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# LINEAR DIOPHANTINE EQUATIONS WITH CYCLIC COEFFICIENT MATRICES AND ITS APPLICATIONS TO RIEMANN SURFACES

*by* NOBUMASA TAKIGAWA

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## 1.

Let $c_0, c_1, \ldots, c_{n-1}$ be the nonzero complex numbers and let $C=(c_{u+1,v+1})=(c_{n+u-v})$, $0 \leqq u, v \leqq n-1$, be a cyclic matrix, where $n+u-v$ is taken modulo $n$. In this paper we shall give the solution of the linear equations

$$\sum_{v=0}^{n-1} c_{n+u-v} y_{n-v} = L_u \quad (0 \leqq u \leqq n-1), \tag{1}$$

where $L_u$ $(0 \leqq u \leqq n-1)$ is a fixed complex number. In Theorem 1 we shall give a necessary and sufficient condition for (1) to have an integral solution.

As an application we shall give a nonnegative integral solution $\{t(v)\}$ of the linear Diophantine equations

$$\sum_{v=1}^{p-1} a(u,v)t(v) = p\{n(u)+1-g'\} \quad (1 \leqq u \leqq p-1), \tag{2}$$

where $a(u,v)=([uv/p]+1)p-uv$, $p$ is an odd prime number and [ ] denotes the Gaussian symbol. The linear equations (2) have first been introduced in [**12**] and it has been shown that nonhyperelliptic compact Riemann surfaces $S$ of genus $g \geqq 3$ with an automorphism group $\langle h \rangle$ of order $p$ can be characterized by nonnegative integral solutions of (2), where $\langle h \rangle$ is a cyclic group generated by $h$.

More precisely it is well known that there exists a Fuchsian surface group $K$ such that $S$ can be represented by an orbit space $D/K$ ($D$ is the open unit disk) and a Fuchsian group $\Gamma$ containing $K$ as a normal subgroup such that $\langle h \rangle \simeq \Gamma/K$ (c.f. [**3**] and [**8**]). When we consider the representation of $\langle h \rangle$ as linear transformations of the space of Abelian differentials of the first kind on $S$, $n(u)(0 \leqq u \leqq p-1)$ denotes the multiplicities of $\exp(2\pi ui/p)$ as an eigenvalue of the diagonal form of that representation matrix, where $n(0)=g'$ (the genus of the quotient space $S/\langle h \rangle$, J. Lewittes [**6**]) and $i=\sqrt{-1}$.

Consider the exact sequence

$$1 \to K \to \Gamma \overset{\theta_p}{\to} Z_p \to 1,$$

where $Z$ is the ring of rational integers and $\Gamma/K \simeq Z_p = Z/(pZ)$. If $\Gamma$ has a presentation



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of the form

generators: $X_1, X_2, \ldots, X_T;\ U_1, V_1, \ldots, U_{g'}, V_{g'}$

relations: $X_1^p = X_2^p = \cdots = X_T^p = \prod_{l=1}^{T} X_l \prod_{k=1}^{g'} U_k V_k U_k^{-1} V_k^{-1} = 1$

satisfying $2(g'-1)+(1-1/p)T>0$, then $t(v)$ denotes the number of generators in $\Gamma$ whose image under a surface kernel epimorphism $\theta_p$ is equal to $v(1 \leqq v \leqq p-1)$.

E. K. Lloyd [7] asked the question: For a fixed Fuchsian group and a fixed cyclic group, how many such epimorphisms are there? He gave an answer to this question for cyclic $p$-groups (c.f. [7, Chapter 5]). In this paper, we restrict our attention to the cyclic group of order $p$ and the following question is asked:

(I) Determine all sets $\{n(u), 1 \leqq u \leqq p-1\}$ explicitly for a fixed $T>4$, and construct $\theta_p$ concretely for such $\{n(u)\}$.

If a surface $S$ is given, then we see $\{t(v)\}$ and so $\{n(u)\}$ could be computed by making use of (2). Conversely, if there exists a nonnegative integral solution $\{t(v)\}$ of (2) for a given $\{n(u), T\}$, then the Riemann surface (and so $\theta_p$) could be constructed from $\{t(v)\}$.

If $g'=0$ and $T>4$, then the Weierstrass gap sequences at the fixed points of $h$ is completely determined by $\{n(u)\}$ (c.f. [12]). By making use of the solution for (I), we can determine all types of the Weierstrass gap sequences which appear at the fixed points of $h$. The case $p=3$, the above problem (I) has already been solved by C. Maclachlan [9].

## 2.

In our study the following lemma is essential.

