Bull. Aust. Math. Soc. **109** (2024), 541–551
doi:10.1017/S0004972723000667

ON SEPARATE CONTINUITY AND SEPARATE CONVEXITY: A SYNTHETIC TREATMENT FOR FUNCTIONS AND SETS

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(Received 12 May 2023; accepted 27 May 2023; first published online 17 July 2023)

Abstract

This paper relies on nested postulates of *separate, linear* and *arc-continuity* of functions to define analogous properties for sets that are weaker than the requirement that the set be open or closed. This allows three novel characterisations of open or closed sets under convexity or separate convexity postulates: the first pertains to separately convex sets, the second to convex sets and the third to arbitrary subsets of a finite-dimensional Euclidean space. By relying on these constructions, we also obtain new results on the relationship between separate and joint continuity of separately quasiconcave, or separately quasiconvex functions. We present examples to show that the sufficient conditions we offer cannot be dispensed with.

2020 *Mathematics subject classification*: primary 26B05; secondary 52A99, 54B99.

Keywords and phrases: convexity, separate convexity, continuity, separate continuity, linear continuity, arc-continuity, restricted continuity.

> *The notion of continuity depends upon that of order, since continuity is merely a particular type of order. Mathematics has, in modern times, brought order into greater and greater prominence.* Bertrand Russell [21]

1. Introduction

This paper is motivated by the following basic questions concerning a set A in a finite-dimensional Euclidean space $\mathbb{R}^n$. Can we tell if A is open or closed just by considering its intersection with lines? We already know that the intersection of an open (or closed) set with a line L in $\mathbb{R}^n$ that is parallel to a coordinate axis must be an open (or closed) subset of L, but the question arises as to whether the converse holds. This is to ask whether A is open (or closed) if the intersections of A with all lines parallel to the axes are open (or closed) subsets of those lines. Furthermore, what can we say regarding whether the intersection of A with all possible lines is open (or closed)? Is that sufficient to conclude A is open (or closed)? These questions are related to matters of separate continuity of a function: is a function $f : \mathbb{R}^n \to \mathbb{R}$ necessarily continuous if all possible single-variable functions of the form $g(z) = f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$ are continuous?

These questions are hardly novel. In [12], Halkin provides a characterisation of a closed and convex set A in a finite-dimensional Euclidean space $\mathbb{R}^n$ by imposing assumptions on the supporting hyperplanes of A. In another development, Azagra and Ferrera [1] provide a characterisation of closed and convex sets in a separable Banach space by seeing them as null sets, and thereby minimisers or maximisers, of C^∞-smooth, real-valued convex functions. What is novel to this note are possibly new characterisations of closed and of open sets in $\mathbb{R}^n$ by weakening or completely dropping the convexity assumption, and thereby generalising the result in [12]. We are motivated by separate, linear and arc-continuity of functions.

Furthermore, Young [25] and Kruse and Deely [16] show that for separately monotone functions defined on an open set in $\mathbb{R}^n$, separate continuity is equivalent to joint continuity. On using the characterisation of sets referred to above, we move from sets to functions and reset our characterisations of sets to functions in a more general setting that substitutes separate quasiconvexity or separate quasiconcavity for separate monotonicity, and by drawing on weaker notions of continuity obtain the results in [16, 25] as corollaries.

Section 2 presents three results each offering necessary and sufficient conditions for a set to be open or closed: the first on open sets, and the other two on sets that can be open or closed. Four supplementary examples illustrate that the assumptions in these results are not redundant. Section 3 presents a theorem and an example concerning necessary and sufficient conditions for the semicontinuity of functions. Section 4 presents three remarks: the first suggesting how the separate convexity postulate can be replaced by piecewise separate convexity, the second relating to the intermediate-value property, and the third pointing to the relevance of the results to correspondences and binary relations and to applications in mathematical economics. All in all, this interdisciplinary investigation mirrors and consolidates the different approaches available in the antecedent literature. Section 5 is devoted to the proofs.

2. On closed sets and on open sets in $\mathbb{R}^n$

A *straight line in* $X \subseteq \mathbb{R}^n$ is defined as the intersection of a one-dimensional subset of the affine hull of X. Next, we introduce a topological property that is motivated by *separate continuity* of a function that imposes continuity restricted to straight lines parallel to a coordinate axis; see [9, 16, 25] for classic results and [4, 10] for recent surveys on different continuity postulates.

DEFINITION 2.1. A set $A \subseteq \mathbb{R}^n$ is separately closed (open) if for any straight line L in $\mathbb{R}^n$ that is parallel to a coordinate axis, $L \cap A$ is closed (open) in the subspace L.

