



EQUIVARIANT K -THEORY OF GRASSMANNIANS

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Abstract

We address a unification of the Schubert calculus problems solved by Buch [A Littlewood–Richardson rule for the K -theory of Grassmannians, *Acta Math.* **189** (2002), 37–78] and Knutson and Tao [Puzzles and (equivariant) cohomology of Grassmannians, *Duke Math. J.* **119**(2) (2003), 221–260]. That is, we prove a combinatorial rule for the structure coefficients in the torus-equivariant K -theory of Grassmannians with respect to the basis of Schubert structure sheaves. This rule is positive in the sense of Anderson *et al.* [Positivity and Kleiman transversality in equivariant K -theory of homogeneous spaces, *J. Eur. Math. Soc.* **13** (2011), 57–84] and in a stronger form. Our work is based on the combinatorics of *genomic tableaux* and a generalization of Schützenberger’s [Combinatoire et représentation du groupe symétrique, in *Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976*, Lecture Notes in Mathematics, 579 (Springer, Berlin, 1977), 59–113] *jeu de taquin*. Using our rule, we deduce the two other combinatorial rules for these coefficients. The first is a conjecture of Thomas and Yong [Equivariant Schubert calculus and jeu de taquin, *Ann. Inst. Fourier (Grenoble)* (2013), to appear]. The second (found in a sequel to this paper) is a *puzzle* rule, resolving a conjecture of Knutson and Vakil from 2005.

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1. Introduction

1.1. Overview. Let $X = \text{Gr}_k(\mathbb{C}^n)$ denote the Grassmannian of k -dimensional subspaces of \mathbb{C}^n . The action of $\text{GL}_n(\mathbb{C})$ on X restricts to an action of the Borel subgroup \mathbb{B} of invertible upper triangular matrices and its subgroup \mathbb{T} of invertible diagonal matrices. The \mathbb{T} -fixed points $e_\lambda \in X$ are naturally indexed by

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Young diagrams λ contained in the rectangle $k \times (n - k)$. The *Schubert varieties* are the orbit closures $X_\lambda = \overline{B_- e_\lambda}$. The Poincaré duals $[X_\lambda]$ of their classes form a \mathbb{Z} -basis of the cohomology ring $H^*(X, \mathbb{Z})$.

The (classical) *Schubert structure constants* $c_{\lambda, \mu}^v$ are defined by

$$[X_\lambda] \cdot [X_\mu] = \sum_v c_{\lambda, \mu}^v [X_v].$$

In Schubert calculus, one interprets $c_{\lambda, \mu}^v \in \mathbb{Z}_{\geq 0}$ as the number of points (when finite) in a generic triple intersection of Schubert varieties. Combinatorially, $c_{\lambda, \mu}^v$ is computed, in a manifestly nonnegative manner, by Littlewood–Richardson rules. The first such rule was stated by Littlewood and Richardson in the 1930s [LiRi34] in their study of the representation theory of the symmetric group. The first rigorous proof of a rule was given by Schützenberger [Sc77] in the 1970s. These rules describe $c_{\lambda, \mu}^v$ as a count of certain Young tableaux.

In the modern Schubert calculus, significant attention is paid to the problem of generalizing the above work to richer cohomology theories. In the early part of the last decade, two problems of this type were solved. Buch [Bu02] found the first rule for the multiplication of the Schubert structure sheaves in K -theory. His rule is positive after accounting for a predictable alternation of sign. Separately, Knutson and Tao [KnTa03] introduced *puzzles* to give the first rule for equivariant Schubert calculus that is positive in the sense of [Gr01].

We turn to a unification of these problems. Let $K_T(X)$ denote the Grothendieck ring of T -equivariant vector bundles over X . This ring has a natural $K_T(\text{pt})$ -module structure and an additive basis given by the classes of Schubert structure sheaves. For background, we refer the reader to, for example, [KoKu90, AnGrMi11] and the references therein. The analogues of Littlewood–Richardson coefficients are the Laurent polynomials

$$K_{\lambda, \mu}^v \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \cong K_T(\text{pt}),$$

defined by

$$[\mathcal{O}_{X_\lambda}] \cdot [\mathcal{O}_{X_\mu}] = \sum_{v \subseteq k \times (n-k)} K_{\lambda, \mu}^v [\mathcal{O}_{X_v}],$$

where $[\mathcal{O}_{X_\lambda}]$ is the class of the structure sheaf of X_λ . These coefficients may be algebraically computed using *double Grothendieck polynomials*; see [LaSc82, FuLa94]. The problem addressed in this paper is to prove a combinatorial rule for $K_{\lambda, \mu}^v$.

We summarize past contributions to the problem. Knutson and Vakil conjectured a formula for $K_{\lambda, \mu}^v$ in terms of puzzles (reported in [CoVa09, Section 5]). Kreiman [Kr05] proved a rule for the case $\lambda = v$, corresponding to a certain localization (see Section 4). Lenart and Postnikov [LePo07] gave a

rule for the case $\lambda = (1)$ (in a broader context applicable to any generalized flag variety); we use this result. Later, Graham and Kumar [GrKu08] determined the coefficients in the case $X = \mathbb{P}^{n-1}$. ‘Positivity’ of $K_{\lambda, \mu}^v$ (in a more general context) was geometrically established by Anderson *et al.* [AnGrMi11]. More recently, Knutson [Kn10] obtained a puzzle rule in $K_T(X)$ for the different problem of multiplying the class of a Schubert structure sheaf by that of an *opposite* Schubert structure sheaf. Finally, Thomas and the second author conjectured the first Young tableau rule for $K_{\lambda, \mu}^v$ [ThYo13, Conjecture 4.7]; they showed that their conjectural rule is [AnGrMi11]-positive; see [ThYo13, Section 4.1]. No combinatorial rule for structure coefficients of $K_T(X)$ with respect to any fixed basis had earlier been proved.

This paper introduces and proves an [AnGrMi11]-positive rule for the structure coefficients $K_{\lambda, \mu}^v$ (Theorem 1.3). In fact, our rule exhibits a further property of the coefficients which seems at present not to have a geometric explanation. The rule allows us to deduce the aforementioned conjecture of [ThYo13]. Indeed, we complete the strategy set out in [ThYo13], and our Theorem 1.3 is a generalization of the rule of [ThYo13] for T -equivariant cohomology. The first step of our proof is to relate our combinatorial rule to a K -theoretic generalization of a recurrence proven by Molev and Sagan [MoSa99] and Knutson and Tao [KnTa03] (who also credit Okounkov). A similar step was employed by Buch [Bu15], who gave a rule for the equivariant quantum cohomology of Grassmannians; see [BuMi11]. (The case of nonequivariant quantum cohomology had previously been handled geometrically by [Co09] and combinatorially by [BKPT16]; see [BuKrTa03].)

In a sequel to this paper [PeYo17a], we use our new rule to also resolve the 2005 puzzle conjecture of Knutson and Vakil. More precisely, we first show that their conjecture is false by explicit counterexample. On the other hand, our rule suggests a mild correction of their conjecture, which we then prove. (Since the present paper and [PeYo17a] appeared in preprint form, Wheeler and Zinn-Justin [WhZi16] have given an independent proof of a different puzzle rule for these same coefficients.)

