A Simple Approach to Great Circle Sailing: The COFI Method

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An approach formulated by vector algebra is proposed to deal with great circle sailing problems. Using the technique of the fixed coordinates system and relative longitude concept, derivations of formulae for this approach are simpler than those of the conventional methods. Due to fixing the initial great circle course, the great circle track (GCT) is determined. Since the course is fixed (known as "COFI" in this paper), the proposed approach, which we have named the "COFI method", can directly calculate the waypoints along the GCT. It is considered that the COFI method is a more understandable and straightforward method to solve waypoint problems than older approaches in the literature. Based on the COFI method, a program has been developed for the navigator. In addition, the spherical triangle method with respect to the equator crossing point (STM-E) is developed by supplemental theorem. Several examples are demonstrated to validate the proposed COFI method and STM-E.

KEY WORDS

1. INTRODUCTION. The shortest distance between any two non-antipodal points on a sphere's surface is the only great circle track (GCT) passing through them. The Earth can be considered as a sphere for obtaining the waypoints on the GCT by using great circle sailing (GCS). However, because of the Earth's rotation, the Earth is approximately an oblate spheroid (or ellipsoid of revolution). Consequently, the Mercator chart or an Electronic Chart Display and Information System (ECDIS) usually uses the WGS 84 (World Geodetic System ellipsoid of 1984). A great circle other than a meridian or the equator is a curved line whose true direction changes continually, thus navigators do not usually attempt to follow it exactly. Instead, they select a number of waypoints along the GCT, construct rhumb lines between the waypoints on the Mercator chart or in the ECDIS, and then steer along these rhumb lines (Bowditch, 2002). In practical navigation, the waypoints on the GCT are entered into the ECDIS, GPS, or a fully integrated navigation system. Then, the vessel is programmed to follow the GCT by sailing from waypoint to waypoint by rhumb line, with allowances made for wind and current effects.

When the Earth is regarded as a sphere, the navigator has to give initial conditions for obtaining the waypoints along the GCT. Generally speaking, these given initial conditions include: giving the longitudes of the waypoints to obtain their latitudes (Condition 1); giving the great circle distances to yield the latitudes and longitudes of the waypoints (Condition 2). Once all the waypoints on the GCT are available, the navigator needs to take the Earth as an oblate spheroid for practical navigation. Because the GCT is composed of legs of rhumb lines, the course and distance of the rhumb line between two adjacent waypoints can be determined by using the Mercator sailing.

The spherical triangle method (STM) with a reference point at the vertex (called STM-V) has been developed to solve the waypoints on the GCT for many years. It can handle the waypoints problem under Conditions 1 or 2 (Holm, 1972; Bowditch, 1981, 2002; Keys, 1983; Cutler, 2004; Chen et al., 2004). The advantage of this method is that the solved formulae are simplified because the method uses Napier's rules of right-angled spherical triangles. Finding the equator crossing point of the GCT is easier than finding the vertex of the GCT. In addition, when the supplemental theorem is introduced, the right-angled spherical triangles can be converted into quadrantal spherical triangles (Clough-Smith, 1966). Therefore, a method with the reference point at the equator crossing point (called STM-E) should be available. The STM-E has the same advantage as the STM-V and can also deal with the waypoints problem under Condition 1 or 2. However, a common disadvantage of the STM-V and STM-E is that the reference point should be determined in advance. Owing to this disadvantage, both methods are usually considered as a type of indirect approach (IA).

To overcome this shortcoming, some researchers take the departure point as the reference point. This means one can replace the Greenwich meridian by the meridian of the departure point and this is usually called the relative longitude concept. Similarly, Jofeh (1981) constructed a linear equation (LE) of the GCT, which appears as a straight line on the polar gnomonic chart. According to his method, the latitudes of the waypoints are determined only under Condition 1. Unfortunately, when the departure and destination points are located in different hemispheres, the method fails. Later, Miller et al. (1991) first used the technique of the fixed coordinates system to construct a vector expression of the waypoints and then adopted linear combination (LC) of a vector basis to formulate another vector expression. Comparing the components of the different vector expressions yields three key formulae, that is, five-parts formula (x-component), five-parts formula (y-component) and side cosine law (z-component). Then, a combination of the five-parts formula (x-component) and the side cosine law (z-component) can handle the waypoints problem but only under Condition 2. Thereafter, Chen et al. (2004) first combined the technique of the fixed coordinates system with the relative longitude concept (FCRL) and then proposed the great circle equation method (GCEM), in which the great circle equation is formulated by vector algebra. It is found that the GCEM can deal with the waypoints problem only under Condition 1. In addition, like Miller et al. (1991), Nastro and Tancredi (2010) adopted linear combination (LC) of a different vector basis with the FCRL, also reaching three key formulae. That is, five-parts formula (x-component), sine law (y-component) and side cosine law (z-component). Then, the sine law divided by the five-parts formula obtains the four-parts formula. A combination of this formula and the yielded side cosine law can handle the waypoints problem but only under Condition 2. However, tedious derivations make their solutions hard to understand.

Table 1. A comparison of different methods for solving the GCS.

Authors	Category	Condition 1	Condition 2
Holm, 1972	IA (STM-V)	available	NA
Jofeh, 1981	DA (LE)	Available*	NA
Bowditch, 1981 and 2002	IA (STM-V)	available	available
Keys, 1983	IA (STM-V)	available	available
Miller et al., 1991	DA (LC)	NA	available
Cutler, 2004	IA (STM-V)	available	available
Chen et al., 2004	IA (STM-V)	available	available
	DA (GCEM)	available	NA
Nastro and Tancredi, 2010	DA (LC)	NA	available
Chen et al., 2013 (the current paper)	IA (STM-E)	available	available
	DA (COFI)	NA	available

* This method fails when departure and destination points are located in different hemispheres.