For any strictly positive natural number n, let $[n] = \{1, \ldots, n\}$. For all $x \in A$, define $A_i(x) = \{z \in \mathbb{R} \mid (z, x_{-i}) \in A\}$, where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. For each i and x, $A_i(x)$ determines the straight line passing through x that is parallel to the coordinate axis i. Hence, a separately closed (open) set A is equivalently defined as $A_i(x)$ *is closed (open) in* $\mathbb{R}$ *for all* $x \in A$ *and all* $i \in [n]$. It is clear that if a set is open, then it is

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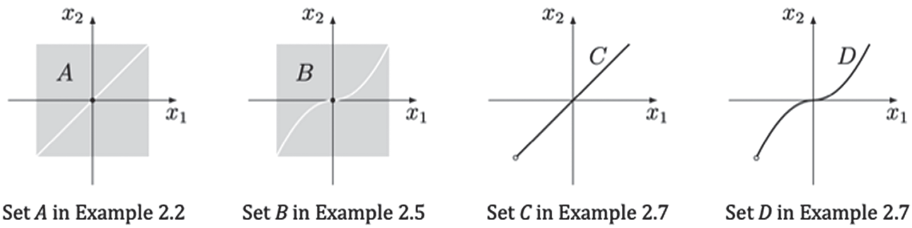

FIGURE 1. Illustration of examples in Section 2.

separately open. The following example illustrates a set that is separately open but not open. (The examples are illustrated in Figure 1.)

EXAMPLE 2.2. Let $A = (0, 1)^2 \backslash \{x \in (0, 1)^2 \mid x_1 = x_2, x \neq 0\}$. The intersection of a line parallel to a coordinate axis and A is either the $(0, 1)$ interval or the union of two open intervals, and hence A is separately open. However, A is not open since every open ball containing 0 contains a point in the complement of A.

Our first result shows that under a weak convexity assumption, separate openness is a necessary and sufficient condition for a set to be open.

THEOREM 2.3. *Let $A \subseteq \mathbb{R}^n$ and J be an $(n-1)$-element subset of $[n]$ such that $A_i(x)$ is convex for all $x \in A$ and all $i \in J$. Then, A is open if and only if it is separately open.*

Note that the conclusion of Theorem 2.3 can be re-stated as follows: *A is open in $\mathbb{R}^n$ if and only if for any straight line L in $\mathbb{R}^n$ that is parallel to a coordinate axis, $L \cap A$ is open in the subspace L.* The convexity assumption in Theorem 2.3 is related to the literature on sets with convex sections as $A_i(x)$ denotes the ith section of A at x (see for example [7]). A set $A \subseteq \mathbb{R}^n$ is *separately convex* if $A_i(x)$ is convex for all $i \in [n]$ and all $x \in A$. Example 2.2 illustrates that the convexity assumption in Theorem 2.3, which is weaker than separate convexity, is not redundant.

Next, we introduce a topological property of a set by imposing assumptions on straight lines in the set that is motivated by the *linear continuity* of a function (see for example [4, 9, 23]).

DEFINITION 2.4. A set $A \subseteq \mathbb{R}^n$ is linearly closed (open) if for any straight line L in $\mathbb{R}^n$, $L \cap A$ is closed (open) in the subspace L.

Example 2.2 illustrates a set that is separately open but not open and also not linearly open. The following example illustrates a set that is not open but both separately open and linearly open.

EXAMPLE 2.5. Let $B = (0, 1)^2 \backslash \{x \in (0, 1)^2 \mid x_2 = x_1^2 \text{ for } x_1 > 0, x_2 = -x_1^2 \text{ for } x_1 < 0, x \neq 0\}$. The intersection of a straight line and B excludes at most finitely many points of $L \cap (0, 1)^2$. Hence, B is linearly open. However, B is not open since every open ball containing 0 contains a point in the complement of B.

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It is clear that every separately closed (open) set is linearly closed (open), and if A is convex, then $A_i(x)$ is convex for all $x \in A$ and all $i \in [n]$. The following theorem presents a necessary and sufficient condition for a set to be closed under a convexity and a topological assumption stronger than those in Theorem 2.3.

THEOREM 2.6. *A convex set $A \subseteq \mathbb{R}^n$ is closed (open) if and only if it is linearly closed (open).*

Since Theorem 2.6 imposes stronger convexity and topological assumptions than Theorem 2.3, for open sets, it is a corollary of Theorem 2.3. Theorem 2.6 complements the results of Azagra and Ferrera [1] and Halkin [12] who provide characterisations of closed and convex sets by using the minimisers of a class of convex functions in [1] and by using the supporting hyperplanes of the set in [12].

The following example illustrates that for closed sets, the convexity and topological assumptions in Theorem 2.6 are not redundant.

