Appendix A: Hardy space lemmas

A.1 Multipliers in h^1

We recall that ω is a modulus of continuity if $\omega : [0,\infty) \longrightarrow \mathbb{R}^+$ is continuous, increasing, $\omega(0) = 0$ and $\omega(2t) \le C\omega(t)$, $0 < t < 1$. A modulus of continuity determines the Banach space $C_\omega(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$|f|_{C_\omega} \doteq \sup_{x \neq y} \frac{|f(y) - f(x)|}{\omega(|x-y|)} < \infty,$$

equipped with the norm $\|f\|_{C_\omega} = \|f\|_{L^\infty} + |f|_{C_\omega}$. Note that C_ω is only determined by the behavior of $\omega(t)$ for values of t close to 0. Consider a modulus of continuity $\omega(t)$ that satisfies

$$\frac{1}{h^n} \int_0^h \omega(t) t^{n-1}\, \mathrm{d}t \le K \left(1 + \log \frac{1}{h}\right)^{-1}, \quad 0 < h < 1, \tag{A.1}$$

and the corresponding space $C_\omega(\mathbb{R}^n)$.

Lemma A.1.1. *Let $b \in C_\omega(\mathbb{R}^n)$ and $f \in h^1(\mathbb{R}^n)$. Then $bf \in h^1(\mathbb{R}^n)$ and there exists $C > 0$ such that*

$$\|bf\|_{h^1} \le C\|b\|_{C_\omega}\|f\|_{h^1}, \quad b \in C_\omega(\mathbb{R}^n), f \in h^1(\mathbb{R}^n).$$

Proof. Let $b(x) \in C_\omega$. It is enough to check that $\|bf\| \le C\|b\|_{C_\omega}$ for every h^1-atom a with C an absolute constant. This fact is obvious for atoms supported in balls B with radius $\rho \ge 1$ without moment condition because b is bounded so $ba/\|b\|_{L^\infty}$ is again an atom without moment condition. If $B = B(x_0, \rho)$, $\rho < 1$, we may write $a(x)b(x) = b(x_0)a(x) + (b(x) - b(x_0))a(x) = \beta_1(x) + \beta_2(x)$. Then $\beta_1(x)/\|b\|_{L^\infty}$ is again an atom while $\beta_2(x)$ is supported in B and satisfies

$$\|\beta_2\|_{L^\infty} \le 2\|b\|_{L^\infty}\|a\|_{L^\infty} \le \frac{C}{\rho^n},$$

$$\|\beta_2\|_{L^1} \le C\|a\|_{L^\infty} \int_B \omega(|x - x_0|)\, \mathrm{d}x \le \frac{C'}{(1 + |\log \rho|)}.$$

We wish to conclude that $\|m_\Phi \beta_2\|_{L^1} < \infty$. Let $B^* = B(x_0, 2\rho)$. Since $m_\Phi \beta_2(x) \leq \|\beta_2\|_{L^\infty}$, we have

$$J_1 = \int_{B^*} m_\Phi \beta_2(x)\,\mathrm{d}x \leq C|B^*|\rho^{-n} \leq C'.$$

It remains to estimate

$$J_2 = \int_{\mathbb{R}\setminus B^*} m_\Phi \beta_2(x)\,\mathrm{d}x = \int_{2\rho\leq|x-x_0|\leq 2} m_\Phi \beta_2(x)\,\mathrm{d}x \tag{A.2}$$

(observe that $m_\Phi \beta_2$ is supported in $B(x_0, 2)$ because $\operatorname{supp}\Phi \subset B(0,1)$). If $0 < \varepsilon < 1$ and $\Phi_\varepsilon * \beta_2(x) \neq 0$ for some $|x - x_0| \geq 2\rho$ it is easy to conclude that $\varepsilon \geq |x - x_0|/2$, which implies

$$|\Phi_\varepsilon * \beta_2(x)| \leq \left|\int \Phi_\varepsilon(y)\beta_2(x-y)\,\mathrm{d}y\right| \leq \frac{C\|\beta_2\|_{L^1}}{\varepsilon^n} \leq \frac{C'|x-x_0|^{-n}}{(1+|\log\rho|)}$$

so

$$m_\Phi \beta_2(x) \leq \frac{C'}{|x-x_0|^n(1+|\log\rho|)} \qquad \text{for} \quad |x-x_0| \geq 2\rho. \tag{A.3}$$

It follows from (A.2) and (A.3) that

$$J_2 \leq \int_{2\rho\leq|x-x_0|\leq 2} \frac{C'}{|x-x_0|^n(1+|\log\rho|)}\,\mathrm{d}x \leq C''$$

which leads to

$$\|ba\|_{h^1} \leq \|\beta_1\|_{h^1} + \|\beta_2\|_{h^1} \leq C_1 + J_1 + J_2 \leq C_2.$$