In contrast to the IA, the methods mentioned above all belong to a type of direct approach (DA). The comparison of the mentioned methods is listed in Table 1. In this table, it is found that the DA can solve the waypoints problem under either Condition 1 or Condition 2; while the IA can deal with the problems under both Conditions 1 and 2.

To overcome the complex derivations (Miller et al., 1991; Nastro and Tancredi, 2010), the concise derivation of the formulae by using multiple products of vector algebra (VA-MP) with the FCRL is proposed to solve the waypoints problem under Condition 2. Once the initial great circle course is fixed (COFI), the GCT can be determined. With this characteristic, the proposed approach is named the "COFI method". Further, to tackle the waypoints problem covering Conditions 1 and 2, a program, based on the COFI method and the simplified GCEM, has been developed for the practical navigator. In addition, because the STM-E can deal with the waypoints problem under two given conditions, derivations of the method are also included in this article.

Theoretical backgrounds of the STM-E, the COFI method and the simplified GCEM are presented in Section 2. Section 3 describes the computation procedures of the COFI method and the simplified GCEM. Validated examples are given in Section 4. Finally, the work is summarised and concluded in Section 5.

2. THEORETICAL BACKGROUNDS.

2.1. *Deriving Formulae for the STM-E.* As mentioned in the previous section, the supplemental theorem can be used to derive the formulae of the STM-E for solving the waypoints problem. The supplemental theorem describes (Clough-Smith, 1966):

"*The angles in the polar triangle are supplements of the corresponding sides in the primitive triangle, and the sides in the polar triangle are supplements of the corresponding angles in the primitive triangle.*"

Due to this property, those formulae used in right-angled spherical triangles can also work in quadrantal spherical triangles. In addition, because the great circle arc from the equator crossing point to the pole should be 90°, the equator crossing point,
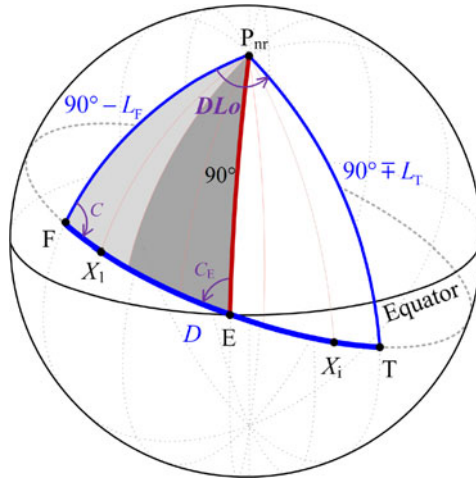
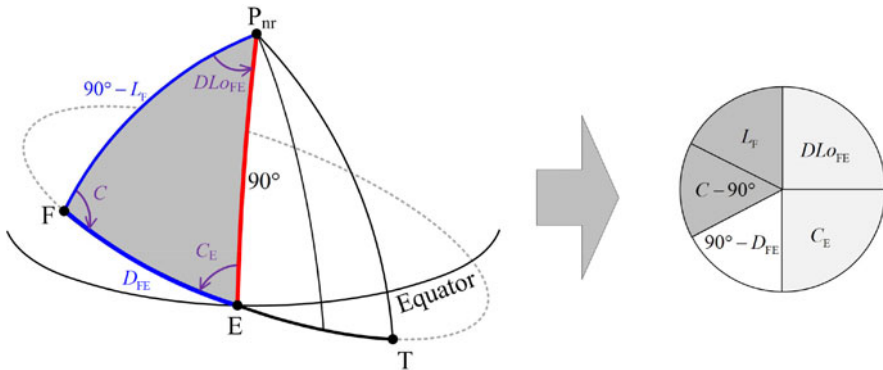Figure 1. An illustration of the STM-E for solving the problem of GCS.

Figure 2. An illustration of finding the equator crossing point on the GCT by using Napier's rule of quadrantal spherical triangles.

the pole nearer the departure and waypoints along the GCT can form numerous quadrantal spherical triangles as shown in Figure 1. Consequently, the solving steps of the STM-E are presented as follows. All the symbols used below are listed in the Appendix.

Step 1. Finding the great circle distance (D) and the initial great circle course angle (C) as shown in Figure 1. The great circle distance and the initial course angle can be calculated by the side cosine law and the four-parts formula of the spherical trigonometry, respectively as (Chen et al., 2004):

$$\cos D = \sin L_F \sin L_T + \cos L_F \cos L_T \cos DLo, \tag{1}$$

$$\tan C = \frac{\sin DLo}{(\cos L_F \tan L_T) - (\sin L_F \cos DLo)}. \tag{2}$$

Step 2. Finding the longitude of the equator crossing point, λ_E, as shown in Figure 2. By using Napier's rules of quadrantal spherical triangles, the following two formulae
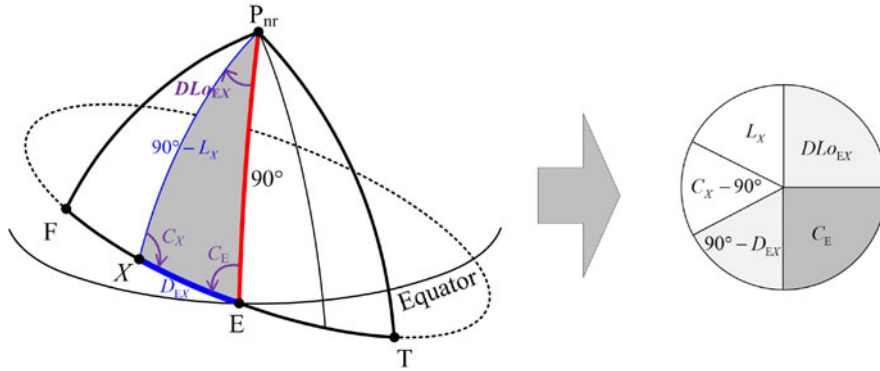
Figure 3. An illustration of finding the waypoints on the GCT by using Napier's rule of quadrantal spherical triangles.

can be yielded as:

$$\tan DLo_{FE} = -\sin L_F \tan C, \tag{3}$$

$$\sin C_E = \cos L_F \sin C. \tag{4}$$

Note that if DLo_{FE} has the same name as DLo (i.e. both West or both East) and λ_F is available, λ_E can be obtained by Equation (3). In addition, C_E, can be obtained by Equation (4) and it will be used in the following step.