EXAMPLE 2.7. Let $C = \{x \in \mathbb{R}^2 \mid x_1 = x_2\} \cap (-1, 1]^2$. It is clear that C is convex and fails the topological assumption in Theorem 2.6, and it is not closed (in $\mathbb{R}^2$). Now let $D = \{x \in \mathbb{R}^2 \mid x_1^2 = x_2\} \cap (-1, 1]^2$. It is clear that D is not convex and the intersection of any straight line L in $\mathbb{R}^2$ with D contains at most finitely many elements, hence it is closed in L, and hence linearly closed. However, D is not closed (in $\mathbb{R}^2$). Note that both C and D are separately closed, and satisfy the convexity assumption in Theorem 2.3. Hence, the weak convexity assumed in Theorem 2.3 is not enough to guarantee a separately closed set to be closed.

Next, we show that the convexity assumptions in the theorems above can be dispensed with by imposing a topological assumption on a class of selections of A that is larger than the set of straight lines, following the restriction continuity of [20] for functions. An *arc* in $\mathbb{R}^n$ is a continuous injective function $m : [0, 1] \to \mathbb{R}^n$, where $m(\lambda) = (m_1(\lambda), \ldots, m_n(\lambda))$. An arc is called *smooth* if m_i is continuously differentiable for all i and $m'(\lambda) = (m'_1(\lambda), \ldots, m'_n(\lambda)) \neq 0$ for all $\lambda \in [0, 1]$. A *curve* in $\mathbb{R}^n$ is the image of an arc and a *smooth curve* is the image of a smooth arc. Since an arc m is continuous and injective, it is a bijection from $[0, 1]$ to its image $m([0, 1])$. Since $[0, 1]$ is compact and $m([0, 1])$ is Hausdorff, m is a homeomorphism between $[0, 1]$ and $m([0, 1])$ (see for example [5, Theorem 2.1, page 226]. Therefore, $[0, 1]$ and the curve induced by an arc are homeomorphic. The following topological property of a set imposes assumptions on smooth curves in the set that is motivated by the arc-continuity of a function (see for example [4, 20, 23, 24]).

DEFINITION 2.8. A set $A \subseteq \mathbb{R}^n$ is arc-closed (open) if for any smooth curve C in $\mathbb{R}^n$, $C \cap A$ is closed (open) in the subspace C.

The following theorem drops the convexity assumption on a set by imposing a topological assumption on restrictions of the set on smooth curves.

THEOREM 2.9. *A set $A \subseteq \mathbb{R}^n$ is closed (open) if and only if it is arc-closed (open).*

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3. On continuity of functions on $\mathbb{R}^n$

In this section, we study the relationship between separate and joint continuity of a function under quasiconvexity and quasiconcavity. A real-valued function $f(x_1, \ldots, x_n)$ defined on a convex subset of $\mathbb{R}^n$ is *quasiconcave (quasiconvex) in* x_i if for all $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $f(\cdot, x_{-i})$ is quasiconcave (quasiconvex). (The general reader may want to note an abuse of notation whereby $x \in \mathbb{R}^n$ is conceived as (x_i, x_{-i}) or, alternatively, (a, x_{-i}) refers to an element of $\mathbb{R}^n$ whereby $a \in \mathbb{R}$ is substituted in the ith place in $x \in \mathbb{R}^n$. Unfortunately, this notation is standard in mathematical economics and game theory, and we submit to it.) Moreover, f is *separately quasiconcave (separately quasiconvex)* if it is quasiconcave (quasiconvex) in each variable.

THEOREM 3.1. *Let $Y \subseteq \mathbb{R}^n$ be a convex and open set. If $f : Y \to \mathbb{R}$ is quasiconvex in $n-1$ variables, then it is upper semicontinuous if and only if it is separately upper semicontinuous. Moreover, if f is quasiconcave in $n-1$ variables, then it is lower semicontinuous if and only if it is separately lower semicontinuous.*

The two classical counterexamples of Genocchi and Peano [9] show that the separate continuity of a function is strictly weaker than its linear continuity, and that is strictly weaker than its joint continuity. Young [25] and Kruse and Deely [16] show that these three continuity postulates are equivalent under a weak monotonicity assumption. Rosenthal [20] shows that imposing continuity of a function restricted to any smooth curve is enough to obtain its joint continuity. Uyanik and Khan [23] show that under quasiconcavity, or quasiconvexity, linear continuity and joint continuity are equivalent. Theorem 3.1 contributes to this literature by studying the relationship between separate and joint continuity of functions under convexity, or concavity, an assumption that is weaker than monotonicity. See also [3, 14, 17] for relationships between separate and joint continuity in infinite-dimensional spaces, and [4, 10, 18] for surveys. Moreover, lower semicontinuity under concavity and upper semicontinuity under convexity of functions is studied in [6, 8]. The following example shows that joint continuity cannot be obtained from separate continuity under separate quasiconcavity or separate quasiconvexity.