**Lemma 1.** *Let* $V(x_0, x_1, \ldots, x_{n-1}) = \sum_{u=0}^{n-1}(-1)^{u+v}\Delta(u+1, v+1)x_v^u (n>1, 0 \leqq v \leqq n-1)$ *be the Vandermond's determinant. Putting* $F(x) = \prod_{u=0}^{n-1}(x-x_u)$, $F(x)/(x-x_v) = \sum_{u=0}^{n-1}\psi(u,v)x^u$ *and* $W_v = \prod_{0 \leqq k < l \leqq n-1}(x_k - x_l)(k \neq v \neq l)$, *we have* $\Delta(u+1, v+1) = (-1)^{n(n-1)/2+v}\psi(u,v)W_v$ *and* $V(x_0, x_1, \ldots, x_{n-1}) = (-1)^{n(n-1)/2+v}W_v F'(x_v)$, *where* $F'(x) = dF(x)/dx$.

**Proof.** We see that $V(x_0, x_1, \ldots, x_{n-1})$

$$= \begin{vmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ x_0 & \cdots & x_{v-1} & x_v & x_{v+1} & \cdots & x_{n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_0^{u-1} & \cdots & x_{v-1}^{u-1} & x_v^{u-1} & x_{v+1}^{u-1} & \cdots & x_{n-1}^{u-1} \\ x_0^{u} & \cdots & x_{v-1}^{u} & x_v^{u} & x_{v+1}^{u} & \cdots & x_{n-1}^{u} \\ x_0^{u+1} & \cdots & x_{v-1}^{u+1} & x_v^{u+1} & x_{v+1}^{u+1} & \cdots & x_{n-1}^{u+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_0^{n-1} & \cdots & x_{v-1}^{n-1} & x_v^{n-1} & x_{v+1}^{n-1} & \cdots & x_{n-1}^{n-1} \end{vmatrix}.$$

Then $V(x_0, x_1, \ldots, x_{n-1}) = (-1)^{n(n-1)/2+v} W_v(\sum_{u=0}^{n-1} \psi(u,v) x_v^u)$ and $F'(x) = \sum_{u=0}^{n-1} \psi(u,v) x_v^u$. Thus the assertions hold.

Suppose that $\det C \neq 0$. Since $C$ is a cyclic matrix, its eigenvalues are given by

$$\lambda_u = \sum_{v=0}^{n-1} c_{n-v} \omega_n^{uv}, \tag{3}$$

where $\omega_n$ is a primitive $n$-th root of unity. Observing $\det C = \prod_{u=0}^{n-1} \lambda_u \neq 0$ (see [**11**, p. 343 (2)]), we see that (1) reduces to

$$\sum_{v=0}^{n-1} \omega_n^{uv} y_{n-v} = \left( \sum_{w=0}^{n-1} L_w \omega_n^{uw} \right) \Big/ \lambda_u \quad (0 \leqq u \leqq n-1). \tag{4}$$

Consider $x_v = \omega_n^v (0 \leqq v \leqq n-1)$ in Lemma 1. Then we have

$$y_{n-u} = \sum_{v=0}^{n-1} \sum_{w=0}^{n-1} (\psi(u,v) \omega_n^{vw} L_w / \lambda_v F'(\omega_n^v)) \quad (0 \leqq u \leqq n-1). \tag{5}$$

**Lemma 2.** *From* $x_v = \omega_n^v (0 \leqq v \leqq n-1)$ *in Lemma* 1, *follows that*

(i) $F'(\omega_n^v) = n \omega_n^{v(n-1)} \quad (0 \leqq v \leqq n-1)$ *and*

(ii) $\psi(u,0) = 1 \quad (0 \leqq u \leqq n-1)$.

**Proof.** Since $F(x) = \prod_{j=0}^{n-1} (x - \omega_n^j) = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1) = x^n - 1$, the assertions follow at once from

$$F(x)/(x-1) = \sum_{u=0}^{n-1} \psi(u,0) x^u = x^{n-1} + x^{n-2} + \cdots + x + 1.$$

**Lemma 3.** *Assume that* $\det C \neq 0$ *and that* $L_w = c = \text{constant}\,(0 \leqq w \leqq n-1)$. *Then* (1) *has the solution* $y_{n-u} = c/\lambda_0 \, (0 \leqq u \leqq n-1)$, *where*

$$\lambda_0 = \sum_{v=0}^{n-1} c_{n-v}.$$

**Proof.** Since $\sum_{w=0}^{n-1} \omega_n^{vw} = 0$ for $1 \leqq v \leqq n-1$, (5) is reduced to $y_{n-u} = nc\psi(u,0)/\lambda_0 F'(1)$ $(0 \leqq u \leqq n-1)$. Thus the assertion follows from Lemma 2.