Step 3. Finding the latitudes and longitudes of the waypoints along the GCT is shown in Figure 3. Since only C_E is available, the given condition is necessary for obtaining the waypoints. By using Napier's rules of quadrantal spherical triangles, we can yield the following formulae under both Conditions 1 and 2.

Condition 1. When λ_X is given, DLo_{EX} can be obtained. Then, L_X can be calculated from the following formula

$$\tan L_X = \pm \cot C_E \sin DLo_{EX}. \tag{5}$$

Note that if DLo_{EX} is the contrary name to DLo (i.e. one East and one West), the right-hand side of Equation (5) should take the positive sign. It means L_X and L_F are located in the same hemisphere. Conversely, if DLo_{EX} has the same name as DLo, the right-hand side of Equation (5) should be treated as negative sign. It means L_X and L_F are located in different hemispheres.

Condition 2. When the D_{EX} is given, the waypoints can be obtained from the formulae,

$$\sin L_X = \cos C_E \sin D_{EX}, \tag{6}$$

$$\tan DLo_{EX} = \sin C_E \tan D_{EX}. \tag{7}$$

In Equation (6), note that when L_X is smaller than L_F, the waypoints are on the GCT. This means L_X and L_F are located in the same hemisphere. Thus, L_X is taken as the positive. Similarly, when L_X is smaller than L_T, the waypoints are on the GCT but L_X and L_T are located in different hemispheres. In this regard, L_X is taken as the negative. In Equation (7), if the value of $\tan DLo_{EX}$ is negative, $(180° - DLo_{EX})$ should replace $(-DLo_{EX})$ for satisfying the definition of DLo_{EX}. As for the designated
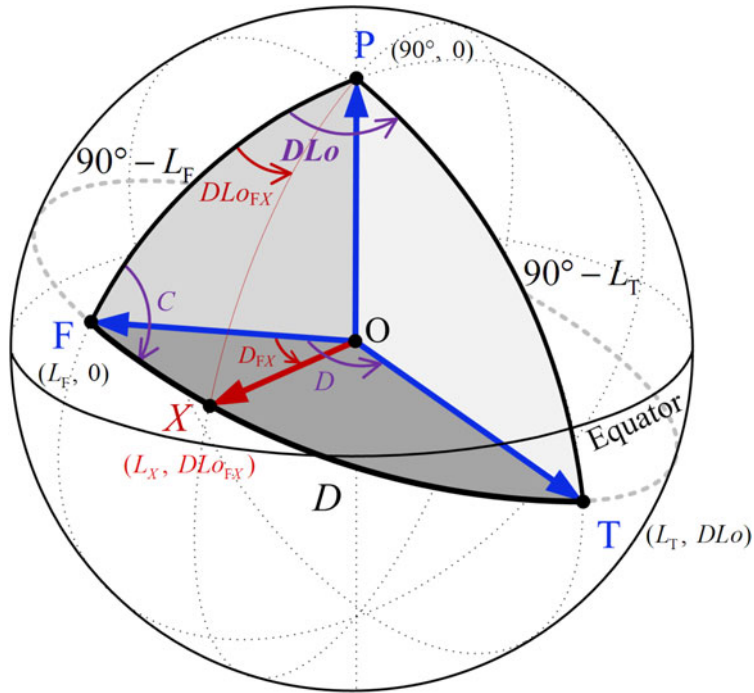
Figure 4. An illustration of four position vectors.

east or west of DLo_{EX}, it depends on the sign of L_X. If L_X is positive, DLo_{EX} is contrary name to DLo. However, if L_X is negative, DLo_{EX} has the same name as DLo. Obviously, too many judgments of sign conventions arise in the solving procedures of the STM-E and this makes use of this method hard work for the navigator. However, the STM-E method still offers a way to solve the waypoints problem.

2.2. *Deriving Formulae for the COFI method.* As the Earth is treated as a unitary sphere, the vector expression of any point $G(L, \lambda)$ on the Earth's surface in a Cartesian coordinates system can be written as:

$$\vec{G} = [\cos L \cos \lambda,\ \cos L \sin \lambda,\ \sin L],\ L = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \lambda = [0, 2\pi). \qquad (8)$$

To avoid an additional judgment of sign convention, the technique of the fixed coordinates system is first considered, that is, the north latitude is treated as a positive value and the south latitude is taken as a negative one. As shown in Figure 4, introducing the relative longitude concept, the unit vectors of the North Pole (P), the departure (F), the destination (T) and the waypoints (X) on a GCT can be expressed as:

$$\vec{P} = [0, 0, 1], \qquad (9)$$

$$\vec{F} = [\cos L_F, 0,\ \sin L_F], \qquad (10)$$

$$\vec{T} = [\cos L_T \cos DLo,\ \cos L_T \sin DLo,\ \sin L_T], \qquad (11)$$

$$\vec{X} = [\cos L_X \cos DLo_{FX},\ \cos L_X \sin DLo_{FX},\ \sin L_X]. \qquad (12)$$

When the vector algebra is introduced, derivations of the formulae used for the COFI will be simpler and clearer than those of the DA. Therefore, we adopt multiple products of the vector algebra to yield the great circle distance, the initial course and the waypoints on the GCT (Spiegel, 2009; Chen et al., 2004).