EXAMPLE 3.2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x) = 0$ if $x_1 \leqslant 0$ or $x_2 \leqslant 0$ and $f(x) = 2xy/(x^2 + y^2)$ otherwise. For all i and all x_i, $f(\cdot, x_i)$ is increasing in x_j for $x_j < x_i$ and decreasing in x_j for $x_j > x_i$, and hence f is quasiconcave in each variable. It is easy to see that f is separately continuous. However, f is not jointly continuous (and also not linearly continuous). Note that f is not upper semicontinuous, and hence this example also illustrates that separate quasiconcavity is not sufficient for the equivalence between separate upper semicontinuity and upper semicontinuity. Analogously, $-f$ illustrates that separate quasiconvexity is not sufficient for the equivalence between separate lower semicontinuity and lower semicontinuity.

Functions that are quasiconcave in some variables and quasiconvex in other variables are commonly used in mathematics such as minimax theorems. The following simple corollary of Theorem 3.1 shows that on $\mathbb{R}^2$, separate continuity is equivalent to the joint continuity of a function.

COROLLARY 3.3. *Let (a_1, b_1) and (a_2, b_2) be two open intervals in $\mathbb{R}$ and let f be a function defined on $(a_1, b_1) \times (a_2, b_2)$ that is quasiconcave in one variable and quasiconvex in the other variable. Then, f is jointly continuous if and only if it is separately continuous.*

A function $f(x_1, \ldots, x_n)$ on a set $A \subseteq \mathbb{R}^n$ is *monotone in x_i* if for all x_{-i}, $f(\cdot, x_{-i})$ is either increasing or decreasing. It is easy to see that if a function is both quasiconcave and quasiconvex in x_i, then it is monotone in x_i. Therefore, for convex sets, the following result of [16, 25] is a corollary of Theorem 3.1.

COROLLARY 3.4. *Let $Y \subseteq \mathbb{R}^n$ be a convex and open set, and $f : Y \to \mathbb{R}$ be a function that is monotone in $n - 1$ variables. Then, f is jointly continuous if and only if it is separately continuous.*

4. Remarks

REMARK 4.1. A real valued function $f(x_1, \ldots, x_n)$ on $A \subseteq \mathbb{R}^n$ is *piecewise monotone in x_i* if there exist $m_i \in \mathbb{N}$ and $a_i^1 < a_i^2 < \cdots < a_i^{m_i}$ such that f is monotone on $[a_i^k, a_i^{k+1}]$ for all $k = 1, \ldots, m_i - 1$, and $A_i(x) \subseteq [a_i^1, a_i^{m_i}]$ for all $x \in A$. This monotonicity property is a generalisation of the piecewise monotonicity concept of Sohrab [22, page 156] from $\mathbb{R}$ to $\mathbb{R}^n$. We show in the next section that the monotonicity assumption in Corollary 3.4 can be replaced by the weaker piecewise monotonicity assumption. Note that in $\mathbb{R}^n$, $n \geqslant 2$, the piecewise monotonicity concept neither implies nor is implied by separate quasiconcavity or separate quasiconvexity properties. To see this, on $\mathbb{R}$, the sine function is piecewise monotone, but it is neither quasiconcave nor quasiconvex. Conversely, Example 3.2 illustrates a function on $\mathbb{R}^2$ that is separately quasiconcave but not piecewise monotone in either coordinate.

REMARK 4.2. If a function $f(x_1, \ldots, x_n)$ defined on $A \subseteq \mathbb{R}^n$ is piecewise monotone in x_i (or monotone in x_i), then the arguments in [22, page 156] imply that separate continuity is equivalent to the following *intermediate value property in the variable x_i*: for all $x \in A$ and all $[a_i, b_i] \subseteq A_i(x)$, $a_i < b_i$, and all y_i between $f(a_i)$ and $f(b_i)$, there exists $c_i \in (a_i, b_i)$ such that $f(c_i) = y_i$. See [10] for an investigation of the intermediate value theorem in two applied registers.