The main innovations of this paper are *genomic tableaux* and a generalization of Schützenberger’s *jeu de taquin* [Sc77]. We anticipate additional applications of these ideas. Monical has reported the use of genomic tableaux in the study of *Lascoux polynomials* (see, for example, [RoYo15] and references therein) and K -theoretic analogues of *Demazure atoms*, extending results of [HLMvW11]. Gillespie and Levinson [GiLe17] have applied genomic tableaux to the real geometry of Schubert curves. These tableaux also give a new rule for (nonequivariant) K -theory of Grassmannians; the announcement [PeYo15]

outlines applications to analogous problems when X is replaced by Lagrangian or maximal orthogonal Grassmannians. A full development of these results within a theory of genomic tableaux is found in [PeYo17b]. Moreover, closely related to the equivariant Schubert calculus of X , the combinatorial rule of Molev and Sagan [MoSa99] solves a *triple Schubert calculus* problem in $H^*(\mathrm{GL}_n(\mathbb{C})/\mathbb{B} \times X \times \mathrm{GL}_n(\mathbb{C})/\mathbb{B})$; see [KnTa03, Section 6]. Our methods should extend to give a K -theoretic analogue; see [KnTa03, Section 6.2]. Finally, we remark that Buch (private communication) has shown us a short argument that turns our rule into an [AnGrMi11]-positive rule for the structure coefficients with respect the basis of $K_T(X)$ dual to $\{\{\mathcal{O}_\lambda\}\}$.

1.2. Genomic tableaux. A *genomic* tableau is a Young diagram filled with (subscripted) labels i_j , where $i \in \mathbb{Z}_{>0}$, and the j s that appear for each i form an initial segment of $\mathbb{Z}_{>0}$. It is *edge-labeled* of shape ν/λ if each horizontal edge of a box weakly below the southern border of λ (viewed as a lattice path from $(0, 0)$ to $(k, n - k)$) is filled with a subset of $\{i_j\}$.

Let x^\rightarrow be the box immediately east of x , and let x^\uparrow be the box immediately north of x , and so on. For a box x , let \bar{x} denote the upper horizontal edge of x , and let \underline{x} denote the lower horizontal edge. We write $\text{family}(i_j) = i$. We distinguish two orders on subscripted labels. We say that $i_j < k_\ell$ if $i < k$. We write $i_j < k_\ell$ if $i < k$ or $i = k$ with $j < \ell$. It should be noted that $<$ is a total order, while $<$ is not.

A genomic tableau T is *semistandard* if the following four conditions hold.

(S.1) $\text{label}(x) < \text{label}(x^\rightarrow)$.

(S.2) Every label is $<$ -strictly smaller than any label South in its column. (Throughout, we write ‘West’, ‘west’ and ‘NorthWest’ to mean ‘strictly west’, ‘weakly west’ and ‘strictly north and strictly west’, respectively, and so on.)

(S.3) If i_j and k_ℓ appear on the same edge, then $i \neq k$.

(S.4) If i_j is West of i_k , then $j \leq k$.

We refer to the set of all labels i_j (for fixed i and j) as a *gene*. The *content* of T is (c_1, c_2, c_3, \dots) , where c_i is the number of genes of family i . Suppose that x is in row r . A label i_j is *too high* if $i \geq r$ and $i_j \in \bar{x}$, or alternatively if $i > r$ and $i_j \in x$ or $i_j \in \underline{x}$.

EXAMPLE 1.1. For $\lambda = (4, 2, 2, 1)$ and $\nu = (6, 5, 4, 3, 2)$, consider the genomic tableau T :

				1 ₂	1 ₃
		1 ₂	2 ₁	2 ₂	
		2 ₁	3 ₂		
	1 ₁	3 ₂			
2 ₁	2 ₁ 3 ₂	4 ₂			
3 ₁	3 ₂	4 ₁			

The content of T is $(3, 2, 2, 2)$. The tableau T is not semistandard, since the second column from the left fails (S.2). If we deleted the 3_2 from the edge, the result would be semistandard. No label is too high.

□

1.3. The ballot property. A genotype G of T is a choice of one label from each gene of T . Let $\text{word}(G)$ be obtained by reading G down columns from right to left. (If there are multiple labels on an edge, we read them from smallest to largest in \prec -order.) Then, G is *ballot* if in every initial segment of $\text{word}(G)$, there are at least as many labels of family i as of family $i + 1$, for each $i \geq 1$. We say that T is *ballot* if all of its genotypes are ballot. Let $\text{BallotGen}(\nu/\lambda)$ be the set of ballot, semistandard, edge-labeled genomic tableaux of shape ν/λ where no label is too high.

EXAMPLE 1.2. Let $T = \begin{array}{|c|c|} \hline \square & 1_2 \\ \hline 1_1 & 2_1 \\ \hline \end{array}$, and let $U = \begin{array}{|c|c|} \hline \square & 1_1 \\ \hline 1_1 & 2_1 \\ \hline \end{array}$. Then, T is ballot: the one genotype (itself) has reading word $1_2 2_1 1_1$, which is a ballot sequence. However, U is not ballot: it has two genotypes $\begin{array}{|c|c|} \hline \square & \square \\ \hline 1_1 & 2_1 \\ \hline \end{array}$ and $\begin{array}{|c|c|} \hline \square & 1_1 \\ \hline \square & 2_1 \\ \hline \end{array}$, and the word for the former is $2_1 1_1$, which is not ballot.

□

1.4. Tableau weights and the main theorem. Let $T \in \text{BallotGen}(\nu/\lambda)$. For a box x , $\text{Man}(x)$ is the ‘Manhattan distance’ from the southwest corner (point) of $k \times (n - k)$ to the northwest corner (point) of x (the length of any north-east lattice path between the corners).

For a gene \mathcal{G} , let $N_{\mathcal{G}}$ be the number of genes \mathcal{G}' with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$ and $\mathcal{G}' \succ \mathcal{G}$. For instance, in Example 1.1, $N_{1_1} = 2$, since the genes 1_2 and 1_3 are of the same family as 1_1 (namely family 1) but $1_1 \prec 1_2, 1_3$.

If $\ell = i_j \in \underline{x}$, and x is in row r , then

$$\text{edgefactor}(\ell) := \text{edgefactor}_{\underline{x}}(i_j) := 1 - \frac{t_{\text{Man}(x)}}{t_{r-i+N_{i_j+1}+\text{Man}(x)}}. \tag{1.1}$$

The *edge weight* $\text{edgewt}(T)$ is $\prod_{\ell} \text{edgefactor}(\ell)$; the product is over edge labels of T .

A nonempty box x in row r is *productive* if $\text{label}(x) < \text{label}(x^\rightarrow)$ or if $x^\rightarrow \notin v/\lambda$. If $i_j \in x$, set

$$\text{boxfactor}(x) := \frac{t_{\text{Man}(x)+1}}{t_{r-i+N_{i_j}+1+\text{Man}(x)}}. \tag{1.2}$$

The *box weight* of a tableau T is $\text{boxwt}(T) := \prod_x \text{boxfactor}(x)$, where the product is over all productive boxes of T . The *weight* of T is $\text{wt } T := (-1)^{d(T)} \times \text{boxwt}(T) \times \text{edgewt}(T)$. Here, $d(T) = \sum_{\mathcal{G}} (|\mathcal{G}| - 1)$, where the sum is over all genes \mathcal{G} , and $|\mathcal{G}|$ is the (multiset) cardinality of \mathcal{G} . Set

$$L_{\lambda, \mu}^v := \sum_T \text{wt } T,$$

where the sum is over all $T \in \text{BallotGen}(v/\lambda)$ that have content μ .

