RANGES OF LYAPUNOV TRANSFORMATIONS FOR HILBERT SPACE

by J. KYLE

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1. Introduction. Interest in the ranges of Lyapunov transformations began with Taussky in [**5**]. Recently in a series of papers, Loewy has studied the ranges of Lyapunov transformations on matrices. In particular in [**2**] and [**3**], the following result was obtained.

THEOREM. *Let $A, B \in \mathbb{C}^{n,n}$ and suppose that L_A is invertible. Then the following are equivalent.*

(i) $B = (\alpha I + i\beta A)(\gamma A + i\delta I)^{-1}$ for some real scalars with $\alpha\gamma + \beta\delta = 1$.

(ii) $L_A(\mathrm{PSD}(n)) = L_B(\mathrm{PSD}(n))$.

Here L_A is the Lyapunov transformation corresponding to A i.e. $L_A(X) = AX + XA^*$, and $\mathrm{PSD}(n)$ denotes the set of all positive semi-definite $n \times n$ matrices.

In this short note we shall obtain a generalization of the above result to the case when A and B are bounded linear operators on any Hilbert space.

2. Mappings which preserve the positive cone. Let H be any Hilbert space and let $L(H)$ denote the set of all bounded linear operators on H. $L(H)^{sa}$ and $L(H)^+$ will denote the self-adjoint and positive parts of $L(H)$ respectively.

2.1 LEMMA. *Suppose α is an invertible bounded linear mapping of $L(H)$ onto itself which takes $L(H)^+$ onto itself, and maps I to I. Then α is either a *-automorphism of $L(H)$ or a *-anti-automorphism of $L(H)$.*

Proof. Let $H \geq 0$. Then $\|H\|\, I - H \geq 0$, so that $\alpha(H) \leq \alpha(\|H\|\, I) = \|H\|\, I$. Thus $\|\alpha(H)\| \leq \|H\|$.

The same argument applied to α^{-1} shows that $\|H\| \leq \|\alpha(H)\|$. Hence α is isometric on $L(H)^+$. α therefore preserves the extreme points of the positive part of the unit ball, i.e. α maps projections onto projections.

From here we may argue as Kadison in [**1**] to show that α is (in the terminology of [**1**]) a C*-isomorphism of $L(H)$. The proof is completed by noting that it is shown also in [**1**] that such a map on $L(H)$ must be either a *-automorphism or a *-anti-automorphism of $L(H)$.

2.2 LEMMA. *Suppose L_A and L_B are two invertible Lyapunov transformations of $L(H)$ and that $L_A(L(H)^+) = L_B(L(H)^+)$. Then there is some non-singular V in $L(H)$ such that $L_A^{-1}L_B(X) = VXV^*$ for all X in $L(H)$.*

Proof. $L_A^{-1}L_B(L(H)^+) = L(H)^+$. Let $T \in L(H)^+$ be such that $L_A^{-1}L_B(T) = I$. Then $\|T\|^{-1}T \leq I$, so that $\|T\|^{-1}I = L_A^{-1}L_B(\|T\|^{-1}T) \leq L_A^{-1}L_B(I)$. Thus $L_A^{-1}L_B(I)$ is invertible and equals S^2 for some non-singular S in $L(H)^{sa}$.

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Let $\alpha(X)=S^{-1}L_A^{-1}L_B(X)S^{-1}$. Then α satisifies the conditions of Lemma 2.1 and so is either a *-automorphism or a *-anti-automorphism.

If α is a *-automorphism, we can find a unitary U in $L(H)$ such that $\alpha(X)=UXU^*$ (see [4 Ch. 4], for example). Taking $V=SU$ gives the required result.

Now if α is a *-anti-automorphism of $L(H)$, by the same token we can find a conjugate linear isomorphism U of H onto itself such that $\alpha(X)=UX^*U^{-1}$ for all X in $L(H)$. Thus $L_B(X)=L_A(SUX^*U^{-1}S)$. Now if X is given by $X\xi=\langle\xi,\sigma\rangle\tau$ (for some σ, τ in H), $X^*\xi=\langle\xi,\tau\rangle\sigma$ and evaluating both sides at σ gives

$$\langle\sigma,\sigma\rangle B\tau+\langle B^*\sigma,\sigma\rangle\tau=\langle U^{-1}S\sigma,\tau\rangle ASU\sigma+\langle U^{-1}SA^*\sigma,\tau\rangle SU\sigma.$$

The L.H.S. is linear in τ, the R.H.S. is conjugate linear in τ. Thus both sides are identically zero, from which one can deduce that A and B are both scalar multiples of I. S can then be chosen to be a positive multiple of I, so that α is the identity map. This contradicts the assumption that α is an anti-automorphism.