REMARK 4.3. The separate convexity assumption in Theorem 2.3 can be replaced by the following weaker convexity property: a set $A \subseteq \mathbb{R}^n$ is *piecewise convex in x_i* if there exist $m_i \in \mathbb{N}$ and $a_i^1 < a_i^2 < \cdots < a_i^{m_i}$ such that $A_i(x) \cap [a_i^k, a_i^{k+1}]$ is convex for all $x \in A$ and all $k = 1, \ldots, m - 1$, and $A_i(x) \subseteq [a_i^1, a_i^m]$ for all $x \in A$. This piecewise separate convexity property is motivated by the piecewise separate monotonicity concept for functions defined above, and is also stronger than (separate) local convexity of a set; see [15] for a detailed discussion of locally convex sets. We leave it for the interested

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reader to check that the construction of the proof in Theorem 2.3 is essentially the same.

REMARK 4.4. If a set $A \subseteq \mathbb{R}^n$ is piecewise convex in x_i, then, as in Remark 4.2, $A_i(x)$ is open for all $x \in A$ if and only if it satisfies the following separate intermediate value property for sets: if $a_i, b_i \in A_i(x)$, then there exists $c_i \in (a_i, b_i)$ such that $c_i \in A_i(x)$.

REMARK 4.5. Different continuity concepts of a binary relation or a correspondence, including the properties of straight lines and curves, have been extensively used in economics and psychology; see for example [2, 10, 11, 13, 23, 24]. We leave it for future research to study the extensions of the results in this paper to infinite-dimensional spaces, and their implications to the continuity of correspondences and binary relations.

5. Proof of the results

Let $\{X_i\}_{i\in I}$ be an indexed family of nonempty sets. For any nonempty $J \subseteq I$, let $X_J = \prod_{j\in J} X_j$ and $X_{-J} = \prod_{j\in J^c} X_j$. The subscript is omitted for $J = I$ and we use X_i and X_{-i} if $J = \{i\}$ for some $i \in I$. For all $x \in X$ and all nonempty $J \subset I$, let $x = (x_J, x_{-J})$. For $A \subseteq X$, all $x \in A$ and all nonempty $J \subseteq I$, let $A_J(x) = \{z \in X_J \mid (z, x_{-J}) \in A\}$.

PROOF OF THEOREM 2.3. We prove the theorem by induction. First, let $A \subseteq \mathbb{R}^2$. Assume that A is separately open and, without loss of generality, that $A_2(x)$ is convex for all $x \in A$. By separate openness, for all $x \in A$, $A_2(x) = \{x'_2 \in \mathbb{R} \mid (x_1, x'_2) \in A\}$ is open and $A_1(x) = \{x'_1 \in \mathbb{R} \mid (x'_1, x_2) \in A\}$ is open.

Pick $x \in A$. By separate openness, there exists a neighbourhood $[a_2, b_2]$ of x_2 such that for all $x'_2 \in [a_2, b_2]$, $(x_1, x'_2) \in A$. That is, $[a_2, b_2] \subset A_2(x)$. By separate openness, $A_1(x_1, a_2) = \{x'_1 \in \mathbb{R} \mid (x'_1, a_2) \in A\}$ and $A_1(x_1, b_2) = \{x'_1 \in \mathbb{R} \mid (x'_1, b_2) \in A\}$ are open. Then, there exists a neighbourhood $[a_1, b_1]$ of x_1 such that $x'_1 \in [a_1, b_1]$ implies $(x'_1, a_2) \in A$ and $(x'_1, b_2) \in A$. That is, $[a_1, b_1] \subset A_1(x_1, a_2) \cap A_1(x_1, b_2)$. By separate convexity, $x'_2 \in [a_2, b_2]$ implies $[a_1, b_1] \subset A_1(x_1, x'_2)$. Hence, for all $x'_1 \in [a_1, b_1]$ and all $x'_2 \in [a_2, b_2]$, $(x'_1, x'_2) \in A$. Since $[a_1, b_1] \times [a_2, b_2] \subset A$ is a neighbourhood of x and x is arbitrarily picked, it follows that A is open.

Now, let $A \subseteq \mathbb{R}^n$, $n > 2$. Assume that A is separately open and, without loss of generality, that $A_i(x)$ is convex for all $x \in A$ and $i \geqslant 2$. Then, by the induction hypothesis,

$$A_{-n}(x) = \{x'_{-n} \in \mathbb{R}^{n-1} \mid (x'_{-n}, x_n) \in A\} \quad \text{is open for all } x \in A. \tag{5.1}$$