THEOREM 1.3 (Main Theorem). *We have $K_{\lambda, \mu}^v = L_{\lambda, \mu}^v$.*

This provides the first proved rule for $K_{\lambda, \mu}^v$ that is manifestly [AnGrMi11]-positive. That is, let $z_i := t_i/t_{i+1} - 1$. For $j > i$, we have

$$\frac{t_i}{t_j} = \prod_{k=i}^{j-1} (z_k + 1) \quad \text{and} \quad 1 - \frac{t_i}{t_j} = - \left(\prod_{k=i}^{j-1} (z_k + 1) - 1 \right). \tag{1.3}$$

Therefore, $(-1)^{\#\text{edge labels}} \times \text{boxwt}(T) \times \text{edgewt}(T)$ is z -positive. Since clearly $d(T) = |v| - |\lambda| - |\mu| + \#\text{edge labels}$, we have that $(-1)^{|v|-|\lambda|-|\mu|} L_{\lambda, \mu}^v = \sum_T (-1)^{|v|-|\lambda|-|\mu|} \text{wt } T$ is z -positive. This positivity is the same as that of [AnGrMi11, Corollary 5.3] after the substitution $z_i \mapsto e^{\alpha_i} - 1$, where α_i is the i th simple root for the root system A_{n-1} .

EXAMPLE 1.4. To compute $K_{(2),(2,1)}^{(2,2)}$ for $\text{Gr}_2(\mathbb{C}^4)$, the required tableaux are

$$T_1 = \begin{array}{|c|c|} \hline & \\ \hline 1_1 & 1_2 \\ \hline 2_1 & \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|} \hline & \\ \hline 1_1 & 1_2 \\ \hline & 2_1 \\ \hline \end{array}, \quad T_3 = \begin{array}{|c|c|} \hline & \\ \hline 1_1 & 1_2 \\ \hline 2_1 & 2_1 \\ \hline \end{array}, \quad T_4 = \begin{array}{|c|c|} \hline & 1_2 \\ \hline 1_1 & 2_1 \\ \hline \end{array}, \quad T_5 = \begin{array}{|c|c|} \hline & 1_2 \\ \hline 1_1 & 2_1 \\ \hline 2_1 & \\ \hline \end{array}$$

Then,

- $\text{edgewt}(T_1) = 1 - t_1/t_2$, $\text{boxwt}(T_1) = t_3/t_4$ and $d(T_1) = 0$;
- $\text{edgewt}(T_2) = 1 - t_2/t_3$, $\text{boxwt}(T_2) = t_3/t_4$ and $d(T_2) = 0$;
- $\text{edgewt}(T_3) = (1 - t_1/t_2)(1 - t_2/t_3)$, $\text{boxwt}(T_3) = t_3/t_4$ and $d(T_3) = 1$;

- $\text{edgewt}(T_4) = (1 - t_3/t_4)$, $\text{boxwt}(T_4) = t_2/t_4$ and $d(T_4) = 0$; and
- $\text{edgewt}(T_5) = (1 - t_1/t_2)(1 - t_3/t_4)$, $\text{boxwt}(T_5) = t_2/t_4$ and $d(T_5) = 1$.

Hence,

$$K_{(2),(2,1)}^{(2,2)} = \left(1 - \frac{t_1}{t_2}\right) \frac{t_3}{t_4} + \left(1 - \frac{t_2}{t_3}\right) \frac{t_3}{t_4} - \left(1 - \frac{t_1}{t_2}\right) \left(1 - \frac{t_2}{t_3}\right) \frac{t_3}{t_4} + \left(1 - \frac{t_3}{t_4}\right) \frac{t_2}{t_4} - \left(1 - \frac{t_1}{t_2}\right) \left(1 - \frac{t_3}{t_4}\right) \frac{t_2}{t_4}.$$

We observe that, after rewriting using (1.3), each term is z -negative, in agreement with the discussion above. That is,

$$\begin{aligned} &(-1)^{|(2,2)|-|(2)|-|(2,1)|} K_{(2),(2,1)}^{(2,2)} \\ &= -(-z_1)(z_3 + 1) - (-z_2)(z_3 + 1) + (-z_1)(-z_2)(z_3 + 1) \\ &\quad - (-z_3)(z_2 + 1)(z_3 + 1) + (-z_1)(-z_3)(z_2 + 1)(z_3 + 1) \\ &= z_1(z_3 + 1) + z_2(z_3 + 1) + z_1z_2(z_3 + 1) \\ &\quad + z_3(z_2 + 1)(z_3 + 1) + z_1z_3(z_2 + 1)(z_3 + 1) \end{aligned}$$

is z -positive (without any cancelation needed). □

There is a stronger positivity property exhibited by the rule of Theorem 1.3. The work of [AnGrMil1] generalizes the positivity of Graham [Gr01]: the equivariant Schubert structure coefficients are polynomials with nonnegative integer coefficients in the simple roots α_i . In [Kn10], Knutson observes that Graham’s geometric argument further implies that the coefficients can be expressed as polynomials with nonnegative integer coefficients in the positive roots *such that each monomial is square-free*. Moreover, Knutson raises the issue of finding a ‘proper analogue’ in equivariant K -theory for this square-free property. For X , we have the following.

COROLLARY 1.5 (Strengthened [AnGrMil1]-positivity). *Let $z_{ij} := t_i/t_j - 1$. Then, $(-1)^{|\nu|-|\lambda|-|\mu|} K_{\lambda,\mu}^\nu$ is expressible as a polynomial with nonnegative integer coefficients in the z_{ij} s such that each monomial is square-free.*

Proof. It should be noted that

$$\frac{t_i}{t_j} = z_{ij} + 1 \quad \text{and} \quad 1 - \frac{t_i}{t_j} = -z_{ij}. \tag{1.4}$$

Thus, with a completely analogous argument to that immediately following (1.3), we see that $(-1)^{|\nu|-|\lambda|-|\mu|} L_{\lambda,\mu}^\nu$ is positive in the z_{ij} s (using the direct

substitution (1.4)). It remains to show each monomial in *this* expression for $L_{\lambda,\mu}^v$ in the z_{ij} s is square-free.

We consider a $T \in \text{BallotGen}(v/\lambda)$. Every $\text{edgefactor}(\ell)$ is of the form $-z_{ij}$, while every $\text{boxfactor}(x)$ is of the form $z_{ij} + 1$. We define an (i, j) -label to be either an edge label with $\text{edgefactor}(\ell) = -z_{ij}$ or a label ℓ in a productive box x with $\text{boxfactor}(x) = z_{ij} + 1$.

Suppose that ℓ and ℓ' are (i, j) -labels of T . Say, $\ell \in x$ or \bar{x} , and $\ell' \in y$ or \bar{y} . Since both are $(i, -)$ -labels, $\text{Man}(x) = \text{Man}(y)$. Hence, x and y are boxes of the same diagonal. We may assume x to be northwest of y . Let ℓ be an instance of m_n , and let ℓ' be an instance of p_q . Since both are $(-, j)$ -labels,

$$\text{row}(x) - m + N_{m_n} = \text{row}(y) - p + N_{p_q}.$$

By (S.1) and (S.2),

$$m + r(y) - r(x) \leq p,$$

so

$$N_{m_n} = r(y) - r(x) + m - p + N_{p_q} \leq p - p + N_{p_q} = N_{p_q}.$$

Hence, by ballotness of T , $x = y$ and moreover $m = p$. Therefore, by (S.2) and (S.3), $\ell = \ell'$, and thus T contains at most one (i, j) -label, and each monomial in the stated expression for $(-1)^{|\nu|-|\lambda|-|\mu|} L_{\lambda,\mu}^v$ is square-free. Now, we conclude by appealing to Theorem 1.3. □

We do not know a geometric explanation for Corollary 1.5. However, based on this result, one speculates that for any G/P , if for each positive root α we set $z_\alpha := e^\alpha - 1$, then the corresponding Schubert structure coefficients for $K_T(G/P)$ may be expressed in a square-free manner with nonnegative coefficients in the z_α s.

