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ON THE FREIHEITSSATZ IN CERTAIN ONE-RELATOR FREE PRODUCTS. III

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Abstract We study one-relator free products in which the relator has free-product length 4. We find conditions for such presentations to have a Freiheitssatz and classify all non-aspherical presentations under certain conditions.

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1. Introduction

Let G_1 and G_2 be groups. Let $r = abcd \in G_1 * G_2$, where $a, c \in G_1$, and $b, d \in G_2$. Let $G = G_1 * G_2/\langle\langle r\rangle\rangle$, where we denote by $\langle\langle r\rangle\rangle$ the normal closure of r. We would like to classify the cases when the natural mappings $G_1 \to G$ and $G_2 \to G$ are embeddings. In other words, we would like to classify all the triples (G_1, G_2, r) for which the Freiheitssatz holds (the classical case when G_1 and G_2 are both free being due to Magnus [**13**]).

The Freiheitssatz does not always hold, as the following example shows.

Example 1.1. $r = a^2b^2a^{-1}b^{-1}$, $|a| = 5$ and $|b| = 7$, where we denote by $|x|$ the order of the element x in the group G.

If $r = a^2b^2a^{-1}b^{-1}$, then we have $ba = a^2b^2$ in G, and then $ba^2 = a^2b^2a = a^2bba = a^2ba^2b^2$. Since $ba^2 = a^2ba^2b^2$, we get $ba^2 = a^{2m}ba^2b^{2m}$, for every integer m. Since $|a| = 5$, we get for $m = 5$, $ba^2 = a^{10}ba^2b^{10} = ba^2b^{10}$. Hence $b^{10} = b^7 = 1$ so $b = 1$.

In [**7**] and [**8**], Howie showed that the Freiheitssatz holds true for the case $r = w^n$, where $n \geqslant 4$.

In [**1**], Duncan and Howie proved the Freiheitssatz for $r = w^3$, when w does not contain a letter of order 2.

In [**9**], Howie and Shwartz proved the Freiheitssatz for $r = (UaU^{-1}b)^3$, where U is a word in $G_1 * G_2$, and a and b are letters in $G_1 \cup G_2$.

In [**5**], Edjvet and Juhasz have a classification of the cases where the Freiheitssatz holds if $r = a^2b^2a^{-1}b$, $|a| > 10$ and $|b| > 10$ or $r = a^2b^2a^{-1}b^{-1}$.

In [14, 16], Shwartz has a classification for the cases of the Freiheitssatz for length-four one-relator free products, where the relator r has the specific forms $r = a^2bad$ and $r = a^2ba^{-1}d$.

This paper is the extension of the Freiheitssatz for length-four one-relator free products, where the relator has the generalized form $abcd$.

We note here that one of the main applications of our work is for solving equations over groups. It turns out that the Freiheitssatz holds for (G_1, G_2, r), where $r = abcd$ if and only if the equation $atbt^{-1}ctdt^{-1}$ has a solution over $G_1 * G_2$. For more information on this subject the reader is referred to [2–4, 6, 10, 11].

2. Preliminaries

We will make the following assumptions.

Let $G = G_1 * G_2/\langle\langle r\rangle\rangle$, where $r = abcd$, $a, c \in G_1$, and $b, d \in G_2$. Let A be the subgroup of G_1 generated by a and c, and let B be the subgroup of G_2 generated by b and d, and let R_A be $\mathrm{Ker}(\langle a, c|-\rangle \to A)$, and let R_B be $\mathrm{Ker}(\langle b, d|-\rangle \to B)$.

Assumption 2.1. R_A contains none of a^e, c^e, a^f, c^f, ac^e, ac^f, a^2c^e, and R_B contains none of b^e, d^e, b^f, d^f, bd^e, bd^f, b^2d^e, where $e \in \{-1, +1\}$ and $f \in \{-2, +2\}$.