2.2.1. *Obtaining the great circle distance.* As shown in Figure 4, there are two ways to obtain the great circle distance. One is to yield the great circle arc (D) of the spherical triangle by the dot product of two unit vectors. That is,

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{T}} = \cos D$$
$$= \cos L_F \cos L_T \cos DLo + \sin L_F \sin L_T, \qquad (13)$$

in which the first row of the above equation uses the geometric definition of vector product, while the second row uses the algebraic operation of vector product.

Another method is to adopt the dot product of two normal vectors, $(\vec{\mathbf{P}} \times \vec{\mathbf{T}})$ and $(\vec{\mathbf{P}} \times \vec{\mathbf{F}})$, to yield the dihedral angle (DLo) of the spherical triangle. Therefore,

$$(\vec{\mathbf{P}} \times \vec{\mathbf{T}}) \cdot (\vec{\mathbf{P}} \times \vec{\mathbf{F}}) = \cos L_T \cos L_F \cos DLo$$
$$= \begin{vmatrix} (\vec{\mathbf{P}} \cdot \vec{\mathbf{P}}) & (\vec{\mathbf{P}} \cdot \vec{\mathbf{F}}) \\ (\vec{\mathbf{T}} \cdot \vec{\mathbf{P}}) & (\vec{\mathbf{T}} \cdot \vec{\mathbf{F}}) \end{vmatrix} = \cos D - \sin L_F \sin L_T. \qquad (14)$$

Similarly, two rows of Equation (14) represent the same mathematical meanings as those of Equation (13). After arranging Equations (13) or (14), we can write the same governing equation as:

$$\cos D = \sin L_F \sin L_T + \cos L_F \cos L_T \cos DLo. \qquad (15)$$

Equation (15) is the well-known side cosine law of spherical trigonometry.

2.2.2. *Obtaining the initial great circle course angle.* To yield the initial great circle course angle, that is, the dihedral angle (C) of the spherical triangle, we adopt the dot product of two normal vectors, $(\vec{\mathbf{F}} \times \vec{\mathbf{P}})$ and $(\vec{\mathbf{F}} \times \vec{\mathbf{T}})$. Hence,

$$(\vec{\mathbf{F}} \times \vec{\mathbf{P}}) \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{T}}) = \cos L_F \sin D \cos C$$
$$= \sin L_T - \sin L_F \cos D, \qquad (16)$$

in which the first row uses the geometric definition of vector products, while the second row uses the algebraic operation of vector products. Rearranging Equation (16) obtains the governing equation as:

$$\cos C = \frac{\sin L_T - \sin L_F \cos D}{\cos L_F \sin D}. \qquad (17)$$

Equation (17) is another form of the side cosine law of the spherical trigonometry.

2.2.3. *Obtaining the latitudes of waypoints along the GCT.* As the initial great circle course is fixed, the GCT can be determined. Then, giving the great circle distance from the departure point (Condition 2) we can obtain every waypoint along the GCT. Replacing the parameter vector $\vec{\mathbf{T}}$ of Equation (16) by the variable

vector $\vec{\mathbf{X}}$ yields

$$(\vec{\mathbf{F}} \times \vec{\mathbf{P}}) \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{X}}) = \cos L_F \sin D_{FX} \cos C \tag{18}$$
$$= \sin L_X - \sin L_F \cos D_{FX}.$$

Rearranging Equation (18) obtains

$$\sin L_X = \sin L_F \cos D_{FX} + \cos L_F \sin D_{FX} \cos C. \tag{19}$$

The above equation is also the side cosine law of the spherical trigonometry.

    2.2.4.   *Obtaining the longitudes of waypoints along the GCT.*   Replacing parameter vector $\vec{\mathbf{T}}$ of Equation (14) by variable $\vec{\mathbf{X}}$ vector yields

$$(\vec{\mathbf{P}} \times \vec{\mathbf{X}}) \cdot (\vec{\mathbf{P}} \times \vec{\mathbf{F}}) = \cos L_X \cos L_F \cos DLo_{FX} \tag{20}$$
$$= \cos D_{FX} - \sin L_F \sin L_X.$$

Rearranging Equation (20), we have

$$\cos DLo_{FX} = \frac{\cos D_{FX} - \sin L_F \sin L_X}{\cos L_F \cos L_X}. \tag{21}$$

The above equation is another form of the side cosine law of spherical trigonometry.

    Note that Equations (19) and (21) are a set of the governing equations to obtain the latitudes and longitudes of the waypoints along the GCT under Condition 2. In summary, formulae used in the COFI method are only a form of the side cosine law of spherical trigonometry. Therefore, introducing the vector algebra into derivations of the COFI method makes this method simpler and clearer than the conventional approaches.

    2.3.   *Reformulating the formulae used for the GCEM.*   To solve the waypoints problem covering Conditions 1 and 2, the GCEM and the COFI method should be combined for the practical navigator. Therefore, revisiting and simplifying the formulae of the GCEM are described as follows.