1.5. Organization. The first key to the proof is to reformulate Theorem 1.3 in terms of the more technical *bundled* tableaux that are appropriate for the inductive argument; this is presented in Section 2. In Section 3, we outline this inductive argument that the rule of Theorem 1.3 satisfies the key recurrence alluded to above. The base case is given in Section 4. Both the plan of induction and the base case may be considered to be routine.

The core of the argument lies in Sections 5–12. The central innovation of this paper is a genomic generalization of Schützenberger’s *jeu de taquin*. This permits us to establish a combinatorial map of formal sums of tableaux. This part of the argument is developed as a sequence of four main ideas.

- (1) To show well-definedness of the map, we identify and characterize the class of *good tableaux* that arise via genomic *jeu de taquin* (Sections 5, 6 and 7).

- (2) To establish surjectivity, we develop reverse genomic *jeu de taquin* (Sections 8 and 9).
- (3) To prove that the map respects the coefficients of the key recurrence, we define and prove properties of a *reversal tree* (Sections 10 and 11).
- (4) The map is weight-preserving. However, a significant subtlety is that it is not generally weight-preserving on *individual* tableaux. To establish this property of the map, we need involutions that pair tableaux (Section 12).

In Section 13, we recall the conjecture of [ThYo13] and prove it from Theorem 1.3; this argument is essentially independent of the rest of the paper. The three appendices isolate mostly straightforward but long technical proofs of important propositions.

2. Bundled tableaux and a reformulation of Theorem 1.3

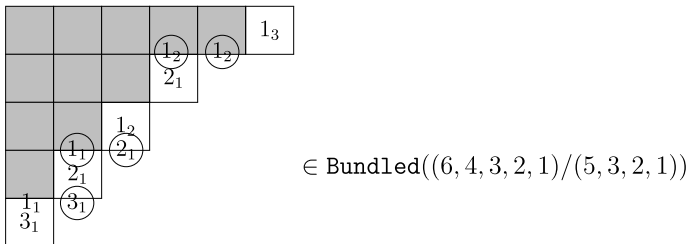
A tableau $T \in \text{BallotGen}(v/\lambda)$ is *bundled* if every edge label is the westmost label of its gene. For example, in Example 1.4, only T_3 is *not* bundled (the eastmost 2_1 is to blame). We denote the set of bundled tableaux of shape v/λ by $\text{Bundled}(v/\lambda)$.

We define a surjection

$$\text{Bun} : \text{BallotGen}(v/\lambda) \rightarrow \text{Bundled}(v/\lambda).$$

This sends T to $\text{Bun}(T)$ by deleting each edge label of T that is not maximally west in its gene. If $B \in \text{Bundled}(v/\lambda)$, then any $T \in \text{Bun}^{-1}(B)$ differs from B by having (possibly 0) additional edge labels. Let E_{i_j} be the edges where i_j appears in some $T \in \text{Bun}^{-1}(B)$ but not in B , that is, the set of edges of B where adding an i_j would yield an element of $\text{BallotGen}(v/\lambda)$. We say that B has a *virtual label* i_j on each edge of E_{i_j} . We denote a virtual label i_j by $\textcircled{i_j}$.

EXAMPLE 2.1. All virtual labels are depicted below:



□

For $B \in \text{Bundled}(v/\lambda)$, let

$$\text{wt}_{\text{Bundled}}(B) = \sum_{T \in \text{Bun}^{-1}(B)} \text{wt}(T). \quad (2.1)$$

For simplicity, we drop the subscript from $\text{wt}_{\text{Bundled}}$ when no confusion will arise. In particular, ‘wt’ means ‘ $\text{wt}_{\text{Bundled}}$ ’ in all of the remaining sections except Section 13. Let $B_{\lambda, \mu}^v$ denote the set of tableaux in $\text{Bundled}(v/\lambda)$ with content μ .

PROPOSITION 2.2. We have $L_{\lambda, \mu}^v = \sum_{B \in B_{\lambda, \mu}^v} \text{wt}_{\text{Bundled}}(B)$.

Proof. The proof is immediate from (2.1) and the definition of $L_{\lambda, \mu}^v$. \square

For a bundled tableau B , we compute $\widetilde{\text{wt}}(B)$ as a product: an edge label ℓ contributes a factor of $\text{edgefactor}(\ell)$ and each productive box x contributes a factor of $\text{boxfactor}(x)$. Each virtual label $\textcircled{\ell} \in \underline{x}$ contributes $1 - \text{edgefactor}_{\underline{x}}(\ell)$ (where the latter is calculated as if $\textcircled{\ell}$ were instead ℓ). We multiply by $(-1)^{d(T)}$, where $d(T) = \sum_{\mathcal{G}} (|\mathcal{G}| - 1)$, and here $|\mathcal{G}|$ is interpreted to be the multiset cardinality of nonvirtual \mathcal{G} in T .

EXAMPLE 2.3. For B from Example 2.1, $\widetilde{\text{wt}}(B) = (-1)^1 \cdot (1 - t_2/t_8) \cdot (t_2/t_4)(t_4/t_6)(t_6/t_9)(t_8/t_8)(t_{11}/t_{11}) \cdot (t_3/t_5)(t_4/t_9)(t_5/t_7)(t_8/t_{10})(t_9/t_{11})$. \square

LEMMA 2.4. For a bundled tableau B , $\text{wt}_{\text{Bundled}}(B) = \widetilde{\text{wt}}(B)$.

Proof. It should be noted that for any subset of the virtual labels of B , there is a unique $T \in \text{Bun}^{-1}(B)$ in which exactly those virtual labels are realized as (additional) edge labels. Let m be the number of virtual labels in B , and let a_i be the nonvirtual weight of the i th virtual label (listed in some given order). By the definition of the weights, the lemma follows from the ‘inclusion–exclusion’ identity $\prod_{i \in [m]} a_i = \sum_{S \subseteq [m]} (-1)^{|S|} \prod_{i \in S} (1 - a_i)$. \square

3. Structure of the proof of Theorem 1.3

Let $\lambda^+ := \{\rho \supseteq \lambda : \rho/\lambda \text{ has no two boxes in the same row or column}\}$, and let $\nu^- := \{\delta \subsetneq \nu : \nu/\delta \text{ has no two boxes in the same row or column}\}$. For a set D of boxes, let $\text{wt } D := \prod_{x \in D} (t_{\text{Man}(x)}/t_{\text{Man}(x)+1})$.