Inspection of the proof shows that C_2 may be estimated by $C\|b\|_{C_\omega}$. □

Example A.1.2. Suppose that a modulus of continuity $\omega(t)$ satisfies:

$$\omega(t)/t^n \quad \text{is a decreasing function of } t \tag{A.4}$$

and

$$D = \int_0^1 \frac{\omega(t)}{t}\,\mathrm{d}t < \infty. \tag{A.5}$$

A short and elegant argument shows (*cf.* [**Ta**], page 25) that under these conditions $h^1(\mathbb{R}^n)$ is stable under multiplication by elements of $C_\omega(\mathbb{R}^n)$. On the other hand, (A.5) alone already implies that

$$\omega(h)\log\frac{1}{h} = \int_h^1 \frac{\omega(h)}{t}\,\mathrm{d}t \leq \int_h^1 \frac{\omega(t)}{t}\,\mathrm{d}t \leq D, \quad 0 < h < 1,$$

which keeping in mind the obvious estimate

$$\frac{1}{h^n}\int_0^h \omega(t)t^{n-1}\,\mathrm{d}t \leq \frac{\omega(h)}{n}$$

shows that the modulus of continuity ω satisfies (A.1) and Lemma A.1.1 can be applied, proving the mentioned stability of $h^1(\mathbb{R}^n)$ under multiplication by elements of $C_\omega(\mathbb{R}^n)$.

Consider now a modulus of continuity $\omega(t)$ such that

$$\omega(t) = \frac{1 - n\log t}{\log^2 t}, \quad \text{for } 0 < t < 1/2.$$

Since $\omega(t) \geq |\log t|^{-1}$ it follows that $\int_0^{1/2}(\omega(t)/t)\,\mathrm{d}t = \infty$ and the Dini condition (A.5) is not satisfied. On the other hand,

$$\frac{1}{h^n}\int_0^h \omega(t)t^{n-1}\,\mathrm{d}t = \left(\log\frac{1}{h}\right)^{-1} \approx \left(1 + \log\frac{1}{h}\right)^{-1}, \quad \text{as } h \to 0,$$

so criterion (A.1) is satisfied. This shows that (A.5) is strictly more stringent than (A.1).

A.2 Commutators

We consider now a bounded smooth function $\psi(\xi)$, $\xi \in \mathbb{R}$, such that

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}x^k}\psi(\xi)\right| \leq C_k \frac{1}{(1+|\xi|)^k}, \qquad \xi \in \mathbb{R}, \quad k = 0, 1, 2, \ldots$$

Then $\psi(\xi)$ is a symbol of order zero and defines the pseudo-differential operator

$$\psi(D)u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{ix\xi}\psi(\xi)\widehat{u}(\xi)\,\mathrm{d}\xi, \quad u \in \mathcal{S}(\mathbb{R}).$$

In particular, $\psi(D)$ is bounded in $h^1(\mathbb{R})$. The Schwartz kernel of $\psi(D)$ is the tempered distribution $k(x-y)$ defined by $\widehat{k}(\xi) = \psi(\xi)$ which is smooth outside the diagonal $x \neq y$. Moreover, $k(x-y)$ may be expressed as

$$k(x-y) = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int \mathrm{e}^{i(x-y)\xi - \varepsilon|\xi|^2}\psi(\xi)\,\mathrm{d}\xi = \lim_{\varepsilon\to 0} k_\varepsilon(x-y),$$

where the limit holds both in the sense of $\mathcal{S}'$ and pointwise for $x \neq y$. Furthermore, the approximating kernels $k_\varepsilon(x-y)$ satisfy uniformly in $0 < \varepsilon < 1$ the pointwise estimates

$$|k_\varepsilon(x-y)| \leq \frac{C_j}{|x-y|^j}, \quad j = 1, 2, \ldots \tag{A.6}$$

which of course also hold for $k(x-y)$ itself when $x \neq y$.