    2.3.1.   *Revisiting the GCEM.*   Those formulae used for the GCEM are briefly revisited here (Chen et al., 2004). As shown in Figure 4, if three vectors are coplanar, the scalar triple product is equal to zero. That is,

$$(\vec{\mathbf{F}} \times \vec{\mathbf{T}}) \cdot \vec{\mathbf{X}} = 0. \tag{22}$$

Now, assuming

$$\vec{\mathbf{F}} \times \vec{\mathbf{T}} = [a, b, c], \tag{23}$$

and substituting Equations (10) and (11) into Equation (23) yield

$$a = -\sin L_F \cos L_T \sin DLo, \tag{24}$$

$$b = \sin L_F \cos L_T \cos DLo - \cos L_F \sin L_T, \tag{25}$$

$$c = \cos L_F \cos L_T \sin DLo. \tag{26}$$

Finally, the Great Circle Equation can be formulated as

$$a \cos L_X \cos DLo_{FX} + b \cos L_X \sin DLo_{FX} + c \sin L_X = 0. \tag{27}$$

Note that Equation (27) implies the information of the great circle, for example, the waypoints along the GCT, the equator crossing point and the vertex.

1. The waypoints along the GCT: When λ_X is given, DLo_{FX} can be obtained. Rearranging Equation (27) yields

$$\tan L_X = \frac{a \cos DLo_{FX} + b \sin DLo_{FX}}{-c}. \tag{28}$$

2. The equator crossing point: Since $L_E = 0$, substituting it into Equation (28) yields

$$\tan DLo_{FE} = -\frac{a}{b}. \tag{29}$$

3. The vertex: When the vertex is the highest latitude for the great circle, the first derivative of Equation (28) must be zero. Therefore, we have

$$\tan DLo_{FV} = \frac{b}{a}. \tag{30}$$

Substituting the above result into Equation (28) yields

$$\tan L_V = \frac{a \cos DLo_{FV} + b \sin DLo_{FV}}{-c}. \tag{31}$$

2.3.2. *Simplifying formulae used for the GCEM.*   Because Equation (25) used for obtaining the parameter, b, is complex, we need to simplify it for a practical use. First, Equations (15) and (17) can be rewritten as

$$\cos DLo = \frac{\cos D - \sin L_F \sin L_T}{\cos L_F \cos L_T}, \tag{32}$$

$$\sin L_T = \sin L_F \cos D + \cos L_F \sin D \cos C. \tag{33}$$

Then, substituting Equations (32) and (33) into Equation (25) and rearranging it yield

$$b = -\sin D \cos C. \tag{34}$$

Introducing the sine law of spherical trigonometry, that is,

$$\cos L_T \sin DLo = \sin D \sin C, \tag{35}$$

and substituting Equation (35) into Equations (24) and (26), respectively yield

$$a = -\sin L_F \sin D \sin C, \tag{36}$$

$$c = \cos L_F \sin D \sin C. \tag{37}$$

Therefore, the concise formulae used for the simplified GCEM are as follows.

1. The waypoints along the GCT: Substituting Equations (34), (36) and (37) into Equation (28) and rearranging it yield

$$\tan L_X = \frac{\cos C \sin DLo_{FX} + \sin L_F \sin C \cos DLo_{FX}}{\cos L_F \sin C}. \tag{38}$$

2. The equator crossing point: Substituting Equations (34) and (36) into Equation (29) yields

$$\tan DLo_{FE} = -\sin L_F \tan C. \tag{39}$$

3. The vertex: Substituting Equations (34) and (36) into Equation (30) yields

$$\tan DLo_{\mathrm{FV}} = \frac{1}{\sin L_{\mathrm{F}} \tan C}. \tag{40}$$

Further, substituting Equations (34), (36) and (37) into Equation (31) yields

$$\tan L_V = \frac{\cos C \sin DLo_{FV} + \sin L_{\mathrm{F}} \sin C \cos DLo_{FV}}{\cos L_{\mathrm{F}} \sin C}. \tag{41}$$

3. COMPUTATION PROCEDURES AND NUMERICAL PRO-GRAM. The great circle sailing problem is first to obtain the waypoints along the GCT. Then, the course and distance of the rhumb line between two adjacent waypoints can be determined by using the Mercator sailing. All required formulae used for the numerical program are listed as

$$M = a_e \ln\left[\tan\left(45^\circ + \frac{L}{2}\right) \times \left(\frac{1 - e \sin L}{1 + e \sin L}\right)^{\frac{e}{2}} \right], \tag{42}$$

in which $e = 0.081819190842622$ for WGS 84 (NIMA, 2000) and $a_e = 3437.74677078$ nautical miles (nm) (Bowditch, 2002).

According to the formulae of the Mercator sailing,

$$\ell = L_{\mathrm{X}_{i+1}} - L_{\mathrm{X}_i}, \ dlo = \lambda_{\mathrm{X}_{i+1}} - \lambda_{\mathrm{X}_i}, \ m = M_{\mathrm{X}_{i+1}} - M_{\mathrm{X}_i}, \tag{43}$$

$$\tan cm = \frac{dlo \times 60'}{m}, \tag{44}$$

$$dm = \begin{cases} \ell \sec cm, & cm \neq 90^\circ \\ dlo \cos L_{\mathrm{X}_i}, & cm = 90^\circ \end{cases} \tag{45}$$

3.1. *Computation procedures for the simplified GCEM and the COFI method.*

- *Step 1.* Great circle information.
  1. Calculate the great circle distance and initial course angle by using Equations (15) and (17), respectively.
  2. Calculate the equator crossing point by using Equation (39).
  3. Calculate the vertex by using Equations (40) and (41).
- *Step 2.* Waypoints along the GCT.
  Condition 1. Use the longitudes of the waypoints to obtain the latitudes by using Equation (38).
  Condition 2. Use the great circle distances from the departure to yield the latitudes and longitudes of the waypoints by using Equations (19) and (21).
- *Step 3.* According to Mercator sailing, the Mercator course and distance of every rhumb line between two adjacent waypoints can be determined by using Equations (42), (43), (44) and (45).