Applying the above Lemma 2, we have

$$y_{n-u} = \sum_{v=0}^{n-1} \sum_{w=0}^{n-1} \{\psi(u,v) \omega_n^{v(w+1)} L_w / n\lambda_v\} \quad (0 \leqq u \leqq n-1). \tag{5'}$$

If $y_{n-j} = 1$ for a certain $j (0 \leqq j \leqq n-1)$ and $y_{n-v} = 0$ for all $v (v \neq j, 0 \leqq v \leqq n-1)$, then

$L_w = c_{n+w-j} (0 \leqq w \leqq n-1)$ follows from (1). We can conclude from (5′) that the identities

$$\sum_{v=0}^{n-1} \sum_{w=0}^{n-1} \{c_{n+w-j} \psi(u, v) \omega_n^{v(w+1)} / n\lambda_v\} = \delta_{ju} \quad (0 \leqq u \leqq n-1) \tag{6}$$

hold, where $\delta_{ju}$ is the Kronecker symbol.

**Theorem 1.** *Let*

$$\sum_{v=0}^{n-1} Zc_{n-v} = \left\{ \sum_{v=0}^{n-1} b_{n-v} c_{n-v}; b_{n-v} \in Z \quad (0 \leqq v \leqq n-1) \right\}.$$

*The linear equations* (1) *have an integral solution* $\{y_{n-v}\}$ *if and only if*

(i) $L_w \in \sum_{v=0}^{n-1} Zc_{n-v}$ *for every* $w (0 \leqq w \leqq n-1)$ *and*

(ii) $\sum_{w=0}^{n-1} L_w \in Z\lambda_0$.

**Proof.** From $u=0$ in (3) and (4), follows that $\lambda_0 \sum_{v=0}^{n-1} y_{n-v} = \sum_{w=0}^{n-1} L_w$. Thus if there exists an integral solution of (1), then (i) and (ii) hold. Conversely, if $\{L_w\}$ satisfy the conditions (i) and (ii), then they can be written in the form $L_w = \sum_{j=0}^{n-1} d_{n-j} c_{n+w-j} (d_{n-j} \in Z)$. (5′) and (6) yield

$$y_{n-u} = \sum_{v=0}^{n-1} \sum_{w=0}^{n-1} \left\{ \psi(u, v) \omega_n^{v(w+1)} \cdot \sum_{j=0}^{n-1} d_{n-j} c_{n+w-j} / n\lambda_v \right\}$$

$$= \sum_{j=0}^{n-1} \left( \sum_{v=0}^{n-1} \sum_{w=0}^{n-1} \{\psi(u, v) \, \omega_n^{v(w+1)} c_{n+w-j} / n\lambda_v\} \right) d_{n-j}$$

$$= \sum_{j=0}^{n-1} \delta_{ju} d_{n-j} = d_{n-u} \quad (0 \leqq u \leqq n-1).$$

**Remark.** It can happen that, for the condition (i) only, all $L_w (0 \leqq w \leqq n-1)$ have the same common value. And then, as can be seen from Lemma 3, (1) does not necessarily have an integral solution.

Let $\mathbb{Z}_0^+$ and $\mathbb{R}$ denote the set of all nonnegative integers $(0 \in \mathbb{Z}_0^+)$ and the field of real numbers, respectively.

**Corollary 1.** *Suppose that* $0 < c_{n-v} \in \mathbb{R}$ *and* $0 < L_w \in \mathbb{R}$ $(0 \leqq v, w \leqq n-1)$. *The linear equations* (1) *have a nonnegative integral solution* $\{y_{n-u}\}$ *if and only if* $L_w \in \sum_{v=0}^{n-1} \mathbb{Z}_0^+ c_{n-v}$ *for every* $w (0 \leqq w \leqq n-1)$ *and* $\sum_{w=0}^{n-1} L_w \in \mathbb{Z}_0^+ \lambda_0$.

**Corollary 2.** *Suppose that* $0 < L_w \in \mathbb{R} (0 \leqq w \leqq n-1)$. *Let* $c_{n-v} = m_{n-v}/l_{n-v}$ $(m_{n-v}, l_{n-v} \in \mathbb{Z}_0^+, \; l_{n-v} \neq 0, \; (m_{n-v}, l_{n-v}) = 1$ *for* $0 \leqq v \leqq n-1)$. *Then the linear equations* (1) *have a nonnegative integral solution* $\{y_{n-u}\}$ *if and only if* $L_w \in \mathbb{Z}_0^+ (1/l)$ *for every* $w (0 \leqq w \leqq n-1)$ *and* $\sum_{w=0}^{n-1} L_w \in \mathbb{Z}_0^+ l$, *where* $l$ *is the least common multiple of* $\{l_{n-v}\}$.

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## 3.