We consider a function $b(x)$ of class $C^{1+\sigma}$, $0 < \sigma < 1$, and wish to prove that the commutator $[\psi(D), b\partial_x]$ is bounded in $h^1(\mathbb{R})$. A simple standard computation that includes an integration by parts gives

$$[\psi(D), b\partial_x]u(x) = \int k'(x-y)(b(y) - b(x))u(y)\,\mathrm{d}y - \psi(D)(b'u)$$

where the integral should be interpreted as the pairing

$$\langle < k'(x - \cdot)(b(\cdot) - b(x)), u(\cdot)\rangle$$

between a distribution depending on the parameter x and a test function u. Since multiplication by b' is bounded in $h^1(\mathbb{R})$ with norm controlled by $\|b'\|_{C^\sigma}$, we need only worry about the remaining integral term that can be rewritten as

$$\begin{aligned} Tu(x) &= \int (y-x)k'(x-y)\frac{b(x)-b(y)}{x-y}u(y)\,\mathrm{d}y \\ &= \int k_1(x-y)\,\beta(x,y)u(y)\,\mathrm{d}y \end{aligned} \tag{A.7}$$

where

$$\beta(x,y) = \int_0^1 b'(\tau x + (1-\tau)y)\,\mathrm{d}\tau \quad \text{and} \quad k_1(x) = -xk'(x).$$

Observe that $\beta \in C^\sigma(\mathbb{R}^2)$.

Lemma A.2.1. *Assume T is given by* (A.7) *with kernel*

$$K(x,y) = k_1(x-y)\,\beta(x,y).$$

Then T is bounded in $h^1(\mathbb{R})$.

Proof. It follows that $\widehat{k_1}(\xi) = (\xi k(\xi))' = \psi(\xi) + \xi\,\psi'(\xi)$. In other words, $\widehat{k_1}(\xi) = \psi_1(\xi)$ is a symbol of order zero and T has kernel $k_1(x-y)\,\beta(x,y)$. We may write $\beta(x,y) = b'(x) + |x-y|^\sigma r(x,y)$ with $r(x,y) \in L^\infty(\mathbb{R}^2)$ so

$$\begin{aligned} Tu(x) &= b'(x)\psi_1(D)u(x) + \int k_1(x-y)\,|x-y|^\sigma r(x,y)u(y)\,\mathrm{d}y, \\ &= T_1u(x) + T_2u(x). \end{aligned}$$

The first operator T_1 is obviously bounded in h^1 because it is the composite of $\psi_1(D)$ with multiplication by a C^σ function. To analyze T_2 we check—writing $k_1 = \lim_{\varepsilon\to 0} k_{1,\varepsilon}$ and using (A.6) for $k_{1,\varepsilon}$—that its Schwartz kernel is a locally integrable distribution given by the integrable function $k_2(x,y) = k_1(x-y)\,|x-y|^\sigma r(x,y)$. Hence, $|k_2(x,y)| \leq C_1|k_1(x-y)|\,|x-y|^\sigma = k_3(x-y)$. Observe that $k_3(x) \leq C\min(|x|^{\sigma-1}, |x|^{-2})$ so $k_3 \in L^1(\mathbb{R})$. We will now show that

$$m_\Phi k_3(x) = \sup_{0<\varepsilon<1} |\Phi_\varepsilon * k_3(x)| \in L^1(\mathbb{R}),$$

where $\Phi \geq 0 \in C_c^\infty([-1/2, 1/2])$, $\int \Phi\mathrm{d}z = 1$, $\Phi_\varepsilon(x) = \varepsilon^{-1}\Phi(x/\varepsilon)$. Since $m_\Phi k_3$ is pointwise majorized by the restricted Hardy–Littlewood maximal function

$$mk_3(x) = \sup_{0<\varepsilon<1} \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} k_3(t)\,\mathrm{d}t$$

we start by observing that

$$\sup_{0<\varepsilon<1} \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |t|^{\sigma-1}\,\mathrm{d}t \leq \frac{|x|^{\sigma-1}}{\sigma}. \tag{A.8}$$

In doing so we may assume that $x > 0$. If $0 < \varepsilon \leq x$ we have

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |t|^{\sigma-1}\,\mathrm{d}t = \frac{(x+\varepsilon)^\sigma - (x-\varepsilon)^\sigma}{2\varepsilon\sigma} \leq \frac{(x+\varepsilon)^{\sigma-1}}{\sigma} \leq \frac{x^{\sigma-1}}{\sigma}$$

where we have used the elementary inequality

$$\frac{b^\sigma - a^\sigma}{b-a} \le b^{\sigma-1}, \quad 0 \le a < b, \quad 0 < \sigma < 1.$$

Similarly, if $0 < x < \varepsilon$,

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |t|^{\sigma-1}\,\mathrm{d}t = \frac{(x+\varepsilon)^\sigma + (x-\varepsilon)^\sigma}{2\varepsilon\sigma} \le \frac{(x+\varepsilon)^{\sigma-1}}{\sigma} \le \frac{x^{\sigma-1}}{\sigma}.$$