PROPOSITION 3.1 (Key recurrence).

$$\sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} K_{\rho, \mu}^v = K_{\lambda, \mu}^v (1 - \text{wt } \nu/\lambda) + \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} K_{\lambda, \mu}^\delta \text{wt } \delta/\lambda. \quad (3.1)$$

Proof. The Chevalley formula in equivariant K-theory [LePo07, Corollary 8.2] implies

$$[\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\square}] = [\mathcal{O}_{X_\lambda}](1 - \text{wt } \lambda) + \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} [\mathcal{O}_{X_\rho}] \text{wt } \lambda.$$

Thus, the coefficient of $[\mathcal{O}_{X_\nu}]$ in $([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\square}])[\mathcal{O}_{X_\mu}]$ is

$$K_{\lambda, \mu}^v (1 - \text{wt } \lambda) + \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} K_{\rho, \mu}^v \text{wt } \lambda.$$

On the other hand, the coefficient of $[\mathcal{O}_{X_\nu}]$ in $([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\mu}])[\mathcal{O}_{X_\square}]$ is

$$K_{\lambda, \mu}^v (1 - \text{wt } \nu) + \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} K_{\lambda, \mu}^\delta \text{wt } \delta.$$

The proposition then follows from associativity and commutativity:

$$([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\square}])[\mathcal{O}_{X_\mu}] = ([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\mu}])[\mathcal{O}_{X_\square}]. \quad \square$$

To prove $K_{\lambda, \mu}^v = L_{\lambda, \mu}^v$, we induct on $|\nu/\lambda|$. Proposition 4.1 is the base case: $K_{\lambda, \mu}^\lambda = L_{\lambda, \mu}^\lambda$; this is proved using the description of $L_{\lambda, \mu}^\lambda$ from Section 1.

The remaining cases use the description of $L_{\lambda, \mu}^v$ from Proposition 2.2. We assume that $K_{\theta, \mu}^\tau = L_{\theta, \mu}^\tau$ when $|\tau/\theta| \leq h$. Suppose that we are given λ, ν with $|\nu/\lambda| = h + 1$. We show that $L_{\lambda, \mu}^v$ satisfies (3.1). Since Proposition 3.1 asserts that $K_{\lambda, \mu}^v$ also satisfies (3.1), we will be done by induction.

We fix λ, μ and ν , with $\lambda \subsetneq \nu$. We define the formal sum

$$\Lambda^+ := \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} \sum_{T \in B_{\rho, \mu}^v} T.$$

We similarly define

$$\Lambda := (1 - \text{wt } \nu/\lambda) \sum_{T \in B_{\lambda, \mu}^v} T \quad \text{and} \quad \Lambda^- := \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} (\text{wt } \delta/\lambda) \sum_{T \in B_{\lambda, \mu}^\delta} T.$$

In Section 7.2, we define an operation $\text{slide}_{\rho/\lambda}$ that takes as input $T \in \Lambda^+$ and returns a formal sum of tableaux with coefficients from $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. The construction of $\text{slide}_{\rho/\lambda}$ and proof of its correctness are found

in Sections 5–7. Specifically, Corollary 7.11 shows that the tableaux in the formal sum are from $B_{\lambda,\mu}^v \cup (\bigcup_{\delta \in v^-} B_{\lambda,\mu}^\delta)$.

In Section 11, we prove that

$$\text{slide}(\Lambda^+) := \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} \sum_{T \in B_{\rho,\mu}^v} \text{slide}_{\rho/\lambda}(T) = \Lambda + \Lambda^-;$$

see Proposition 11.8 for the precise statement. Finally, Proposition 12.3 shows that $\text{wt } \Lambda^+ = \text{wt } \text{slide}(\Lambda^+)$, so $\sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} L_{\rho,\mu}^v = L_{\lambda,\mu}^v (1 - \text{wt } v/\lambda) + \sum_{\delta \in v^-} (-1)^{|v/\delta|+1} L_{\lambda,\mu}^\delta \text{wt } \delta/\lambda$. This completes the proof that the Laurent polynomials $L_{\lambda,\mu}^v$ defined by the rule of Proposition 2.2 equal $K_{\lambda,\mu}^v$. Hence, we have completed our proof of Theorem 1.3. □

4. The base case of the recurrence

A different rule for the case $K_{\lambda,\mu}^\lambda$ was given by Kreiman [Kr05]. We give an independent proof of the following.

PROPOSITION 4.1 (Base case of the recurrence). *We have $K_{\lambda,\mu}^\lambda = L_{\lambda,\mu}^\lambda$.*

Proof. We use the original (unbundled) definition of $L_{\lambda,\mu}^\lambda$ from Section 1.

One says that $\pi \in S_n$ is a *Grassmannian permutation* if there is at most one k such that $\pi(k) > \pi(k + 1)$. The Grassmannian permutation for $\lambda \subseteq k \times (n - k)$ is the (unique) Grassmannian permutation $\pi_\lambda \in S_n$ defined by $\pi_\lambda(i) = i + \lambda_{k-i+1}$ for $1 \leq i \leq k$ and $\pi(i) < \pi(i + 1)$ for $i \neq k$.

Let $w', v' \in S_n$ be the Grassmannian permutations for the conjugate diagrams $\lambda', \mu' \subseteq (n - k) \times k$. The following identity relates $K_{\lambda,\mu}^\lambda$ to the localization of the class $[\mathcal{O}_{X_\lambda}]$ at the \mathbb{T} -fixed point e_μ , expressed as a specialization of a double Grothendieck polynomial.

LEMMA 4.2. *We have $K_{\lambda,\mu}^\lambda = \overline{\mathfrak{G}_{v'}(t_{w'(1)}, \dots, t_{w'(n)}; t_1, \dots, t_n)}$, where $\overline{f(t_1, \dots, t_n)}$ is obtained by applying the substitution $t_j \mapsto t_{n-j+1}$ to $f(t_1, \dots, t_n)$.*

Proof. This lemma is known to experts, but for completeness we give details and references. Suppose that X_w is a Schubert variety in $\text{GL}_n(\mathbb{C})/\text{B}$. We have in $K_{\mathbb{T}}(\text{GL}_n(\mathbb{C})/\text{B})$,

$$[\mathcal{O}_{X_v}][\mathcal{O}_{X_w}] = K_{v,w}^w[\mathcal{O}_{X_w}] + \sum_{\theta \neq w} K_{v,w}^\theta[\mathcal{O}_{X_\theta}]. \tag{4.1}$$

It is known that $K_{v,w}^\theta = 0$ unless $w \leq \theta$ in Bruhat order. This follows, for instance, from the equivariant K -theory localization formula of Willems [Wi06] together with the *mutatis mutandis* modification of the proof of [KnTa03, Proposition 1].

Now, let $[\mathcal{O}_{X_v}]|_{e_w}$ denote the localization of the class $[\mathcal{O}_{X_v}]$ at the T -fixed point $e_w := wB/B$. Localization is a $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ -algebra homomorphism from $K_T(\mathrm{GL}_n(\mathbb{C})/B)$ to $K_T(e_w) \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Application of the localization map to (4.1) gives

$$[\mathcal{O}_{X_v}]|_{e_w} [\mathcal{O}_{X_w}]|_{e_w} = K_{v,w}^w [\mathcal{O}_{X_w}]|_{e_w}.$$

All terms in the summation vanish because $[\mathcal{O}_{X_\pi}]|_{e_\rho} = 0$ unless $\rho \leq \pi$ in Bruhat order. This vanishing condition appears in [Wi06] for generalized flag varieties; it also follows in the case at hand from, for example, the later work [WoYo12, Theorem 4.5] (see specifically the proof). For similar reasons, $[\mathcal{O}_{X_w}]|_{e_w} \neq 0$. Hence, dividing by this shows that $K_{v,w}^w = [\mathcal{O}_{X_v}]|_{e_w}$.

