqslant \mu^{-\gamma}, \qquad 0 \leqslant \nu \leqslant \mu$$

it is known that [**3**, p. 133]

$$(21) \qquad R_{\nu,\mu}(x_0) \equiv | \nu\mu^{-1} - x_0 |^\alpha p_{\nu,\mu}(x_0)$$

$$\cong \frac{| \nu\mu^{-1} - x_0 |^\alpha}{[2\pi x_0(1 - x_0)\mu]^{\frac{1}{2}}} \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (\nu\mu^{-1} - x_0)^2 \right] \equiv P_{\nu,\mu}(x_0)\ ;$$

this is a uniform asymptotic relation, that is, uniformly for all ν satisfying (20),

$$\lim_{\mu \to \infty} \frac{R_{\nu,\mu}(x_0)}{P_{\nu,\mu}(x_0)} = 1.$$

We now obtain by a well-known argument [**5**]

$$R_{\nu,\mu}(x_0) \cong S_{\nu,\mu}(x_0)$$

$$\equiv \mu^{\frac{1}{2}}[2\pi x_0(1 - x_0)]^{-\frac{1}{2}} \int_{\nu\mu^{-1}}^{(\nu+1)\mu^{-1}} | \nu\mu^{-1} - x_0 |^\alpha \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (u - x_0)^2 \right] du$$

uniformly as $\mu \to \infty$ for all ν satisfying (20). Now

$$| \nu\mu^{-1} - x_0 |^\alpha - | u - x_0 |^\alpha = O(| \nu\mu^{-1} - u |^\alpha) = O(\mu^{-\alpha})$$

uniformly in ν and u as $|\nu\mu^{-1} - u| \leqslant \mu^{-1}$, and so we deduce

$$(22) \quad R_{\nu,\mu}(x_0) \cong S_{\nu,\mu}(x_0)$$

$$= \mu^{\frac{1}{2}}[2\pi x_0(1 - x_0)]^{-\frac{1}{2}} \int_{\nu\mu^{-1}}^{(\nu+1)\mu^{-1}} | u - x_0 |^\alpha \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (u - x_0)^2 \right] du$$

$$+ O\left[\mu^{-\alpha} \mu^{\frac{1}{2}}[2\pi x_0(1 - x_0)]^{-\frac{1}{2}} \int_{\nu\mu^{-1}}^{(\nu+1)\mu^{-1}} \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (u - x_0)^2 \right] du \right]$$

uniformly in ν satisfying (20), as $\mu \to \infty$.

Since $|\nu\mu^{-1} - x_0|^\alpha \leqslant 1$ and applying (8) we have

$$\sum_{|\nu\mu^{-1}-x_0|>\delta_0} |\nu\mu^{-1} - x_0|^\alpha p_{\nu,\mu}(x_0) \leqslant \sum_{|\nu n^{-1}-x_0|>\delta_0} p_{\nu,\mu}(x_0) \leqslant C_2\mu^{-l},$$

where $\delta_0 = \mu^{-\gamma}$ for every $l > 0$ with $0 < \gamma < \frac{1}{2}$, and where the constant $C_2 = C_2(\gamma, l)$. We take $l > \frac{1}{2}\alpha$. Then for $\frac{1}{3} < \gamma < \frac{1}{2}$,

$$\begin{aligned}\sum_{\nu=0}^{\mu} R_{\nu,\mu}(x_0) &= \sum_{|\nu u^{-1}-x_0|\leqslant\delta_0} R_{\nu,\mu}(x_0) + \sum_{|\nu\mu^{-1}-x_0|>\delta_0} R_{\nu,\mu}(x_0) \\ &= \sum_{|\nu\mu^{-1}-x_0|\leqslant\delta_0} R_{\nu,\mu}(x_0) + O\,(\mu^{-l}) \\ &= (1+\epsilon_\mu) \sum_{|\nu\mu^{-1}-x_0|\leqslant\delta_0} S_{\nu,\mu}(x_0) + o\,(\mu^{-\frac{1}{2}\alpha})\end{aligned}$$

where $\epsilon_\mu \to 0$ for $\mu \to \infty$. We now obtain

$$\begin{aligned}|f(x_0) - \mathfrak{L}_n^{[2]}(x_0)| &= |\mathfrak{L}_n^{[2]}(x_0)| = |2B_{2n}(x_0) - B_n(x_0)| \geqslant 2B_{2n}(x_0) - B_n(x_0) \\ &= 2\sum_{\nu=0}^{2n} R_{\nu,2n}(x_0) - \sum_{\nu=0}^{n} R_{\nu,n}(x_0) \\ &= (2+\epsilon'_n) \sum_{|\nu(2n)^{-1}-x_0|\leqslant\delta_2} S_{\nu,2n}(x_0) - (1+\epsilon''_n) \sum_{|\nu n^{-1}-x_0|\leqslant\delta_1} S_{\nu,n}(x_0) + o(n^{-\frac{1}{2}\alpha}),\end{aligned}$$