3.2. *Developing the numerical program.* A GCS program, called "GCSPro", covering Conditions 1 and 2, has been developed based on the COFI method and the simplified GCEM. For ease of use, GCSPro uses Visual Basic (VB) with a graphical

Table 2. Results of solving waypoints along the GCT under λ_X by using the STM-E in Example 1.

Process	Item Equation	Input	Output	Solution
1	(1)	$L_F = 40°27\cdot0'$ (N)	$D = 112\cdot867789°$	$D \doteqdot 6772\cdot1'$
2	(2)	$L_T = -34°25\cdot0'$ (S)	$C = -63\cdot479168°$	$N116\cdot5°E$
		$DLo = 092°E$	$= 116\cdot520832*$	$C_n = 116\cdot5°$
3	(3)	$L_F = 40°27\cdot0'$	$DLo_{FE} = 52\cdot43302404°E$	$\lambda_E = 021°24\cdot0'W$
		$C = 116\cdot520832°$		
4	(4)	$\lambda_F = 073°50\cdot0'W$	$C_E = 42\cdot91376773°$	
5	(5)	$DLo_{EX} = 021°24\cdot0'W \sim 060°W$ $= 38°36\cdot0'W$	$L_X = 33°51\cdot8'$	$(33°51\cdot8'N, 060°W)$
		$DLo_{EX} = 28°36\cdot0'W$	$L_X = 27°14\cdot6'$	$(27°14\cdot6'N, 050°W)$
		$DLo_{EX} = 18°36\cdot0'W$	$L_X = 18°56\cdot2'$	$(18°56\cdot2'N, 040°W)$
		$DLo_{EX} = 08°36\cdot0'W$	$L_X = 09°08\cdot2'$	$(09°08\cdot2'N, 030°W)$
		$DLo_{EX} = 01°24\cdot0'W$	$L_X = -01°30\cdot3'$	$(01°30\cdot3'S, 020°W)$
		$DLo_{EX} = 11°24\cdot0'E$	$L_X = -12°00\cdot2'$	$(12°00\cdot2'S, 010°W)$
		$DLo_{EX} = 21°24\cdot0'E$	$L_X = -21°25\cdot7'$	$(21°25\cdot7'S, 000°)$
		$DLo_{EX} = 31°24\cdot0'E$	$L_X = -29°16\cdot0'$	$(29°16\cdot0'S, 010°E)$

* Since $\tan(-\theta) = \tan(180° - \theta)$, $(-\theta)$ is replaced as $(180° - \theta)$.

user interface (GUI) for programming. In addition, for the purpose of choosing reasonable numbers of the waypoints, a diagram of total Mercator distance versus waypoints number (called tMd-n diagram) is provided for the navigator.

4. DEMONSTRATED EXAMPLES AND DISCUSSION.

4.1. *Example 1.* A vessel is proceeding from **New York** (*USA*) *to* **Cape Town** (*South Africa*). *The master desires to use the great circle sailing from L*40°27·0′N, λ073° 50·0′W to *L*34°25·0′S, λ018°10·0′E.

4.1.1. *Required.* Calculate the following cases under different given conditions by using the STM-E.

1. Calculate the great circle distance, initial course and the latitudes and longitudes of the waypoints along the GCT at longitude 060°W and at each 10° of longitude thereafter to longitude 010°E (Condition 1).

2. Calculate the latitudes and longitudes of the waypoints along the GCT at equal interval of great circle distance, 600 (10°) nautical miles (nm), from the equator crossing point (Condition 2).

4.1.2. *Solution.*

1. The STM-E is adopted to solve the waypoints along the GCT under Condition 1. Results and the solving procedures with the suggested formulae are shown in Table 2.

Table 3. Results of solving waypoints along the GCT under D_{EX} by using the STM-E in Example 1.