We consider the natural projection $\mathrm{GL}_n(\mathbb{C})/B \rightarrow X$. The pullback of each Schubert variety in X is a distinct Schubert variety in $\mathrm{GL}_n(\mathbb{C})/B$ (see, for example, [Br05, Example 1.2.3(6)]). Thus, the Schubert basis of X is sent into the Schubert basis of $\mathrm{GL}_n(\mathbb{C})/B$. Hence, we obtain an injection $K_T(X) \hookrightarrow K_T(\mathrm{GL}_n(\mathbb{C})/B)$. Thus, if $\lambda, \mu \subseteq k \times (n - k)$ and $w, v \in S_n$ are respectively their Grassmannian permutations, then $K_{\lambda,\mu}^\lambda = K_{w,v}^w$. The lemma now follows from [WoYo12, Theorem 4.5] (after chasing conventions). \square

Since v' is Grassmannian, by [KnMiYo09, Theorem 5.8], $\mathfrak{S}_{v'}(X; Y) = \sum_T \mathrm{SVSSYTwt}(T)$, where the sum is over all set-valued semistandard Young tableaux T of shape μ' with entries bounded above by $n - k$. Here, $\mathrm{SVSSYTwt}(T) = (-1)^{|L(T)| - |\mu'|} \prod_{\ell \in L(T)} (1 - x_\ell / y_{\ell + \mathrm{col}(x) - \mathrm{row}(x)})$, where $L(T)$ is the set of labels in T and x is the box containing ℓ .

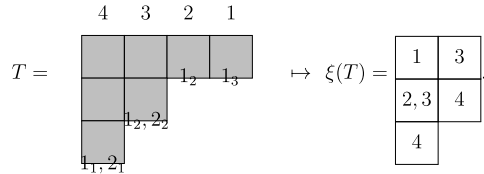
Let $\mathrm{SVSSYTeqwt}(T)$ be the result of the substitution $x_j \mapsto t_{w'(j)}$, $y_j \mapsto t_j$. We define \mathcal{A} to be the set of $T \in \mathrm{BallotGen}(\lambda/\lambda)$ that have content μ . We define \mathcal{B} to be the set of set-valued semistandard tableaux U of shape μ' where $\mathrm{SVSSYTeqwt}(U) \neq 0$.

LEMMA 4.3. *There is a bijection $\xi : \mathcal{A} \rightarrow \mathcal{B}$, with $\mathrm{wt}(T) = \mathrm{SVSSYTeqwt}(\xi(T))$ for all $T \in \mathcal{A}$.*

Proof. Index columns of $k \times (n - k)$ by $1, 2, \dots, n - k$ from right to left. To construct $\xi(T)$, begin with a Young diagram of shape μ' . For each label in T , we add a label to $\xi(T)$ as follows. If i_j appears in column c in T , place a label c in position $(\mu_i + 1 - j, i)$ in $\xi(T)$.

We have a candidate inverse map $\xi^{-1} : \mathcal{B} \rightarrow \mathcal{A}$. For each label c in (matrix) position (r, i) in $U \in \mathcal{B}$, we place an i_{μ_i+1-r} at the bottom of column c of λ/λ .

EXAMPLE 4.4. Let $n = 7, k = 3, \lambda = (4, 2, 1)$ and $\mu = (3, 2, 0)$. Then, T , together with the column labels 1, 2, 3, 4 and $\xi(T)$, is depicted below:



We compute that $\text{wt}(T) = (-1)^1(1-t_1/t_6)(1-t_3/t_6)(1-t_5/t_7)(1-t_6/t_7)(1-t_1/t_4)(1-t_3/t_4)$, where the first four factors correspond to the labels 1_j of T from left to right and the last two factors correspond to the labels 2_j of T from left to right. Now,

$$\begin{aligned} \text{SVSSYTwt}(\xi(T)) &= (-1)^1 \left(1 - \frac{x_4}{y_2}\right) \left(1 - \frac{x_3}{y_2}\right) \left(1 - \frac{x_2}{y_1}\right) \left(1 - \frac{x_1}{y_1}\right) \\ &\quad \times \left(1 - \frac{x_4}{y_4}\right) \left(1 - \frac{x_3}{y_4}\right), \end{aligned}$$

where the factors correspond to the entries of $\xi(T)$ as read up columns from left to right (that is, consistent with the order of factors of $\text{wt}(T)$ above).

Since $\lambda' = (3, 2, 1, 1)$, we have $w' = 2357146$ (one-line notation). Therefore, on substituting, we get

$$\begin{aligned} \text{SVSSYTEqwt}(\xi(T)) &= (-1)^1 \left(1 - \frac{t_7}{t_2}\right) \left(1 - \frac{t_5}{t_2}\right) \left(1 - \frac{t_3}{t_1}\right) \\ &\quad \times \left(1 - \frac{t_2}{t_1}\right) \left(1 - \frac{t_7}{t_4}\right) \left(1 - \frac{t_5}{t_4}\right). \end{aligned}$$

The reader can check $\overline{\text{SVSSYTEqwt}(\xi(T))} = \text{wt}(T)$, in agreement with the lemma. □

(The map ξ^{-1} is well-defined and is weight-preserving): Let $U \in \mathcal{B}$. That $\xi^{-1}(U)$ is an edge-labeled genomic tableau is immediate from the column strictness of U . Ballotness follows from the row increasingness of U .

We now check that no label of $\xi^{-1}(U)$ is too high. Suppose that c is a *bad* label in U in (matrix) position (r, i) ; that is, one such that the label i_{μ_i+1-r} placed in column c of λ/λ is too high. Observe that every label c' North of c and in the

same column of U is also bad; this is since c' corresponds to placing another label of family i in the weakly shorter column c' East of column c (since $c' < c$). Thus, we may assume that c is in the northmost row of U ; that is, $r = 1$. Now, if $i = 1$, then, since c is bad, it must be that $\lambda'_{n-k-c+1} = 0$, so $w'(c) = c + 0$. Now, c contributes a factor of $1 - x_c/y_c$ to $\text{SVSSYTwt}(U)$ and hence a factor of $1 - t_{c+0}/t_c = 0$ to $\text{SVSSYTEqwt}(U)$. That is, $\text{SVSSYTEqwt}(U) = 0$, so $U \notin \mathcal{B}$, which is a contradiction. Otherwise, we may also assume that $i > 1$ is the smallest such that a label in $(r = 1, i)$ is bad. Since no label c' in $(r = 1, i - 1)$ of U is bad, it must be that c is ‘barely’ bad; that is,

$$\lambda'_{n-k-c+1} = i - 1 \tag{4.2}$$

(column c is one box too short). However, c contributes a factor of $1 - x_c/y_{c+i-1}$ to $\text{SVSSYTwt}(U)$ and hence a factor of $1 - t_{c+\lambda'_{n-k-c+1}}/t_{c+i-1}$ to $\text{SVSSYTEqwt}(U)$. This latter factor is 0 precisely by (4.2). Hence, again, $U \notin \mathcal{B}$, which is a contradiction. Thus, U has no bad labels and thus no label of $\xi^{-1}(U)$ is too high, as desired.

The sign appearing in $\text{wt } \xi^{-1}(U)$ records the difference between $|\mu|$ and the number of labels in $\xi^{-1}(U)$, while the sign in $\overline{\text{SVSSYTEqwt}(U)}$ records the difference between $|\mu|$ and number of labels in U . Since the number of labels in U is clearly the same as the number of labels in $\xi^{-1}(U)$, these signs are equal.