Throughout the remainder of this paper the following symbols will be used:

$\mathbb{Q}$: the field of rational numbers

$H_1(p)$: the first factor of the class number of the cyclotomic field $\mathbb{Q}(\exp(2\pi i/p))$

$$\phi = p-1, \quad s=\phi/2 \quad \text{and} \quad \omega_\phi = \exp(2\pi i/\phi)$$

$r$: a primitive root (mod $p$) (In [**1**, p. 266] the notation $g$ is used instead of $r$)

$R(u)$ for $u \in \mathbb{Z}$: the least positive residue of $u(\text{mod}\, p)$

$r_j = R(r^j)$ for $j \in \mathbb{Z}$ (the indices $j$ are taken mod $\phi$)

$$a'(u,v)=\alpha(p-u,v)/p=R(uv)/p(1 \leqq u,v \leqq p-1).$$

We investigate the fundamental properties of the coefficient matrix $A_p=(a(u,v))$ of (2). Replace $A'_p=(a'(u+1,v+1))=(R((u+1)(v+1))/p)$ by $C_p=(c_{u+1,v+1})=I_1A'_pI_2$, where $I_1$ and $I_2$ are the permutation matrices corresponding to the permutation $I_1: r_u \to u+1$ and $I_2: r_{\phi-v} \to v+1$ for $0 \leqq u,v \leqq \phi-1\,(r_0=r_\phi=1)$. Then $c_{u+1,v+1}=R(r_ur_{\phi-v})/p=r_{\phi+u-v}/p$. Hence (2) is reduced to

$$\sum_{v=0}^{\phi-1}(r_{\phi+u-v}/p)t(r_{\phi-v})=n(p-r_u)+1-g' \quad (0 \leqq u \leqq \phi-1). \tag{2'}$$

Since

$$r_v+r_{s+v}=p \quad (0 \leqq v \leqq s-1) \quad ([\mathbf{10}, \text{p. 11 Hilfssatz 2}]), \tag{7}$$

we have

$$T=\sum_{v=0}^{\phi-1}t(r_{\phi-v})=n(p-r_u)+n(p-r_{s+u})+2-2g' \quad (0 \leqq u \leqq s-1). \tag{8}$$

It follows from (2′), (7) and the Riemann–Hurwitz relation that

$$g=pg'+s(T-2)=g'+\sum_{u=0}^{\phi-1}n(p-r_u). \tag{9}$$

For a fixed $T>0$, $T \leqq p\{n(p-r_u)+1-g'\} \leqq (p-1)T$ $(0 \leqq u \leqq \phi-1)$ hold. Since $\{r_0, r_1, \ldots, r_{\phi-1}\}=\{1,2,\ldots,p-1\}$ it follows that

$$\left.\begin{aligned} &T/p \leqq M(p-r_u) \leqq T-T/p \quad \text{if} \quad T \equiv 0\,(\text{mod}\, p),\\ &[T/p]+1 \leqq M(p-r_u) \leqq T-[T/p]-1 \quad \text{if} \quad T \not\equiv 0\,(\text{mod}\, p),\\ &\text{where } M(p-r_u)=n(p-r_u)+1-g'(0 \leqq u \leqq \phi-1) \;([\mathbf{12}, \text{p. 239}]). \end{aligned}\right\} \tag{10}$$

The eigenvalues of the cyclic matrix $C_p=(r_{\phi+u-v}/p)$ are given by

$$\Lambda_u=\sum_{v=0}^{\phi-1}(r_v/p)\omega_\phi^{uv} \quad (0 \leqq u \leqq \phi-1). \tag{3'}$$

**Lemma 4.**

(i) $\Lambda_0 = s$,

(ii) $\Lambda_{2u} = 0 \quad (1 \leqq u \leqq s-1)$,

(iii) $\Lambda_{2u+1} = \left\{ \sum_{v=0}^{s-1} (2r_v - p)\omega_\phi^{(2u+1)v} \right\} \Big/ p \quad (0 \leqq u \leqq s-1)$.

**Proof.** The relations

$$\sum_{v=0}^{s-1} \omega_\phi^{2uv} = 0 \quad (1 \leqq u \leqq s-1) \quad \text{and} \quad \omega_\phi^{(2u+1)v} = -\omega_\phi^{(2u+1)(s+v)} \quad (0 \leqq u, v \leqq s-1) \tag{11}$$

hold ([**10**, p. 15 (3.5), (3.6)]). It follows from (3′), (7) and (11) that

$$\Lambda_0 = \sum_{v=0}^{s-1} (r_v + r_{s+v})/p = s, \Lambda_{2u} = \sum_{v=0}^{s-1} \{(r_v/p)\omega_\phi^{2vu} + ((p-r_v)/p)\omega_\phi^{2(s+v)u}\} = 0$$

and

$$\Lambda_{2u+1} = \sum_{v=0}^{s-1} \{(r_v/p) - (1 - r_v/p)\}\omega_\phi^{(2u+1)v} = \sum_{v=0}^{s-1} \{(2r_v - p)/p\}\omega_\phi^{(2u+1)v}.$$

It is well known that $H_1(p)$ is given by

$$H_1(p) = (-1)^s 2^{1-s} p \prod_{u=0}^{s-1} \left\{ \sum_{v=0}^{s-1} (2r_v - p)\omega_\phi^{(2u+1)v}/p \right\}$$

$$= (-1)^s 2^{1-s} p \prod_{u=0}^{s-1} \Lambda_{2u+1} > 0 \quad ([\mathbf{1}, (2.12)]).$$

Thus the assertions hold.