| Process	Equation	Input	Output	Solution
	Item			
		Process 1–4 is the same as Table 2.		
		Preliminary: $\lambda_E = 021°24\cdot0'W$ $C_E = 42\cdot91376773°$		
5	(6)	$D_{EX} = 00°$	$L_X = 00° = L_E$	$(00°, 021°24\cdot0'W)$
6	(7)		$DLo_{EX} = 00° (\lambda_X = \lambda_E)$	
5	(6)	$D_{EX} = 10°$	$L_X = \pm 07°18\cdot4'$	$(07°18\cdot4'N, 028°14\cdot8'W)$
6	(7)		$DLo_{EX} = 06°50\cdot8'(W/E)$	$(07°18\cdot4'S, 014°33\cdot2'W)$
5	(6)	$D_{EX} = 20°$	$L_X = \pm 14°30\cdot4'$	$(14°30\cdot4'N, 035°19\cdot1'W)$
6	(7)		$DLo_{EX} = 13°55\cdot1'(W/E)$	$(14°30\cdot4'S, 007°28\cdot9'W)$
5	(6)	$D_{EX} = 30°$	$L_X = \pm 21°28\cdot8'$	$(21°28\cdot8'N, 042°51\cdot6'W)$
6	(7)		$DLo_{EX} = 21°27\cdot6'(W/E)$	$(21°28\cdot8'S, 000°03\cdot6'E)$
5	(6)	$D_{EX} = 40°$	$L_X = \pm 28°05\cdot0'$	$(28°0\cdot0'N, 051°08\cdot5'W)$
6	(7)		$DLo_{EX} = 29°44\cdot5'(W/E)$	$(28°05\cdot0'S, 008°20\cdot5'E)$
5	(6)	$D_{EX} = 50°$	$L_X = \pm 34°07\cdot6'$	$(34°07\cdot6'N, 060°27\cdot5'W)$
6	(7)		$DLo_{EX} = 39°03\cdot5'(W/E)$	$(34°07\cdot6'S, 017°39\cdot5'E)$
5	(6)	$D_{EX} = 60°$	$L_X = \pm 39°21\cdot9'$	$(39°31\cdot9'N, 071°06\cdot3'W)$
6	(7)		$DLo_{EX} = 49°42\cdot3'(W)$	
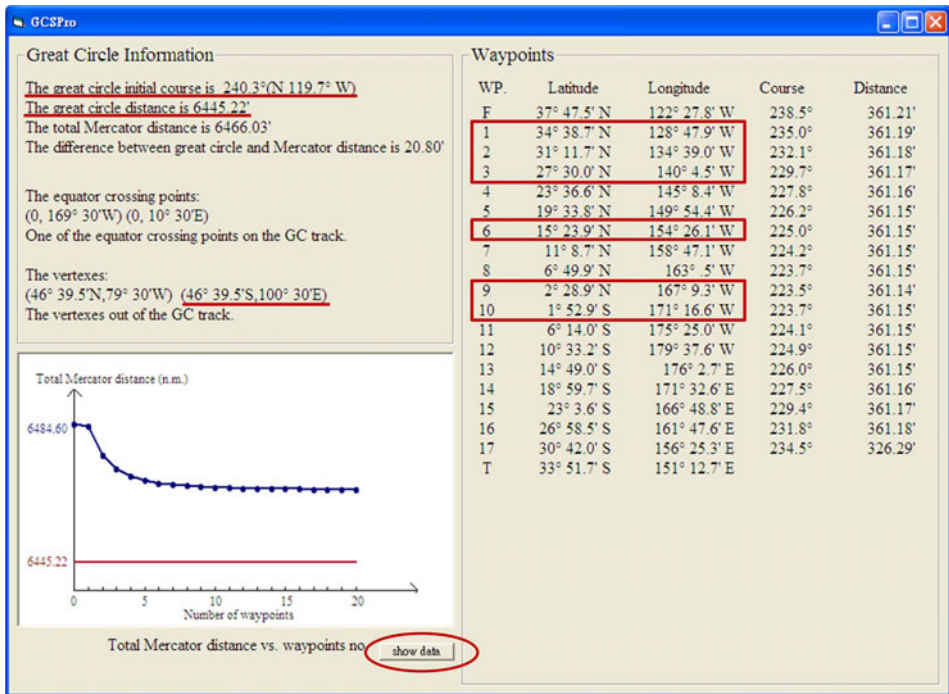
Figure 5. Results of running the GCSPro under Condition 2 in Example 2.

Table 4. A comparison of results obtained by the Ageton method and the COFI method in Example 2.

Item	The Ageton method*	The COFI method
D	644·5′	6445·22′
C	240°17·5′	240·3°
V	L46°39·5′S, λ100°29·7′E	L46°39·5′S, λ100°30·0′E
$D_{FX}=6°$	L34°39·0′N, λ128°48·3′W	L34°38·7′N, λ128°47·9′W
$D_{FX}=12°$	L31°12·0′N, λ134°39·3′W	L31°11·7′N, λ134°39·0′W
$D_{FX}=18°$	L27°30·0′N, λ140°04·3′W	L27°30·0′N, λ140°04·5′W
$D_{FX}=36°$	L15°24·0′N, λ154°26·3′W	L15°23·9′N, λ154°26·1′W
$D_{FX}=54°$	L02°29·0′N, λ167°09·3′W	L02°28·9′N, λ167°09·3′W
$D_{FX}=60°$	L01°52·5′N, λ171°17·3′W	L01°52·9′S, λ171°16·6′W

* Resource: Bowditch, 1981, P.616–618.

Table 5. The relationship between total Mercator distance (nm) and waypoint number on the GCT in Example 2.

Waypoints no.	Total Mercator distance	Waypoints no.	Total Mercator distance
0	6484·60	9	6466·74
1	6483·89	10	6466·56
2	6475·60	11	6466·43
3	6471·67	12	6466·32
4	6469·65	13	6466·24
5	6468·49	14	6466·17
6	6467·77	15	6466·12
7	6467·30	16	6466·07
8	6466·97	17	6466·03

2. The STM-E is adopted to solve the waypoints along the GCT under Condition 2. Results and the solving procedures with the suggested formulae are shown in Table 3.

4.1.3. *Discussion.* In this example, although the STM-E can solve waypoints problems under Conditions 1 and 2, many sign convention judgments arise in the solving procedure of the STM-E. Anyway, it still offers another solving approach for the IA.

4.2. *Example 2.* *A vessel is proceeding from* **San Francisco** (*USA*) *to* **Sydney** (*Australia*)*. The navigator desires to use great circle sailing from* L37°47·5′N, λ122°27·8′W *to* L33°51·7′S, λ151°12·7′E (Bowditch, 1981, P.616–618).

4.2.1. *Required.* Using GCSPro to calculate the latitudes and longitudes of the waypoints on the GCT 360 nm apart (Condition 2), and the information for the great circle (eg. the great circle distance, initial course, the equator crossing point and the vertices).