The special cases when R_A contains at least one of ac^f or a^2c^e or R_B contains at least one of bd^f or b^2d^e have been studied in [5, 14, 16], and the other cases which appear in Assumption 2.1 are trivial.

Let $P = \{A_4, Z_3 \oplus Z_3\}$ and let $R = \{A_4, S_4, A_5, Z_3 \oplus Z_3, Z_9, Z_{12}, Z_{15}, Q_{12}\}$, where A_n and S_n denote the alternating and symmetric group of degree n (respectively), Z_n the cyclic group of order n and Q_{12} the quarternionic group of order 12, where $Q_{12} = \langle a, c|a^3c^2, a^6, acac^{-1}\rangle$.

In this paper we shall prove the following theorems.

Theorem 2.2. *Let $G = G_1 * G_2/\langle\langle r\rangle\rangle$, and let A and B be the subgroups of G_1 and of G_2, respectively, as defined above, and suppose that Assumption 2.1 is satisfied. If $A \notin P$ and $B \notin R$, or if $A \notin R$ and $B \notin P$, then the Freiheitssatz holds for (G_1, G_2, r).*

Theorem 2.3. *Let $G = G_1 * G_2/\langle\langle r\rangle\rangle$, and let A and B be the subgroups of G_1 and of G_2, respectively, as defined above, and suppose that Assumption 2.1 is satisfied. If $A \notin R$ and $B \notin R$, and assume either $R_1 = R_A$ and $R_2 = R_B$ or $R_1 = R_B$ and $R_2 = R_A$, and $xyzw$ is a cyclic conjugate of $abcd$, where $\{x, y, z, w\} = \{a, b, c, d\}$, then G is aspherical, unless one of the following holds.*

(1) *$x^3, z^3 \in R_1$ and $ywyw \in R_2$.*

(2) *$x^3, z^4 \in R_1$ and $ywyw \in R_2$.*

(3) *$x^3, z^5 \in R_1$ and $ywyw \in R_2$.*

In each of these cases there exists a non-trivial spherical van Kampen diagram.

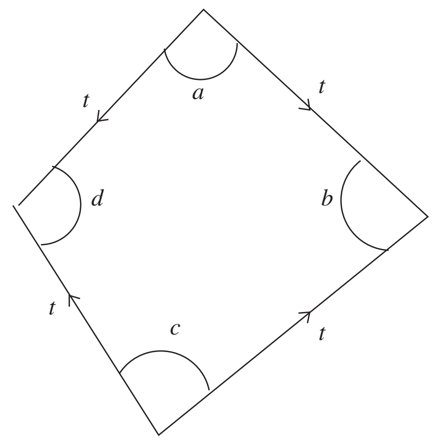

Figure 1. Labelling of a region in a modified relative diagram.

If Theorem 2.2 were false, then there would exist a van Kampen diagram (see Chapter 5 of [**12**] for undefined terms) over the presentation $A*B/\langle\langle r\rangle\rangle$ with boundary label g, where either $g \in A-\{1\}$ or $g \in B-\{1\}$. If Theorem 2.3 were false, then there would be a non-trivial spherical diagram over the presentation for G. Our method of proof in both cases is to show that no such diagrams exist.

Instead of working with van Kampen diagrams we will use a modified relative diagram which we now describe. Consider the presentation $\langle a,b,c,d,t \mid R_A, R_B, atbt^{-1}ctdt^{-1}\rangle$. The diagram M, say, that we study will have regions given by Figure 1 (up to cyclic permutation and inversion).