Pick $x \in A$. By separate openness, there exists a neighbourhood $[a_n, b_n]$ of x_n such that for all $x'_n \in [a_n, b_n]$, $(x_{-n}, x'_n) \in A$. That is, $[a_n, b_n] \subset A_n(x)$. By (5.1) above, $A_{-n}(x_{-n}, a_n) = \{x'_{-n} \in \mathbb{R} \mid (x'_{-n}, a_n) \in A\}$ and $A_{-n}(x_{-n}, b_n) = \{x'_{-n} \in \mathbb{R} \mid (x'_{-n}, b_n) \in A\}$ are open. Then, there exists a neighbourhood $U = \prod_{i=1}^{n-1}[a_i, b_i]$ of x_{-n} such that $x'_{-n} \in U$ implies $(x'_{-n}, a_n) \in A$ and $(x'_{-n}, b_n) \in A$. That is, $U \subset A_{-n}(x_{-n}, a_n) \cap A_{-n}(x_{-n}, b_n)$. By separate convexity, $x'_n \in [a_n, b_n]$ implies $U \subset A_{-n}(x_{-n}, x'_n)$. Hence, for all $x'_{-n} \in U$ and

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all $x'_n \in [a_n, b_n]$, $(x'_{-n}, x'_n) \in A$. Since $U \times [a_n, b_n] \subset A$ is a neighbourhood of x and x is arbitrarily picked, A is open. □

We prove Theorem 2.6 by using an important result in convex analysis due to Rockafellar [19, page 45]. The statement of Rockafellar's theorem requires the following additional concepts. Let X be a subset of $\mathbb{R}^n$. Since any lower dimensional subset of $\mathbb{R}^n$ has an empty interior, it is more convenient to work with the concept of relative interior. A subset X of a (real) vector space is called *affine* if for all $x, y \in X$ and $\lambda \in \mathbb{R}$, $\lambda x + (1 - \lambda)y \in X$. It is clear that A is affine if and only if $A - \{a\}$ is a subspace of X for all $a \in A$. The *affine hull of* X, aff$X$, is the smallest affine set containing X. The *relative interior* of a subset X of $\mathbb{R}^n$ is defined as

$$\text{ri}X = \{x \in \text{aff}X \mid \text{there is an } \varepsilon \text{ neighbourhood}, N_\varepsilon, \text{of } x \text{ such that } N_\varepsilon \cap \text{aff}X \subseteq X\}.$$

That is, the relative interior of X is the interior of X with respect to the smallest affine subspace containing X.

THEOREM 5.1 (Rockafellar). *Let X be a nonempty and convex subset of $\mathbb{R}^n$. Then* ri$X$ *is nonempty, and for all $x \in \text{ri}X, y \in \text{cl}X$ and all $\lambda \in [0, 1)$, $y\lambda x \in \text{ri}X$.*

PROOF OF THEOREM 2.6. For open sets, Theorem 2.6 is a corollary of Theorem 2.3. Moreover, it is clear that if A is closed, then A is linearly closed. Now, assume A is convex and linearly closed. If A is empty or a singleton, then it is closed. Otherwise, pick $x \in \text{cl}A$. Since A is convex, Theorem 5.1 implies that its relative interior is nonempty, and that for all $y \in \text{ri}A$ and all $\lambda \in [0, 1)$, $x\lambda y \in \text{ri}A$. Hence, for any $\lambda^k \to 1$, $x\lambda^k y \in A$ for all k. Pick $y \in \text{ri}A$ and let L denote the straight line in $\mathbb{R}^n$ passing through x and y. Since A is linearly closed, $A \cap L$ is closed in L. Since $x\lambda^k y \in A$ for all k when $\lambda^k \to 1$, it follows that $x \in A$. Therefore, A is closed. □

The proof of Theorem 2.9 we provide below crucially hinges on the work of Rosenthal [20, Theorem 3]. We state a slightly stronger version of the theorem of Rosenthal with the notation of our paper.

THEOREM 5.2 (Rosenthal). *Any bounded infinite set in $\mathbb{R}^n$ contains an infinite subset through which a smooth curve can be laid.*

PROOF OF THEOREM 2.9. The forward direction is obvious. To prove the backward direction, assume A is arc-closed. If A is empty or a singleton, then it is closed. Otherwise, pick $x \in \text{cl}A$ and a sequence x_n in A such that $x_n \to x$. It follows from Theorem 5.2 that there exists a smooth curve containing x and a subsequence x_{n_k} of x_n. Since A is arc-closed, $x \in A$. Hence, A is closed.