4.2.2. *Solution.* GCSPro is used to solve the waypoints on the GCT under Condition 2. Results including of the waypoints, great circle information, and a tMd-n diagram are shown in Figure 5. The comparison of results obtained by the COFI method and those by the Ageton method (tabular method) is shown in Table 4. In addition, when the "show data" button in Figure 5 is clicked, detailed numerical
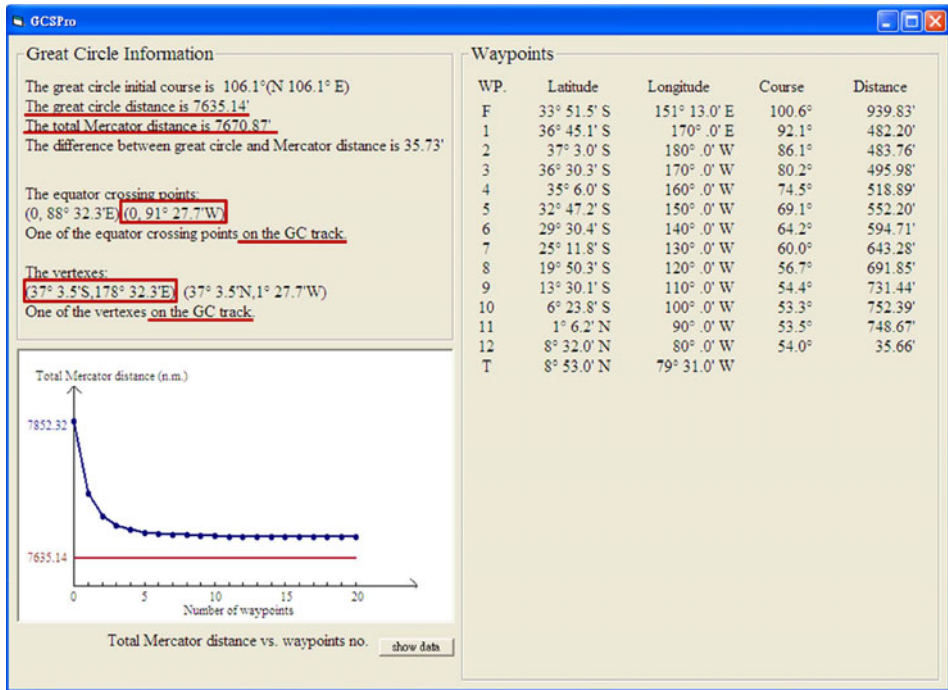
Figure 6. Results of running the GCSPro under Condition 1 in Example 3.

information of total Mercator distance versus numbers of waypoints will display in the format shown in Table 5.

4.2.3. *Discussion*. In this example, the COFI method has been validated successfully. In Table 4, it is found that the COFI method is more accurate than the Ageton method because the former is free of rounding errors, which was also reported in Bowditch (2002). As for Table 5, the total Mercator distance of 16 waypoints is nearly equal to that of 8 waypoints and their distance difference is less than 1 nm. An optimum number of waypoints can be determined from this table for the navigator.

4.3. *Example 3*. *A vessel is proceeding from* **Sydney** *(Australia) to* **Balboa** *(Panama). The master desires to use the great circle sailing from* $L33°51·5'S$, $λ151°$ $13·0'E$, *to* $L08°53·0'N$, $λ079°31·0'W$ (Chen et al, 2004. pp. 317–319).

4.3.1. *Required*. Using GCSPro to calculate the latitudes and longitudes of the waypoints on the GCT at longitude and at each 10 degrees of longitude thereafter to longitude (Condition 1), and the information for the great circle (eg. the great circle distance, initial course, the equator crossing point and the vertices).

4.3.2. *Solution*. GCSPro is used to solve the waypoints along the GCT under Condition 1 successfully. Results including the waypoints, great circle information and a tMd-n diagram are shown in Figure 6.

4.3.3. *Discussion*. Based on the proposed COFI method and the simplified GCEM, the developed GCSPro program has been validated by Examples 2 and 3. It is found that the GCSPro program shows the advantages of completeness and practical application.

5. CONCLUSIONS. In this paper, the COFI method has been developed to calculate the waypoints along the GCT successfully using the multiple products of the vector algebra (VA-MP). Due to fixing the initial great circle course, the GCT can be determined and the waypoints along the GCT can be calculated directly. In addition, without tedious derivations, the COFI method is simpler and more straightforward than the conventional methods. A program, GCSPro, for calculating GCT problems has been validated by the practical examples. It is found that the program can be user friendly and effectively operated by the navigator under two given initial conditions. Finally, the STM-E has been derived and also offers another way to solve the waypoints problem successfully.

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APPENDIX

F	the departure
T	the destination
P_{nr}	the pole nearer the departure in Figures 1, 2 and 3
P	the North Pole
E	the equator crossing point of the great circle
V	the vertex of the great circle
X; Xi	The waypoints along the GCT

L	the latitude
L_F	the latitude of the departure
L_T	the latitude of the destination
L_V	the latitude of the vertex
L_X	the latitudes of the waypoints on the GCT
λ	the longitude
λ_F	the longitude of the departure
λ_T	the longitude of the destination
λ_V	the longitude of the vertex
λ_X	the longitudes of the waypoints on the GCT
λ_E	the longitude of the equator crossing point
DLo	the difference of longitude from the departure to the destination
DLo_{FV}	the difference of longitude from the departure to the vertex
DLo_{FX}	the difference of longitude from the departure to the waypoints
DLo_{FE}	the difference of longitude from the departure to the equator crossing point
DLo_{EX}	the difference of longitude from the equator crossing point to the waypoints
[a, b, c]	the parameter vector of the great circle equation
D	the great circle distance from the departure to the destination
D_{FE}	the great circle distance from the departure to the equator crossing point
D_{EX}	the great circle distance from the equator crossing point to the waypoints
D_{FX}	the great circle distance from the departure to the waypoints
C	the initial great circle course angle from the departure to the destination
C_E	the course angle of the equator crossing point toward the departure
C_X	the course angle of the waypoint toward the destination in Figure 3
dm	the distance between two adjacent waypoints in Mercator sailing
cm	the course angle in Mercator sailing
M	meridional parts
a_e	the equatorial radius of the Earth (nm)
E	first eccentricity of WGS 84
l	the difference of latitude between two adjacent waypoints
dlo	the difference of longitude between two adjacent waypoints
m	the difference of meridional parts