Thus the edges are labelled by t^e, and the corners of the regions are labelled by a^e, b^e, c^e, d^e, where $e \in \{-1,+1\}$. Let v_0 be a distinguished vertex in the boundary of the diagram; the non-distinguished vertices will be called *inner vertices*, and an *inner region* is a region all of whose vertices are inner. The labelling is done in such a way as to ensure that the label $l(v)$, read anticlockwise around each inner vertex v, is a member of R_A or R_B; and reading the labels clockwise around any region will give $(atbt^{-1}ctdt^{-1})^e$, where $e \in \{-1,+1\}$, up to cyclic permutation. If D is a region of M, for $x \in \{a,b,c,d\}$ denote by $v_x(D)$ the vertex of D whose corner is labelled x. If v is a vertex of the region D, then $t_D(v)$ denotes the label of the corner at v in D.

Thus if the Freiheitssatz fails for G, there will be a diagram M as described above with distinguished vertex v_0 such that $l(v_0) \in A-\{1\}$ or $l(v_0) \in B-\{1\}$ (see [**6**]); and if Theorem 2.3 is false, then there exists a spherical M.

To prove that in each case no such diagram M exists we use a curvature argument. If S is a subdiagram of M containing the vertex v, then $\|v\|_s$ denotes the valency of v in S; if $S=M$ or if S is clear from the context, we will use $\|v\|$. For each region D of M having m vertices v_i, define the curvature $K(D)$ of D by $K(D)=(1-m/2)+\sum 1/\|v_i\|$. Let $K(M)=\sum K(D)$. It is a consequence of the Gauss–Bonnet formula that $K(M)=2$. Our strategy will be to show that this cannot happen, thus obtaining our desired contradiction.

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Lemma 2.4. *Suppose that E is a region of M containing the vertex v.*

(a) *If $t_E(v) \in \{a, a^{-1}, c, c^{-1}\}$, then $t_D(v) \in \{a, a^{-1}, c, c^{-1}\}$ for every region D in M that contains v.*

(b) *If $t_E(v) \in \{b, b^{-1}, d, d^{-1}\}$, then $t_D(v) \in \{b, b^{-1}, d, d^{-1}\}$ for every region D in M that contains v.*

(c) *If $\|v\| = 3$ and $v \neq v_0$, then $l(v) \in \{a^3, b^3, c^3, d^3\}$.*

(d) *If $\|v\| = 3$ and $v \neq v_0$, then $\|u\| \geqslant 4$ for every inner vertex u which is adjacent to v.*

Proof.

(a), (b) These follow from the fact that every vertex which is labelled by a or a^{-1} or c or c^{-1} is a source, and every vertex which is labelled by b or b^{-1} or d or d^{-1} is a sink. Hence every inner vertex is labelled by either a word in R_A or a word in R_B.

(c) This is immediate from Assumption 2.1.

(d) By the symmetry between a and b and c and d in the rectangular region, we may assume without loss of generality that $l(v) = a^3$. Let u be a vertex adjacent to v, and suppose that the edge connecting u to v is on the boundary of the regions D_1 and D_2. Then $t_{D_1}(v) = a$, thus $t_{D_1}(u) = b$ and $t_{D_2}(v) = a$, thus $t_{D_2}(u) = d$. Thus bd is a subword of $l(u)$. Since $l(u)$ is mixed in b and d, it follows that $\|u\| \geqslant 4$. □

Remark 2.5. Since every inner region in the diagram is a rectangle, considering the Equation (3.2) in [**12**, p. 243] we have that $p = 4$, and then by the equality $1/p + 1/q = 1/2$, we get $q = 4$ as well. Then the number of vertices in the boundary of the diagram is one. Hence we may assume that we are working in a spherical diagram, and every vertex in the diagram is an inner vertex.

Let E be a region which satisfies $K(E) \geqslant K(D)$, for every $D \in M$.
For $x \in \{a, b, c, d\}$ denote by v_x the vertex v such that $t_E(v) = x$.
Without loss of generality we may make the following assumption.

Assumption 2.6.

(a) $\|v_a\| \leqslant \|v_b\|$.

(b) $\|v_a\| \leqslant \|v_c\|$.

(c) $\|v_b\| \leqslant \|v_d\|$.

Lemma 2.7.