This proves (A.8). Thus,

$$m_\Phi k_3(x) \le C\, mk_3(x) \le C'\,|x|^{\sigma-1}$$

which shows that $m_\Phi k_3$ is locally integrable. For large $|x|$ the inequality $k_3(x) \le C|x|^{-2}$ easily implies $m_\Phi k_3(x) \le C|x|^{-2}$ and we conclude that $m_\Phi k_3 \in L^1$. Finally, we see that

$$|\Phi_\varepsilon * T_2 u(x)| \le \Phi_\varepsilon * k_3 * |u|(x) \le m_\Phi k_3 * |u|(x)$$

so $m_\Phi T_2 u(x) \le m_\Phi k_3 * |u|(x)$, which implies that $\|T_2 u\|_{h^1} \le C\|u\|_{L^1} \le C\|u\|_{h^1}$. This proves that $T = T_1 + T_2$ is bounded in $h^1(\mathbb{R})$. □

Summing up, we have proved:

PROPOSITION A.2.2. *If $\psi(\xi)$, $\xi \in \mathbb{R}$, is a smooth symbol of order 0 and $b(x) \in C^{1+\sigma}(\mathbb{R})$, $0 < \sigma < 1$, the commutator*

$$[\psi(D), b\partial_x]$$

is bounded in $h^1(\mathbb{R})$.

A.3 Change of variables

Consider a diffeomorphism $F: \mathbb{R}^n \to \mathbb{R}^n$ of class C^1, with Jacobian F' such that for some $K \ge 1$

$$K^{-1}|x-y| \le |F(x) - F(y)| \le K|x-y|, \quad x, y \in \mathbb{R}^n. \tag{A.9}$$

Write $H = F^{-1}$, denote by H' the Jacobian matrix of H, and assume that

$$\det H' \in C_\omega, \tag{A.10}$$

where the modulus of continuity $\omega(t)$ satisfies

$$\frac{1}{h^n}\int_0^h \omega(t)t^{n-1}\,\mathrm{d}t \le K\left(1 + \log\frac{1}{h}\right)^{-1}, \quad 0 < h < 1.$$

Notice that if F is a diffeomorphism of Hölder class $C^{1+\varepsilon}$, $\varepsilon > 0$, then (A.9) and (A.10) hold.

PROPOSITION A.3.1 (S. Chanillo, [**Ch2**]). *If F satisfies* (A.9) *and* (A.10)*, the map $h^1(\mathbb{R}^n) \ni g \mapsto g \circ F$ is bounded in $h^1(\mathbb{R}^n)$.*

https://doi.org/10.1017/CBO9780511543067.011 Published online by Cambridge University Press

The main step in the proof of the proposition is

LEMMA A.3.2. *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism such that for some $K \geq 1$*

$$K^{-1}|x-y| \leq |H(x)-H(y)| \leq K|x-y|, \quad x, y \in \mathbb{R}^n. \tag{A.11}$$

Let $\Phi \in C_c^\infty(B(0,1))$, $\Phi_t(x) = t^{-n}\Phi(x/t)$, $u \in H^1(\mathbb{R}^n)$ and set

$$U(x,t) = \int \Phi_t(H(x)-H(z))u(z)\,\mathrm{d}z, \quad 0 < t < 1,$$
$$U^*(x) = \sup_{0<t<1} |U(x,t)|.$$

Then there exists a constant $C > 0$ depending on the dimension n, on K and on Φ but not on u such that

$$\int U^*(x)\,\mathrm{d}x \leq C\|u\|_{h^1}. \tag{A.12}$$

PROOF. In view of the atomic decomposition it is enough to prove (A.12) when $u(x)$ is an atom, that we denote by $a(x)$. We must show that if $a(x)$ is an h^1-atom and

$$A(x,t) = \int \Phi_t(H(x)-H(z))a(z)\,\mathrm{d}z, \quad 0 < t < 1,$$
$$A^*(x) = \sup_{0<t<1} |A(x,t)|,$$

then $\|A^*\|_{L^1} \leq C$ with C independent of $a(x)$. Let $a(x)$ be an atom supported in ball $B = B(z_0, r)$ such that $\|a\|_{L^\infty} \leq |B|^{-1}$. Note that in view of (A.11) and the hypothesis on Φ

$$|x-z| \geq Kt \implies |H(x)-H(z)| \geq t \implies \Phi_t(H(x)-H(z)) = 0$$

for $0 < t < 1$ so

$$|A(x,t)| \leq \|a\|_{L^\infty}\|\Phi\|_{L^\infty} \int_{|z-x|<Kt} \frac{1}{t^n}\,\mathrm{d}z \leq \frac{C}{r^n},$$