Now assume A is arc-open but not open. Then there exists $x \in A \cap \text{cl}(A^c)$. Pick a sequence x_n in A^c such that $x_n \to x$. It follows from Theorem 5.2 that there exists a smooth curve C containing x and a subsequence x_{n_k} of x_n. This implies that $C \cap A$ contains $x \in A$, but every open neighbourhood of x in the subspace C contains $x' \in A^c$. This contradicts the assumption that A is arc-open. Hence, A is open. □

Proof of Theorem 3.1. First, we prove the result for $Y \subseteq \mathbb{R}^2$. Assume without loss of generality that f is quasiconvex in x_2. It follows that, for all $\alpha \in \mathbb{R}$ and all $x \in Y$, $\{x_2' \in \mathbb{R} \mid (x_1, x_2') \in Y \text{ and } f(x_1, x_2) \leqslant \alpha\}$ is convex. Next, we show that the quasiconvexity assumption implies that $\{x_2' \in \mathbb{R} \mid (x_1, x_2') \in Y \text{ and } f(x_1, x_2') < \alpha\}$ is convex for all $\alpha \in \mathbb{R}$ and all $x \in Y$. Assume towards a contradiction that there exists x_1, x_2, y_2 and λ such that $f(x_1, x_2) < \alpha$, $f(x_1, y_2) < \alpha$ and $f(x_1, \lambda x_2 + (1-\lambda)y_2) \geqslant \alpha$. Set $\beta = \max\{f(x_1, x_2), f(x_1, y_2)\} < \alpha$. By the quasiconvexity assumption, it follows that $\{x_2' \in \mathbb{R} \mid (x_1, x_2') \in Y \text{ and } f(x_1, x_2') \leqslant \beta\}$ is convex. Hence, $\alpha \leqslant f(x_1, \lambda x_2 + (1-\lambda)y_2) \leqslant \beta < \alpha$ which yields a contradiction.

By separate upper semicontinuity, $\{x_2' \in \mathbb{R} \mid (x_1, x_2') \in Y \text{ and } f(x_1, x_2') < \alpha\}$ is open and $\{x_1' \in \mathbb{R} \mid (x_1', x_2) \in Y \text{ and } f(x_1', x_2) < \alpha\}$ is open, for all $\alpha \in \mathbb{R}$ and all $x \in Y$. Pick $\alpha \in \mathbb{R}$ and $x \in Y$ such that $f(x) < \alpha$. Hence, pick a neighbourhood $[a_2, b_2]$ of x_2 such that for all $x_2' \in [a_2, b_2]$, $f(x_1, x_2') < \alpha$. By separate upper semicontinuity, $\{x_1' \in \mathbb{R} \mid (x_1', a_2) \in Y \text{ and } f(x_1', a_2) < \alpha\}$ and $\{x_1' \in \mathbb{R} \mid (x_1', b_2) \in Y \text{ and } f(x_1', b_2) < \alpha\}$ are open. Then, there exists a neighbourhood $[a_1, b_1]$ of x_1 such that $f(x_1', a_2) < \alpha$ and $f(x_1', b_2) < \alpha$ for all $x_1' \in [a_1, b_1]$. By the quasiconvexity assumption, $f(x_1', x_2') < \alpha$ for all $x_1' \in [a_1, b_1]$ and all $x_2' \in [a_2, b_2]$. Since $[a_1, b_1] \times [a_2, b_2]$ is a neighbourhood of x, it follows that f is upper semicontinuous.

Now, assume $Y \subseteq \mathbb{R}^n$ with $n > 2$. We proceed by induction. Without loss of generality, assume f is quasiconvex in each variable x_i for $i = 2, \dots, n$. Since f is separately upper semicontinuous and quasiconcave in each x_i for $i = 2, \dots, n-1$, the induction hypothesis implies that $f(\cdot, x_n)$ is upper semicontinuous in $x_{-n} = (x_1, \dots, x_{n-1})$ for each fixed value of x_n. Since f is quasiconvex in x_n, for all $\alpha \in \mathbb{R}$ and all $x \in Y$, the sets $\{x_n' \in \mathbb{R} \mid (x_{-n}, x_n') \in Y \text{ and } f(x_{-n}, x_n') \leqslant \alpha\}$ and, as we showed above, $\{x_n' \in \mathbb{R} \mid (x_{-n}, x_n') \in Y \text{ and } f(x_{-n}, x_n') < \alpha\}$ are convex.

Pick $\alpha \in \mathbb{R}$ and $x \in Y$ such that $f(x) < \alpha$. Since f is upper semicontinuous in x_n, $\{x_n' \in \mathbb{R} \mid (x_{-n}, x_n') \in Y \text{ and } f(x_{-n}, x_n') < \alpha\}$ is open, there exists a neighbourhood $[a_n, b_n]$ of x_n such that for all $x_n' \in [a_n, b_n]$, $f(x_{-n}, x_n') < \alpha$. Since f is upper semicontinuous in x_{-n}, $\{x_{-n}' \in \mathbb{R}^{n-1} \mid (x_{-n}', a_n) \in Y \text{ and } f(x_{-n}', a_n) < \alpha\}$ and $\{x_{-n}' \mid (x_{-n}', b_n) \in Y \text{ and } f(x_{-n}', b_n) < \alpha\}$ are open. Then, there exists a neighbourhood U_{-n} of x_{-n} such that $f(x_{-n}', a_n) < \alpha$ and $f(x_{-n}', b_n) < \alpha$ for all $x_{-n}' \in U_{-n}$. By the quasiconvexity assumption, $f(x_{-n}', x_n') < \alpha$ for all $x_{-n}' \in U_{-n}$ and all $x_n' \in [a_n, b_n]$. Since $U_{-n} \times [a_n, b_n]$ is a neighbourhood of x, f is upper semicontinuous.