where $\delta_2 = (2n)^{-\gamma}$, $\delta_1 = n^{-\gamma}$, and $\epsilon'_n \to 0$ and $\epsilon''_n \to 0$ as $n \to \infty$. Applying (22), this expression is seen to be equal to

$$\begin{aligned}&\frac{2+\epsilon'_n}{\sqrt{\pi}} \frac{[x_0(1-x_0)]^{\frac{1}{2}\alpha}}{n^{\frac{1}{2}\alpha}} \int_0^{2^{-\gamma}n^{\frac{1}{2}-\gamma}[x_0(1-x_0)]^{-\frac{1}{2}}} v^\alpha \exp(-v^2)\,dv \\ &\quad + O\left[\frac{1}{(2n)^\alpha}\int_0^{2^{-\gamma}n^{\frac{1}{2}-\gamma}[x_0(1-x_0)]^{-\frac{1}{2}}} \exp(-v^2)\,dv\right] \\ &\quad - \frac{1+\epsilon''_n}{\sqrt{\pi}} \frac{[2x_0(1-x_0)]^{\frac{1}{2}\alpha}}{n^{\frac{1}{2}\alpha}} \int_0^{n^{\frac{1}{2}-\gamma}[2x_0(1-x_0)]^{-\frac{1}{2}}} v^\alpha \exp(-v^2)\,dv \\ &\quad + O\left[n^{-\alpha}\int_0^{n^{\frac{1}{2}-\gamma}[2x_0(1-x_0)]^{-\frac{1}{2}}} \exp(-v^2)\,dv\right] + o(n^{-\frac{1}{2}\alpha}),\end{aligned}$$

where $\frac{1}{3} < \gamma < \frac{1}{2}$. But the second and fourth terms need not be considered, as they are of order $O(n^{-\alpha})$; the integrals in the remaining two terms converge to the same positive limit, and the difference of the factors outside these integrals is of the form

$$C_3 n^{-\frac{1}{2}\alpha} + o(n^{-\frac{1}{2}\alpha})$$

where C_3 is positive as $0 < \alpha \leqslant 1$. So we deduce

$$|f(x_0) - \mathfrak{L}_n^{[2]}(x_0)| \geqslant C_4 n^{-\frac{1}{2}\alpha}$$

where C_4 is a strictly positive constant, proving the theorem.

We have constructed linear combinations of the Bernstein polynomials $B_n(x)$, namely

$$\mathfrak{L}_n^{[2k]}(x),$$

of degree $2^k n$, which under conditions imposed on the corresponding function, approach $f(x)$ more closely than

$$B_{2^k n}(x).$$

The order of approximation of a function by polynomials of best approximation is generally better than that given by the

$$\mathfrak{L}_n^{[2k]}(x).$$

For instance, if $f^{(2k)}(x) \in \text{Lip } \alpha$, $0 < \alpha \leqslant 1$, there are polynomials $P_n(x)$ of degree n such that

$$| P_n(x) - f(x) | \leqslant M' n^{-2k-\alpha}$$

[**4**, p. 18]. For the

$$\mathfrak{L}_n^{[2k]}(x)$$

of degree $2^k n$ we have

$$| \mathfrak{L}_n^{[2k]}(x) - f(x) | \leqslant M n^{-k-\frac{1}{2}\alpha}.$$

It remains an open question whether there are other linear combinations of degree not exceeding $2^k n$ approaching $f(x)$ more closely than the combination

$$\mathfrak{L}_n^{[2k]}(x).$$

References

1. S. N. Bernstein, *Complétement a l'article de E. Voronowskaja*, C. R. Acad. Sci. U.R.S.S. (1932), 86–92.

2. P. L. Butzer, *On Bernstein polynomials*, Thesis, University of Toronto Library (1951).

3. W. Feller, *An introduction to probability theory and its applications* (New York, 1950).

4. D. Jackson, *The theory of approximation* (Amer. Math. Soc. Coll. Publ., vol. 11, New York, 1930).

5. G. G. Lorentz, *Bernstein polynomials* (Toronto, 1953).

6. T. Popoviciu, *Sur l'approximation des fonctions convexes d'ordre supérieur*, Mathematica, Cluj, *10* (1935), 49–54.

7. E. Voronowskaja, *Détermination de la forme asymptotique d'approximation des fonctions par les polynomes de M. Bernstein*, C. R. Acad. Sci. U.R.S.S. (1932), 79–85.

The University of Toronto and
McGill University