(a) $\|v_a\| = 3$.

(b) *If E is inner, then $l(v_a) = a^3$.*

(c) *If E is inner, then $\|v_b\| = 4$ or 5, $4 \leqslant \|v_d\| \leqslant 11$.*

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Proof. These observations follow almost immediately from our assumptions, the definition of the curvature, the fact that $K(E) > 0$ and Lemma 2.4. For example, if $\|v_d\| > 11$, then $K(E) \leqslant (1 - \frac{4}{2}) + (\frac{1}{3} + \frac{1}{4} + \frac{1}{3} + \frac{1}{12}) = 0$, a contradiction. $\square$

If $\mathrm{Conf}(E) = (\|v_a\|, \|v_b\|, \|v_c\|, \|v_d\|)$, and E is again our region of maximal curvature, then it follows from Lemma 2.7 that the possibilities for $\mathrm{Conf}(E)$ are

(1) $(3, 4, 3, 11)$ (in this case $K(E) = \frac{1}{132}$);

(2) $(3, 5, 3, 7)$ (in this case $K(E) = \frac{1}{105}$);

(3) $(3, 4, 3, 10)$ (in this case $K(E) = \frac{1}{60}$);

(4) $(3, 4, 3, 9)$ (in this case $K(E) = \frac{1}{36}$);

(5) $(3, 5, 3, 6)$ (in this case $K(E) = \frac{1}{30}$);

(6) $(3, 4, 5, 4)$ (in this case $K(E) = \frac{1}{30}$);

(7) $(3, 4, 4, 5)$ (in this case $K(E) = \frac{1}{30}$);

(8) $(3, 4, 3, 8)$ (in this case $K(E) = \frac{1}{24}$);

(9) $(3, 4, 3, 7)$ (in this case $K(E) = \frac{5}{84}$);

(10) $(3, 5, 3, 5)$ (in this case $K(E) = \frac{1}{15}$);

(11) $(3, 4, 3, 6)$ (in this case $K(E) = \frac{1}{12}$);

(12) $(3, 4, 4, 4)$ (in this case $K(E) = \frac{1}{12}$);

(13) $(3, 4, 3, 5)$ (in this case $K(E) = \frac{7}{60}$); and

(14) $(3, 4, 3, 4)$ (in this case $K(E) = \frac{1}{6}$).

For $1 \leqslant i \leqslant 14$ the statement $\mathrm{Conf}(E) = i'$ will mean that $\mathrm{Conf}(E)$ is case (i) in the above list.

3. The idea of the proof

The proofs of Theorems 2.2 and 2.3 are by case-by-case analysis of the 14 possible cases for $\mathrm{Conf}(E)$.

The idea of the proof is to look in every region E which has positive curvature, and then to show that the average curvature of the diagram is less then 0. We use the negatively curved neighbouring regions of E to give compensation to E, and to other neighbouring positively curved regions. The proof involves repetitive calculations in each of the 14 cases. For this reason we give details of only one case below, and refer the reader to [**15**] for a complete account.

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The idea of the proof of one of the 14 cases is as follows.

Assume E is a region which satisfies

(i) $\mathrm{Conf}(E) = (3, 4, 3, 11)$; and

(ii) every region D in the diagram satisfies $K(D) \leqslant K(E)$.

Lemma 3.1. *Every region D, which satisfies $K(D) > 0$, contains a vertex which has valency* 11.

Proof. Since every positively curved region D should satisfy $K(D) \geqslant \frac{1}{132}$, and every D satisfies $K(D) \leqslant K(E) = \frac{1}{132}$, we get $K(D) = \frac{1}{132}$. Then $\mathrm{Conf}(D) = (3, 4, 3, 11)$. Then D contains a vertex which has valency 11.

Hence, if a region D does not contain a vertex which has valency 11, then $K(D) \leqslant 0$.