The proof of lower semicontinuity under quasiconcavity is analogous. □

Next, we provide the proof of the following claim that we stated in Remark 4.1.

Claim 5.3. If $Y \subseteq \mathbb{R}^n$ is convex and open and $f : Y \to \mathbb{R}$ is piecewise monotone in $n-1$ coordinates, then f is jointly continuous if and only if it is separately continuous.

Proof of Claim 5.3. The proof is analogous to that of Theorem 3.1; we consider only the case when $n = 2$ and leave the generalisation for an arbitrary finite n to the interested reader.

https://doi.org/10.1017/S0004972723000667 Published online by Cambridge University Press

Let $Y \subseteq \mathbb{R}^2$ be a convex and open set. Assume without loss of generality that f is piecewise monotone in x_2. Next, we prove that f is upper semicontinuous at x. Pick $x \in Y$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha$. By separate upper semicontinuity, $\{x_2' \in \mathbb{R} \mid (x_1, x_2') \in Y \text{ and } f(x_1, x_2') < \alpha\}$ is open. Hence, there exists a neighbourhood $[a_2, b_2]$ of x_2 such that for all $x_2' \in [a_2, b_2]$, $f(x_1, x_2') < \alpha$. If $x_2 \in (a_2^k, a_2^{k+1})$ for some $k = 1, \ldots, m_2 - 1$, then set $[a_2, b_2] \subset [a_2^k, a_2^{k+1}]$; if $x_2 = a_2^k$ for some $k = 2, \ldots, m_2 - 1$, then set $[a_2, b_2] \subset [a_2^{k-1}, a_2^{k+1}]$ (note that since Y is open, $x_2 \in (a_2^1, a_2^{m_2})$).

By separate upper semicontinuity, the sets $\{x_1' \in \mathbb{R} \mid (x_1', x_2) \in Y \text{ and } f(x_1', x_2) < \alpha\}$, $\{x_1' \in \mathbb{R} \mid (x_1', a_2) \in Y \text{ and } f(x_1', a_2) < \alpha\}$ and $\{x_1' \in \mathbb{R} \mid (x_1', b_2) \in Y \text{ and } f(x_1', b_2) < \alpha\}$ are open. Then, there exists a neighbourhood $[a_1, b_1]$ of x_1 such that $f(x_1', x_2) < \alpha$, $f(x_1', a_2) < \alpha$ and $f(x_1', b_2) < \alpha$ for all $x_1' \in [a_1, b_1]$. If $x_2 \in (a_2^k, a_2^{k+1})$ for some $k = 1, \ldots, m_2 - 1$, then $[a_2, b_2] \subset [a_2^k, a_2^{k+1}]$, and hence $f(\tilde{x}_1, \cdot)$ is monotonic on $[a_2, b_2]$ for all $\tilde{x}_1$. Therefore, $f(x_1', x_2') < \alpha$ for all $x_1' \in [a_1, b_1]$ and all $x_2' \in [a_2, b_2]$. If $x_2 = a_2^k$ for some $k = 2, \ldots, m_2 - 1$, then $[a_2, b_2] \subset [a_2^{k-1}, a_2^{k+1}]$, and hence $f(\tilde{x}_1, \cdot)$ is either monotonic on $[a_2, b_2]$ or has a unique kink at x_2 for all $\tilde{x}_1$. Therefore, $f(x_1', x_2') < \alpha$ for all $x_1' \in [a_1, b_1]$ and all $x_2' \in [a_2, b_2]$. Since $[a_1, b_1] \times [a_2, b_2]$ is a neighbourhood of x, f is upper semicontinuous.

The proof of lower semicontinuity at x is analogous. Therefore, f is continuous at x. □

Acknowledgements

The authors are grateful, above all, to Ed Scheinerman for a careful reading of the first draft of this paper, and for his many useful suggestions that we hope to consider in future work. They also thank Aniruddha Ghosh, Imran Parvez Khan, Arthur Paul Pederson, Asghar Qadir and Nicholas Yannelis for conversation and continuing collaboration, and an anonymous referee for their suggestions.

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https://doi.org/10.1017/S0004972723000667 Published online by Cambridge University Press