As a consequence of Lemma 4, we get the following

**Proposition 1.** Rank $A_p = s+1$.

Hence (2′) yields

$$\sum_{v=0}^{s-1} \omega_\phi^{(2u+1)v}\{t(r_{\phi-v}) - t(r_{s-v})\} = \left\{ \sum_{w=0}^{\phi-1} \omega_\phi^{(2u+1)w} M(p - r_w) \right\} \Big/ \Lambda_{2u+1} \quad (0 \leqq u \leqq s-1). \tag{2″}$$

Taking into consideration that $n = s$ and $x_v = \omega_\phi^{2v+1}$ $(0 \leqq v \leqq s-1)$ in Lemma 1, we can conclude from (2″) that

$$t(r_{\phi-u}) - t(r_{s-u}) = \sum_{v=0}^{s-1} \left\{ \sum_{w=0}^{\phi-1} \psi(u, 2v+1)\omega_\phi^{(2v+1)w} M(p - r_w) \right\} \Big/ \Lambda_{2v+1} F'(\omega_\phi^{2v+1}) \quad (0 \leqq u \leqq s-1) \tag{5″}$$

Using a similar method as in the proof of (6), we get the following identities.

**Lemma 5.** *Let an integer $j(0\leqq j\leqq s-1)$ be fixed. Then*

$$\left.\begin{aligned}&\sum_{v=0}^{s-1}\sum_{w=0}^{\phi-1}\{\psi(u,2v+1)\omega_\phi^{(2v+1)w}r_{\phi+w-j}/p\Lambda_{2v+1}F'(\omega_\phi^{2v+1})\}=\delta_{ju},\\&\sum_{v=0}^{s-1}\sum_{w=0}^{\phi-1}\{\psi(u,2v+1)\omega_\phi^{(2v+1)w}r_{s+w-j}/p\Lambda_{2v+1}F'(\omega_\phi^{2v+1})\}=-\delta_{ju}\quad(0\leqq u\leqq s-1).\end{aligned}\right\}\quad(6')$$

**Proposition 2.** *If $T\equiv 0\ (\mathrm{mod}\ 2)$ and $T\geqq 2$, then the following statements* (i) *and* (ii) *are equivalent*:

(i) $t(r_{\phi-v})=t(r_{s-v})\quad(0\leqq v\leqq s-1)$

(ii) $n(r_v)=n(r_{s+v})=T/2+g'-1\quad(0\leqq v\leqq s-1)$.

**Proof.** Using (7), we see that (2′) can be written as

$$\sum_{v=0}^{s-1}\{t(r_{\phi-v})+r_{s+u-v}(t(r_{s-v})-t(r_{\phi-v}))\}=n(p-r_u)+1-g'\quad(0\leqq u\leqq\phi-1).$$

Thus if (i) holds, then (ii) follows. Conversely, if $n(r_{\phi-v})=\text{constant}\,(0\leqq v\leqq s-1)$, then (i) follows from (5′). Then (2′) yields $n(r_{\phi-v})=T/2+g'-1\quad(0\leqq v\leqq\phi-1)$.

By a similar method as in Corollary 2 we get the following

**Proposition 3.** *The linear equations* (2) *have an integral solution* $\{t(r_{\phi-u})-t(r_{s-u});\ 0\leqq u\leqq s-1\}$ *if and only if* $M(r_w)\in\mathbb{Z}_0^+(1/p)$ *for every* $w(0\leqq w\leqq\phi-1)$ *and* $\sum_{w=0}^{\phi-1}M(r_w)\in\mathbb{Z}_0^+p$.

**Example 1.** We give an example that (2″) has an integral solution even if $M(r_w)\notin\mathbb{Z}_0^+$. Consider the case $p\nmid H_1(p)$, in which $p$ is a regular prime [**13**, pp. 61–62]. Putting $T=H_1(p)$ and $M(r_w)=r_wH_1(p)/p$ for every $w(0\leqq w\leqq\phi-1)$, we can easily verify that they satisfy the conditions of the above Proposition 3. Then it follows from (5″) and (6′) that (2″) has the solution $t(1)-t(p-1)=H_1(p)$ and $t(r_{\phi-u})-t(r_{s-u})=0\ (1\leqq u\leqq s-1)$.