We check that the weight assigned to a label c of U in position (r, i) is the same as the edgefactor assigned to the corresponding label i_{μ_i+1-r} at the bottom of column c in $\xi^{-1}(U)$. The label c is assigned the weight

$$\text{SSYTEqfactor}_{(r,i)}(c) := 1 - \frac{x_c}{y_{c+i-r}} = 1 - \frac{t_{c+\lambda'_{n-k+1-c}}}{t_{c+i-r}}.$$

Hence, we must show the equality of these two quantities:

$$\begin{aligned} \overline{\text{SSYTEqfactor}_{(r,i)}(c)} &= 1 - \frac{t_{n+1-c-\lambda'_{n-k+1-c}}}{t_{n+1-c+r-i}} \quad \text{and} \\ \text{edgefactor}_{\underline{x}}(i_{\mu_i+1-r}) &= 1 - \frac{t_{\text{Man}(\mathbf{x})}}{t_{\lambda'_{n-k+1-c}-i+r+\text{Man}(\mathbf{x})}}, \end{aligned}$$

where \underline{x} is the southern edge of λ in column c .

Now, counting the rows and columns separating \mathbf{x} from the southwest corner of $k \times (n - k)$, we have

$$\text{Man}(\mathbf{x}) = (n - k - c) + (k - \lambda'_{n-k+1-c} + 1) = n + 1 - c - \lambda'_{n-k+1-c}.$$

Thus, the numerators of the quotients of $\overline{\text{SSYTEqfactor}(c)}$ and $\text{edgefactor}(c)$ are equal. To see that the denominators are also equal,

we observe that

$$\begin{aligned} \text{Man}(\mathbf{x}) + \lambda'_{n-k+1-c} - i + r &= (n + 1 - c - \lambda'_{n-k+1-c}) + \lambda'_{n-k+1-c} - i + r \\ &= n + 1 - c - i + r. \end{aligned}$$

(The map ξ is well-defined and weight-preserving): Let $T \in \mathcal{A}$. We must show that $\xi(T)$ is strictly increasing along columns. This is clear since T satisfies (S.3) and (S.4).

Now, we show that $\xi(T)$ is weakly increasing along rows. Suppose that we have a in position (r, i) and b in position $(r, i + 1)$. This a comes from an i_{μ_i+1-r} in column a in T , while this b comes from an $(i + 1)_{\mu_{i+1}+1-r}$ in column b . By ballotness of T , each i_{μ_i+1-r} must be weakly right of each $(i + 1)_{\mu_{i+1}+1-r}$. Thus, $a \leq b$.

Hence, $\xi(T)$ is a set-valued semistandard tableau of shape μ' . The same computations that show that ξ^{-1} is weight-preserving show that $0 \neq \text{wt}(T) = \text{SSYT}_{\text{eqwt}}(\xi(T))$, and so the desired conclusions hold. □

The proposition now follows immediately from Lemmas 4.2 and 4.3. □

5. Good tableaux

In this section, we give an intrinsic description of the tableaux that will appear during our generalized *jeu de taquin* process (defined in Section 7). We call a label that appears in a box of a tableau a *box label*. Since we will use additional box labels $\bullet_{\mathcal{G}}$, we distinguish labels i_j as *genetic labels*. As a visual aid, we mark genetic labels \mathcal{F} southeast of a $\bullet_{\mathcal{G}}$ with $\mathcal{F} < \mathcal{G}$ as $\mathcal{F}^!$. For a gene \mathcal{G} , let \mathcal{G}^+ (respectively, \mathcal{G}^-) denote the successor (respectively, predecessor) of \mathcal{G} in the total order $<$ on genes. For example, $1_1^+ = 2_1$ if $\mu_1 = 1$, and $1_1^+ = 1_2$ if $\mu_1 > 1$. Let \mathcal{G}_{\max} be the maximum gene that can appear, namely $\ell(\mu)_{\mu_{\ell(\mu)}}$, where $\ell(\mu)$ is the number of nonzero rows of μ . Declare $\mathcal{G}_{\max}^+ := (\ell(\mu) + 1)_1$.

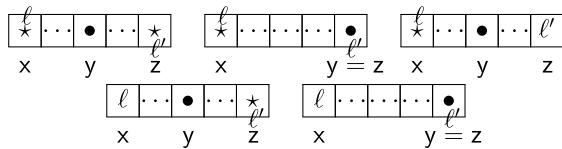
A \mathcal{G} -good tableau is an edge-labeled filling T of ν/λ by genetic labels i_j (such that $i \in \mathbb{Z}_{>0}$ and the j s that appear for each i form an initial segment of $\mathbb{Z}_{>0}$) and box labels $\bullet_{\mathcal{G}}$, satisfying the conditions (G.1)–(G.13).

(G.1) No genetic label is too high.

(G.2) No $\bullet_{\mathcal{G}}$ is southeast of another (in particular, $\bullet_{\mathcal{G}}$ s are in distinct rows and columns).

(G.3) The labels $<$ -increase along rows (ignoring any $\bullet_{\mathcal{G}}$ s), except for possibly three consecutive labels $\boxed{\mathcal{H}} \bullet_{\mathcal{G}} \boxed{\mathcal{F}^!}$ with $\mathcal{H} > \mathcal{F}$.

- (G.4) The labels $<$ -increase down columns (ignoring any \bullet_G s), except that unmarked \mathcal{F} may appear adjacent to and above $\mathcal{F}^!$ when both are box labels.
- (G.5) If i_j, k_ℓ appear on the same edge, then $i \neq k$.
- (G.6) If i_j is West of i_k , then $j \leq k$.
- (G.7) Each edge label is maximally west in its gene.
- (G.8) Each genotype G obtained by choosing one label of each gene of T is ballot in the sense defined in Section 1.3.
- (G.9) If \mathcal{F} appears northwest of \bullet_G , then $\mathcal{F} < \mathcal{G}$.
- (G.10) If $\mathcal{F}^! \in \underline{x}$ or $\mathcal{F}^! \in \underline{x}$, then \bullet_G appears in \underline{x} 's row.
- (G.11) \bullet_G does not appear in a column containing a marked label.
- (G.12) If ℓ and ℓ' are genetic labels of the same family with ℓ NorthWest of ℓ' , then there are boxes $\underline{x}, \underline{z}$ in row r with \underline{x} West of \underline{z} , $\ell \in \underline{x}$ or \bar{x} , and $\ell' \in \underline{z}$ or \bar{z} . Further, \bullet_G appears in some box \underline{y} of r that is East of \underline{x} and west of \underline{z} . Pictorially, the scenarios are



Furthermore, if $\underline{y} = \underline{z} = \underline{x}^{\rightarrow}$ in the last scenario, then $\underline{y}^{\rightarrow}$ does not contain a marked label or another instance of the gene of ℓ' .

We place a *virtual label* \mathcal{H} on each edge \underline{x} where $\mathcal{H} \in \underline{x}$ would

- (V.1) *not* be marked (hence if \mathcal{H} appears southeast of a \bullet_G , then $\mathcal{H} \geq \mathcal{G}$);
- (V.2) *not* be maximally west in its gene (hence violating condition (G.7)); and
- (V.3) satisfy the conditions (G.1), (G.4), (G.5), (G.6), (G.8), (G.9) and (G.12).
- (G.13) If $\mathcal{E}^! \in \underline{x}$ or $\mathcal{E}^! \in \underline{x}$, then there is \mathcal{F} or \mathcal{F} on \underline{x} with $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$.

A tableau is *good* if it is \mathcal{G} -good for some \mathcal{G} .

1_1	\bullet_{2_2}	1_2
2_1		2_2

EXAMPLE 5.1. The tableau is 2_2 -good. Although the labels in the second row do not increase left to right, they satisfy (G.3). Furthermore, it should be noticed that the 1_1 and 1_2 satisfy (G.12), as do the 2_1 and 2_2 .