showing that

$$|A^*(x)| \leq \frac{C}{r^n}. \tag{A.13}$$

If we write $B^* = B(z_0, 2r)$ we see right away that

$$\int_{B^*} A^*(x)\,\mathrm{d}x \leq C$$

and we need only concern ourselves with the integral

$$\int_{\mathbb{R}^n \setminus B^*} A^*(x)\,\mathrm{d}x.$$

We first consider the case $0 < r < 1$ so that $a(x)$ has vanishing mean $\int a(x)\,\mathrm{d}x = 0$. We will initially show that $A(x,t) = 0$ if $x \notin B^*$ and $2Kt \leq |x-z_0|$. Since $|x-z_0| \geq 2r$ and $|z-z_0| \leq r$ implies that $|z-z_0| \leq |x-z_0|/2$ we obtain from the triangular property that $|x-z| \geq |x-z_0|/2$ if $|x-z_0| \geq 2r$ and $|z-z_0| \leq r$. Thus, $2Kt \leq |x-z_0| \leq 2|x-z| \leq 2K|H(x)-H(z)|$. This implies that $|H(x)-H(z)|/t \geq 1$ so $\Phi_t(H(x)-H(z))a(z) = 0$.

Hence, $A(x,t)=0$ if $|x-z_0|\geq 2r$ and $t\leq |x-z_0|/(2K)$ and when we estimate $A^*(x)$ on $\mathbb{R}^n\backslash B^*$ we may take the supremum of $|A(x,t)|$ for t in the range $|x-z_0|/(2K)\leq t<1$. We may write

$$\begin{aligned}|A(x,t)| &= \left|\int (\Phi_t(H(x)-H(z))-\Phi_t(H(x)-H(z_0))\, a(z)\,\mathrm{d}z\right|\\ &\leq \frac{C\,\|a\|_{L^\infty}}{t^{n+1}}\int_{B(z_0,r)}|H(z)-H(z_0)|\,\mathrm{d}z\\ &\leq \frac{Cr}{|x-z_0|^{n+1}}\end{aligned}$$

to conclude that

$$A^*(x)\leq \frac{Cr}{|x-z_0|^{n+1}}\quad \text{for}\quad x\notin B^*$$

and

$$\int_{B^*} A^*(x)\,\mathrm{d}x\leq C.$$

Assume now that $r\geq 1$. Then, for $|z-z_0|\leq r$ and $|x-z_0|\geq (K+1)r$ we have $|x-z|\geq (K+1)r-r=Kr$ so

$$|H(x)-H(z)|\geq r\geq 1 \quad \text{and} \quad \Phi_t(H(x)-H(z))=0.$$

This shows that $\operatorname{supp} A(x,t)\subset B(z_0,(K+1)r)$ and also $\operatorname{supp} A^*\subset B(z_0,(K+1)r)$. Hence, we get

$$\|A^*\|_{L^1}\leq \|A^*\|_{L^\infty}|\operatorname{supp} A^*|\leq C,$$

where we have used (A.13). □

Proof of Proposition A.3.1. Let $g\in h^1(\mathbb{R}^n)$. Choose some test function $0\leq\Phi\in C_c^\infty(B(0,1))$ with $\int\Phi(x)\,\mathrm{d}x=1$ and set $v=g\circ F$. We must show that $v^*(x)=\sup_{0<t<1}|\Phi_t*[g\circ F](x)|$ satisfies $\|v^*\|_{L^1}\leq C\|g\|_{h^1}$. Since

$$\int v^*(y)\,\mathrm{d}y=\int v^*\circ H(x)\,|\det H'(x)|\,\mathrm{d}x\leq C\|v^*\circ H\|_{L^1},$$

it is enough to estimate

$$\|v^*\circ H\|_{L^1}=\int \sup_{0<t<1}\left|\int \Phi_t(H(x)-z)g(F(z))\,\mathrm{d}z\right|\,\mathrm{d}x$$

which after the change of variables $z=H(y)$ may be written as

$$I=\int \sup_{0<t<1}\left|\int \Phi_t(H(x)-H(y))\,g(y)\,|\det H'(y)|\,\mathrm{d}y\right|\,\mathrm{d}x$$

because $H=F^{-1}$. Notice that $u(y)=\pm g(y)\det H'(y)\in h^1(\mathbb{R}^n)$ by Lemma A.1.1 and (A.10); furthermore, $\|u\|_{h^1}\leq C\|g\|_{h^1}$. Using Lemma A.3.2 we get $I\leq C\|u\|_{h^1}\leq C'\|g\|_{h^1}$, as we wished to prove. □