## 4.

We are ready to answer the problem (I). Let $\Omega(p)=\{T,M(r_w);0\leqq w\leqq s-1\}$ be a set of $s+1$ nonnegative integers satisfying the conditions (8), (9) and (10). It should be remarked that the remaining $\{g,M(r_w);\ s\leqq w\leqq\phi-1\}$ is determined by (8) and (9). Putting $\Omega^*(p,g)=\{g,g',T,M(r_w);\ 0\leqq w\leqq\phi-1\}$, we have

**Theorem 2.** *Suppose that a set $\Omega^*(p,g)$ is given. Then the corresponding Riemann surface (and so $\theta_p$) exists if and only if the linear equations* (2) *have a nonnegative integral solution.*

**Proof.** If there exist a nonnegative integral solution $\{t(v)\}$ of (2), then

$$\sum_{v=1}^{p-1} a(p-1,v)t(v)=\sum_{v=1}^{p-1} vt(v)\equiv 0 \pmod p.$$

It follows from the result of W. J. Harvey [4, Lemma 6] that there really exists $\theta_p$. The inverse is obvious.

There does not necessarily exist a nonegative integral solution of (2) corresponding to a $\Omega^*(p,g)$, because $M(r_w)\in\Omega^*(p,g)$ $(0\leqq w\leqq\phi-1)$, does not necessarily imply $M(r_w)\in\mathbb{Z}_0^+(1/p)$ or $\sum_{w=0}^{\phi-1}M(r_w)\in\mathbb{Z}_0^+p$.

Let $W(p,g)=\{\Omega^*(p,g);\ M(r_w)\in\mathbb{Z}_0^+(1/p)$ for $0\leqq w\leqq\phi-1$ and $\sum_{w=0}^{\phi-1}M(r_w)\in\mathbb{Z}_0^+p\}$. Then the above Proposition 3 tells us that there exists a compact Riemann surface corresponding to $\Omega^*(p,g)$ if and only if $\Omega^*(p,g)\in W(p,g)$.

**Theorem 3.** *Let the nonnegative integers $g'$ and $T=\xi p+\zeta>4$ ($\zeta=0$, $\zeta=p+1$ or $2\leqq\zeta\leqq p-1$) be given and let $\xi$ and $\zeta$ have nonnegative partitions $\xi=\sum_{j=0}^{\phi-1}b(r_j)$ and $\zeta=\sum_{j=0}^{\phi-1}b'(r_j)$ respectively. Put*

$$M(r_w)=\sum_{j=0}^{\phi-1}b(r_j)r_{\phi+w-j}+\left\{\sum_{j=0}^{\phi-1}b'(r_j)r_{\phi+w-j}\right\}\Big/p\quad(0\leqq w\leqq\phi-1) \tag{13}$$

*and $g=pg'+s(T-2)$. Then $\Omega^*(p,g)=\{g,g',T,M(r_w);\ 0\leqq w\leqq\phi-1\}\in W(p,g)$ if and only if*

$$\sum_{j=0}^{\phi-1}b'(r_j)r_j\equiv 0 \pmod p\left(\sum_{j=0}^{s-1}b'(r_j)r_j\equiv\sum_{j=0}^{s-1}b'(r_{s+j})r_j \pmod p\right). \tag{14}$$

*Moreover in this case the linear equations (2″) have a nonnegative integral solution*

$$\left.\begin{aligned} t(r_{\phi-u})&=b(r_u)p+b'(r_u)\\ t(r_{s-u})&=b(r_{s+u})p+b'(r_{s+u})\end{aligned}\quad(0\leqq u\leqq s-1).\right\} \tag{15}$$

**Proof.** Since $r_{\phi+w-j}\equiv r_{\phi-j}r_w \pmod p$ for $0\leqq w,\ j\leqq\phi-1$, the conditions $M(r_w)\in\mathbb{Z}_0^+(1/p)$ for every $w$ and $\sum_{w=0}^{\phi-1}M(r_w)\in\mathbb{Z}_0^+p$ are equivalent to (14). Then it follows from (5″) and (6′) that

$$\begin{aligned} t(r_{\phi-u})-t(r_{s-u})&=\sum_{j=0}^{\phi-1}(b(r_j)p+b'(r_j))\sum_{v=0}^{s-1}\\ &\qquad\left\{\sum_{w=0}^{\phi-1}\psi(u,2v+1)\,\omega_\phi^{(2v+1)w}r_{\phi+w-j}/p\Lambda_{2v+1}F'(\omega_\phi^{2v+1})\right\}\\ &=\sum_{j=0}^{s-1}\{(b(r_j)p+b'(r_j))-(b(r_{s+j})p+b'(r_{s+j}))\}\delta_{uj}\\ &=b(r_u)p+b'(r_u)-(b(r_{s+u})p+b'(r_{s+u}))\quad(0\leqq u\leqq s-1).\end{aligned}$$