Let E be any region which satisfies $K(E) = \frac{1}{132}$. Then $\mathrm{Conf}(E) = (3, 4, 3, 11)$. Let E_i, where $1 \leqslant i \leqslant 10$, be the 10 regions other than E that contain the vertex v_d. Let v_i be the vertex adjacent to v_d in E_i for each $1 \leqslant i \leqslant 10$. □

Lemma 3.2. *If one of the E_j has only one vertex which has valency 3, then* $\sum K(E_i) + K(E) \leqslant 0$.

Proof. Let E_j be the region which satisfies the condition that there is only one vertex in E_j of valency 3. Since every E_i has a vertex v_d which has valency 11, then

$$K(E_j) \leqslant \frac{1}{3} + \frac{1}{11} + \frac{1}{4} + \frac{1}{4} - 1 \leqslant -\frac{10}{132}.$$

Since every region E_i satisfies $K(E_i) \leqslant \frac{1}{132}$, then

$$K(E_j) + \sum K(E_i) + K(E) \leqslant -\frac{10}{132} + \frac{10}{132} \leqslant 0.$$

□

Lemma 3.3. $\sum K(E_i) + K(E) \leqslant 0$.

Proof. Assume $K(E) + \sum K(E_i) > 0$. By Lemma 3.2, every E_i contains at least two vertices of valency 3. By Lemma 2.4 (d), no two adjacent vertices are of valency 3. Thus if $K(E) + \sum K(E_i) \geqslant 0$, then every E_i contains two opposite vertices of valency 3. Since $\|v_d\| = 11$, 11 vertices adjacent to v_d each have valency 3. Since every v_i is adjacent to v_d, every v_i is labelled by a word of length 3 in a, a^{-1}, c and c^{-1}. Since $\|v_i\| = 3$, then by Lemma 2.4 (a) $l(v_i) \in \{a^3, c^3\}$. Let v_0 be the vertex v_a. Thus $l(v_0) = a^3$. Now v_1 is opposite to v_0 in E_1, thus $l(v_1) = c^3$. Furthermore, v_i is opposite to v_{i-1} for every $i \leqslant 11$, thus if $l(v_{i-1}) = a^3$, then $l(v_i) = c^3$. Since $l(v_0) = a^3$, thus $l(v_{2k}) = a^3$. Since $v_{10} = v_c$, we have $l(v_{10}) = c^3$, which is a contradiction to $l(v_{2k}) = a^3$. Thus $\sum K(E_i) + K(E) \leqslant 0$. □

Now we turn to the proof of the case $\mathrm{Conf}(E) = (3, 4, 3, 11)$. By Lemma 3.1, every positively curved region D contains a vertex which has valency 11. If a region D contains a vertex which has valency 11, then according to the classification of the configurations, $\mathrm{Conf}(D) = (3, 4, 3, 11)$. In particular, one vertex only in D has valency 11. Then it is enough to prove that if a region D is positively curved, then the 11 neighbouring regions D_i to D, which contain the vertex which has valency 11 in D, are giving enough compensation to D, considering that some of the neighbouring regions contain more then one vertex which has valency 11, and then they should give compensation to regions other than D as well. Assume a region D_1, which is a neighbouring region to D, has two or more vertices which have valency 11. Then

$$K(D_1) \leqslant \frac{1}{11} + \frac{1}{11} + \frac{1}{3} + \frac{1}{3} - 1 \leqslant -\frac{5}{33}.$$

Since every positively curved region has curvature $\frac{1}{132}$, D_1, by itself only, can give compensation to at least 20 regions, which is more than enough. Hence, we may assume that every neighbouring region D_i to D has one vertex only which has valency 11. Then every D_i needs to give compensation to D only. Then by Lemma 3.3, $\sum K(D_i) + K(D) \leqslant 0$, which proves the Freiheitssatz for the case where $\mathrm{Conf}(E) = (3, 4, 3, 11)$. $\square$

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