3. Main result. We first prove:

3.1 LEMMA. *$L_A=L_B$ if and only if $B=A+i\lambda I$ for some $\lambda\in\mathbb{R}$.*

Proof. "if" is easy. Suppose $A=A_1+iA_2$ (with A_i self-adjoint) and suppose that $L_A=0$. Then $0=L_A(I)=2A_1$, and so $A_1=0$. Hence, for all X in $L(H)$, $0=L_A(X)=i(A_2X-XA_2)$ and so $A_2=\lambda I$ for some real λ.

3.2 THEOREM. *Suppose L_A and L_B are two non-singular Lyapunov transformations. Then the following are equivalent.*

(i) $B=(\alpha I+i\beta A)(\gamma A+i\delta I)^{-1}$ *for some real α, β, γ, δ with $\alpha\gamma+\beta\delta=1$.*

(ii) $L_A(L(H)^+)=L_B(L(H)^+)$.

Proof. (i)⇒(ii). As in [2] and [3] we can show that (i) implies that either L_B is a positive multiple of L_A or a positive multiple of $L_{(A+i\lambda I)^{-1}}$ for real λ. From this (ii) follows easily.

(ii)⇒(i). Using Lemma 2.2, we can find some non-singular V in $L(H)$ such that $BX+XB^*=AVXV^*+VXV^*A^*$ for all A in $L(H)$. If X is given by $X\xi=\langle\xi,\sigma\rangle\tau$ where σ, τ are arbitrary vectors of norm one, we get

$$B\tau+\langle B^*\sigma,\sigma\rangle\tau=\langle V^*\sigma,\sigma\rangle AV\tau+\langle V^*A^*\sigma,\sigma\rangle V\tau. \tag{1}$$

Choosing two values of σ and eliminating $B\tau$ from (1) we see that

$$V=(\gamma A+i\delta I)^{-1}$$

for some scalars γ and δ. If $\gamma=0$, V is a multiple of I and so L_B is a multiple (which must be positive) of L_A. Lemma 3.1 then allows us to display B in the form (i). We shall therefore assume that $\gamma\neq 0$. Multiplying V by some scalar of unit modulus leaves the equations unaltered, but allows us to assume that γ is real. Further, using (1) with $\sigma=\tau$ gives

$$2\,\mathrm{Re}\langle B\sigma,\sigma\rangle=2\gamma^{-1}\,\mathrm{Re}\langle V\sigma,\sigma\rangle+i\gamma^{-1}(\bar\delta-\delta)\,|\langle V\sigma,\sigma\rangle|^2$$

and so $\delta=\bar\delta$ as required.

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Returning once more to (1), we see that $B+\lambda I=\nu AV+\mu V$ for some scalars λ, μ, ν. This may be rewritten as $BV^{-1}=(\alpha I+i\beta A)$ for some scalars α, β. Using the original equation we get $BV^{-1}XV^{*-1}+V^{-1}XV^{*-1}B^*=AX+XA^*$ for all X in $L(H)$ which reduces to

$$(\bar{\alpha}\gamma+\beta\delta-1)AX+(\alpha\gamma+\delta\bar{\beta}-1)XA^*=i(\gamma\bar{\beta}-\beta\gamma)AXA^*.$$

This shows that $(\gamma\bar{\beta}-\beta\gamma)=0$, and so $\beta\in\mathbb{R}$, and gives $\gamma L_{\alpha A}+(\beta\delta-1)L_A=0$. Hence, by Lemma 3.1, $\bar{\alpha}A=-\gamma^{-1}(\beta\delta-1)A+i\lambda I$, for some $\lambda\in\mathbb{R}$. Thus $\alpha=-\gamma^{-1}(\beta\delta-1)$ as required, or A is a multiple of I. In this case either L_B or $-L_B$ maps $L(H)^+$ onto $L(H)^+$ and so (as in §2)$L_B(X)=UXU^*$ for some U in $L(H)$. Arguing as above shows that U is a multiple of I, and so (by Lemma 3.1) also is B. This completes the proof.

3.3 Remarks. For finite dimensional H, L_B is non-singular whenever L_A is, and so Theorem 3.2 gives Loewy's theorem. Essentially, this is because any linear mapping taking PSD(n) onto PSD(n) must be non-singular. This may be false, however, in general. For example, when $H=\ell^2(Z^+)$ and U is the unilateral shift, the mapping $\alpha(X)=U^*XU$ takes $L(H)^+$ onto $L(H)^+$ yet has no inverse.

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Department of Mathematics,
University of Technology,
Loughborough.

Present address
Department of Pure Mathematics,
University of Birmingham,
P.O. Box 363
Birmingham B15 2TT.

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