According to Proposition 1, we regard $\{t(r_{s-u});\ 1\leqq u\leqq s-1\}$ as the parameters and take $t(r_{s-u})=b(r_{s+u})p+b'(r_{s+u})$ for $1\leqq u\leqq s-1$. Then we have $t(r_{\phi-u})=b(r_u)p+b'(r_u)$ for $1\leqq u\leqq s-1$. Since

$$T=\sum_{u=0}^{s-1}\{p(b(r_u)+b(r_{s+u}))+b'(r_u)+b'(r_{s+u})\}=\sum_{u=0}^{s-1}\{t(r_{\phi-u})+t(r_{s-u})\},$$

we have

$$t(r_\phi)+t(r_s)=t(1)+t(p-1)=pb(1)+b'(1)+pb(p-1)+b'(p-1).$$

On the other hand

$$t(1)-t(p-1)=pb(1)+b'(1)-\{pb(p-1)+b'(p-1)\}.$$

Hence $t(1)=pb(1)+b'(1)$ and $t(p-1)=pb(p-1)+b'(p-1)$.

**Remark.** It is possible that (15) is not the only solution for (2″), corresponding to (13), but we want to remark here that at least (15) can be given as a solution.

Looking at the above Theorems 2 and 3, we see that our problem (I) is completely solved.

## 5.

Throughout this section we consider a set $\Omega^*(p,g)=\{g,g'=0,\ T>4, M(r_w);\ 0\leqq w\leqq \phi-1\}\in W(p,g)$. Let $\{t(r_w);\ 0\leqq w\leqq\phi-1\}$ be a nonnegative integral solution (2″) corresponding to $\Omega^*(p,g)$. The condition $T>4$ means that every fixed point $Q$ of an automorphism $h$ on $S$ (which is determined by $\{t(r_w)\}$) is a Weierstrass point (see **[6]**). Let $\gamma(Q)$ denote the Weierstrass gap sequence at $Q$. If $t(r_{\phi-v})\neq 0$ i.e., if there exists $X_j\in\Gamma$ satisfying $r_{\phi-v}=\theta_p(X_j)$ for a certain $j$ $(1\leqq j\leqq T)$, then $h^{-1}$ is locally represented as

$$z\to\exp(2\pi i r_v/p)\ \text{at}\ Q(r_{\phi-v}), \tag{16}$$

where $Q(r_{\phi-v})$ is a fixed point on $S=D/K$ corresponding to $t(r_{\phi-v})$ (or $X_j$) (**[4]**, Theorem 7]).

We define the number $J$ as follows:

$$J=\begin{cases}1 & \text{if}\quad \zeta=0,\\ p-1 & \text{if}\quad \zeta=1,\\ p-\zeta+1 & \text{if}\quad 2\leqq\zeta\leqq p-1,\ \text{where}\ T=\xi p+\zeta>4\ \text{and}\ 0\leqq\zeta<p.\end{cases}$$

Let a natural number $r_w (0\leqq w\leqq\phi-1)$ be given, and let $r_{v(k)}$ $(1\leqq k\leqq J, 0\leqq v(k)\leqq\phi-1)$ be the solution of

$$kr_{v(k)}\equiv r_w\ (\text{mod}\ p).$$

We consider the following condition

$$(A_0)\begin{cases} T=\sum_{k=1}^{J} t(r_{\phi-v(k)})>4, \quad \text{and} \\ J-1=\sum_{k=2}^{J}(k-1)t(r_{\phi-v(k)}) \quad \text{if} \quad T\not\equiv \pmod p, \\ p-1=\sum_{k=2}^{J}(k-1)t(r_{\phi-v(k)}) \quad \text{if} \quad T\equiv 1 \pmod p, \text{ [\textbf{12}, p. 240].} \end{cases}$$

Then we have

**Theorem 4.** *Assume* $t(r_{\phi-v})\neq 0$ *for a certain* $v(0\leqq v\leqq \phi-1)$.

(i) *If* $T>p$ *for* $p>3$ *and* $T>4$ *for* $p=3$, *then*

$$\gamma(Q(r_{\phi-v}))=\{lp+r_{\phi+u-v};0\leqq l\leqq n(r_u)-1, 0\leqq u\leqq \phi-1\}. \tag{17}$$

(ii) *If* $4<T\leqq p$ *and the automorphism* $h$ *does not satisfy the condition* $(A_0)$, *then* $\gamma(Q(r_{\phi-v}))$ *is also given by* (17).

(iii) *If* $4<T\leqq p$ *and* $h$ *satisfies the condition* $(A_0)$, *then*

$$\gamma(Q(r_{\phi-u}))=\{lp+r_{\phi+u-v};0\leqq l\leqq n(r_u)-1, \text{ where } u \text{ runs through all } u\ (0\leqq u\leqq \phi-1) \text{ satisfying } n(r_u)\neq 0\}.$$

**Proof.**

(i) Through this assumption we see that $p$ is the first nongap value at $Q(r_{\phi-v})$ [**12**, Prop. 2]. This means that $n(r_u)\neq 0$ for $0\leqq u\leqq \phi-1$. Using the same notation as [**12**, pp. 236–237], we get $\beta_j\equiv r_v$(compare (16) with [**12**, p. 236 (3)]), $\alpha_j(1)=p-\delta_j=r_{\phi-v}$ and $\alpha_j(r_u)\equiv r_u\alpha_j(1)=r_ur_{\phi-v}\equiv r_{\phi+u-v} \pmod p$ ([**12**, (14)]). Then $\beta_j\cdot\alpha_j(r_u)=r_vr_{\phi+u-v}\equiv r_u \pmod p$. Thus (17) follows from [**12**, Lemma 2 (i)].

(ii) The assumption shows that $n(r_u)\neq 0$ for every $0\leqq u=\phi-1$ (see [**12**, Theorem 1] and [**12**, (13)]). By arguments similar to the ones which were used above, we get (ii).

**Example 2.** We will give all sets $\Omega^*(3,g)=\{g,g'=0,T>4,M(r_w);\ w=0,1\}\in W(3,g)$. Then $r_0=1$ and $r=r_1=2$. Put $M(3-r_0)=M(2)=b(1)+2b(2)+\{b'(1)+2b'(2)\}/3$ and $M(1)=2b(1)+b(2)+\{2b'(1)+b'(2)\}/3$. For any natural number $m$ we take

| | $T$ | $b(1)$ | $b(2)$ | $b'(1)$ | $b'(2)$ | | $n(1)$ | $n(2)$ | $g$ |
|---|---|---|---|---|---|---|---|---|---|
| (i) | $3m+2$ | $m-k$ | $k$ | 1 | 1 | | $2m-k$ | $m+k$ | $3m$ |
| (ii) | $3m+3$ | $m-k$ | $k$ | 0 | 3 | i.e. | $2m-k$ | $m+k+1$ | $3m+1$ |
| (iii) | $3m+4$ | $m-k$ | $k$ | 2 | 2 | | $2m+1-k$ | $m+k+1$ | $3m+2$ |

$(0<k\leqq m)$.

In each case (2′) has a solution

| | $t(1)$ | $t(2)$ |
|---|---|---|
| (i) | $3(m-k)+1$ | $3k+1$, |
| (ii) | $3(m-k)$ | $3(k+1)$, |
| (iii) | $3(m-k)+2$ | $3k+2$. |

In each case the gap sequence at a fixed point $Q(j)$ (of $h$) corresponding to $t(j)$ are as follows:

$$\gamma(Q(1))=\{3l+1;0\leqq l\leqq n(1)-1\}\cup\{3l+2;0\leqq l\leqq n(2)-1\},$$

$$\gamma(Q(2))=\{3l+1;0\leqq l\leqq n(2)-1\}\cup\{3l+2;0\leqq l\leqq n(1)-1\}.$$

In this connection see [5, Lemma 6]. *We emphasize that all types of the Weierstrass gap sequences which appear at the fixed points of h are determined explicitly by Theorems 3 and 4.* We give another example.

**Example 3.** Consider the case $T=\xi p+2=p\sum_{j=0}^{\phi-1}b(r_j)+2 \quad (\xi>0)$.

Then $\sum_{j=0}^{\phi-1}b'(r_j)r_j\equiv 0 \pmod p$ and $\sum_{j=0}^{\phi-1}b'(r_j)=2$ have the solution $b'(r_{s+j})=b'(r_j)=1$ for a certain $j$ $(0\leqq j\leqq s-1)$. Hence for

$$n(p-r_w)=\sum_{v=0}^{\phi-1}b(r_v)r_{\phi+w-v} \quad (0\leqq w\leqq\phi-1), \tag{18}$$

(2′) has a solution $t(r_{\phi-j})=b(r_j)p+1$, $t(r_{s-j})=b(r_{s+j})p+1$ and $t(r_{\phi-v})=b(r_v)p$ for every $v(v\neq j, 0\leqq v\leqq s-1)$. All types of the Weierstrass gap sequences which appear at the fixed points of $h$ are determined explicitly by (17) and (18). Indeed, if $j=0$, then

$$\gamma(Q(1))=\{lp+r_u;0\leqq l\leqq n(r_u)-1, 0\leqq u\leqq\phi-1\}$$

and

$$\gamma(Q(r_s))=\gamma(Q(p-1))=\{lp+r_{s+u};0\leqq l\leqq n(r_u)-1, 0\leqq u\leqq\phi-1\}.$$

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Department of Applied Mathematics
Okayama University of Science
Ridai-cho, Okayama, 700
Japan