	1_1	\bullet_{2_1}
\bullet_{2_1}	1_1	2_1

The tableau is also good. Although the label 1_1 appears twice in the same column, the lower instance is marked in accordance with (G.4). □

EXAMPLE 5.2. The following tableaux are *not* good:

	1_1	\bullet_{1_2}	1_2
2_1	2_1		

\bullet_{2_1}	1_1	3_1
2_1		

	\bullet_{2_1}	1_2
\bullet_{2_1}	1_1	2_1

1_1	1_2
\bullet_{1_2}	

The first fails conditions (G.1) and (G.7) because of the edge label 2_1 . The second fails (G.8), as the unique genotype is not ballot, as well as (G.1). Although the marked 1_1 in the third tableau has a label of family 2 on the lower edge of its box, the tableau fails (G.13) as $1 = N_{1_1} \neq N_{2_1} = 0$. It also fails (G.11) by having both a \bullet_{2_1} and a marked label in the second column. The fourth tableau fails (G.12). □

LEMMA 5.3. *If $T \in \text{Bundled}(v/\lambda)$, then T is \mathcal{G} -good for every \mathcal{G} . Moreover, the virtual labels of the \mathcal{G} -good tableau T (as defined by (V.1)–(V.3)) are the same as the virtual labels of the bundled tableau T (as defined in Section 2).*

Proof. Since T is bundled, (S.1), (S.2), (S.3) and (S.4) hold. These conditions respectively imply (G.3), (G.4), (G.5) and (G.6). The conditions (G.1), (G.7) and (G.8) are part of the definition of a bundled tableau. For (G.12), if ℓ is NorthWest of ℓ' and both are from the same family, (S.1) or (S.2) is violated. The remaining conditions are vacuous since T has no $\bullet_{\mathcal{G}}$ s. Hence, T is \mathcal{G} -good.

The claim about virtual labels is then clear from the definitions. □

We now collect various technical lemmas that we will need on the structure of good tableaux.

LEMMA 5.4 (Strong form of (G.10)). *We assume that T is \mathcal{G} -good. Let x be a box of T in row r .*

- (I) *If $\mathcal{F}^1 \in x$, then $\text{label}(x)$ is marked.*
- (II) *If $\mathcal{F}^1 \in x$, then there is a y West of x in r such that $\bullet_{\mathcal{G}} \in y$. Every box label of r between x and y is marked.*

Proof. (I): Since $\mathcal{F}^1 \in \underline{x}$, \underline{x} (and hence also x) is southeast of a \bullet_G . By (G.11), $\bullet_G \notin x$. Hence, some $\mathcal{E} \in x$. By (G.4), $\mathcal{E} < \mathcal{F}$. Therefore, the $\mathcal{E} \in x$ is marked.

(II): Since $\mathcal{F}^1 \in x$, there is a \bullet_G northwest of x . By (G.10), there is a \bullet_G in x 's row. If this latter \bullet_G is East of x , these two \bullet_G s are distinct and violate (G.2). Hence, the \bullet_G in x 's row is in some box y West of x . If \mathcal{E} is a box label between x and y (and in the same row), it is southeast of the $\text{label}(y) = \bullet_G$. By (G.3), $\mathcal{E} < \mathcal{F}$. Hence, this \mathcal{E} is also marked. □

LEMMA 5.5 (Strong form of (G.13)). *Let T be \mathcal{G} -good. Suppose that $\mathcal{E}^1 \in x$ or $\mathcal{E}^1 \in \underline{x}$ with $\text{family}(\mathcal{G}) - \text{family}(\mathcal{E}) = k > 0$. For each $0 < h < k$, there is $\mathcal{H}^1 \in \underline{x}$ with $N_{\mathcal{H}} = N_{\mathcal{E}}$ and $\text{family}(\mathcal{H}) = \text{family}(\mathcal{E}) + h$. Moreover, there is a \mathcal{G}' or $\mathcal{G}' \in \underline{x}$ with $N_{\mathcal{G}'} = N_{\mathcal{E}}$ and $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$.*

Proof. This follows by repeated application of (G.13). It should be noted that none of the \mathcal{H} s of the statement can be virtual since they must be marked. □

LEMMA 5.6. *If $\mathcal{E} < \mathcal{F}$ are genes of a good tableau T with $N_{\mathcal{E}} = N_{\mathcal{F}}$, then no \mathcal{F} or \mathcal{F} is East of any \mathcal{E} .*

Proof. First suppose that some \mathcal{F} is East of some \mathcal{E} . Let G be a genotype of T with $\mathcal{F} \in G$ that is East of some $\mathcal{E} \in G$. Then, \mathcal{F} appears before \mathcal{E} in $\text{word}(G)$. By (G.6), the initial segment W of $\text{word}(G)$ ending at \mathcal{F} contains $N_{\mathcal{F}} + 1$ labels of $\text{family}(\mathcal{F})$ and at most $N_{\mathcal{E}}$ labels of $\text{family}(\mathcal{E})$. Thus, T 's (G.8) is violated for some $\text{family}(\mathcal{E}) \leq i < \text{family}(\mathcal{F})$, which is a contradiction. Finally, if some \mathcal{F} is East of some \mathcal{E} , then, by (V.3), the tableau T' obtained by replacing that \mathcal{F} by \mathcal{F} satisfies (G.6) and (G.8). Now, we derive the same contradiction as before, using T' in place of T . □

LEMMA 5.7. *If \mathcal{E}^1 appears in a good tableau T , then it is maximally west in its gene.*

Proof. Suppose that $\mathcal{E}^1 \in x$ or $\mathcal{E}^1 \in \underline{x}$. By (G.13), there is an \mathcal{F} or $\mathcal{F} \in \underline{x}$ with $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\mathcal{E} < \mathcal{F}$. Thus, we are done by Lemma 5.6. □

LEMMA 5.8. *Suppose that column c of good tableau T contains labels \mathcal{H} and \mathcal{J} with $\mathcal{H} < \mathcal{J}$ and $N_{\mathcal{H}} = N_{\mathcal{J}}$. Then, for every i such that $\text{family}(\mathcal{H}) < i < \text{family}(\mathcal{J})$, there is a label \mathcal{I} of family i in column c such that $N_{\mathcal{H}} = N_{\mathcal{I}}$.*

Proof. Suppose the contrary. By (G.8), there is some $\mathcal{I} \in T$ of family i such that $N_{\mathcal{H}} = N_{\mathcal{J}} = N_{\mathcal{I}}$. If this \mathcal{I} is not in column c , we contradict Lemma 5.6. □

LEMMA 5.9. *Suppose that \mathcal{E} and \mathcal{F} satisfy $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$. Let T be a \mathcal{G} -good tableau with $(\mathcal{F}) \in \underline{x}$ and either $\mathcal{E}^! \in \underline{x}$ or $\mathcal{E}^! \in \underline{x}$. Then, $\bullet_{\mathcal{G}} \in \underline{x}^{\leftarrow}$ and either $\text{family}(\mathcal{G}) = \text{family}(\mathcal{F})$ or $\text{family}(\mathcal{G}) = \text{family}(\mathcal{E})$.*

Proof. If $\bullet_{\mathcal{G}} \notin \underline{x}^{\leftarrow}$, then, by Lemma 5.4, $\mathcal{D}^! \in \underline{x}^{\leftarrow}$. By (G.3) and (G.4), $\mathcal{D} \prec \mathcal{E}$. Moreover, $\mathcal{E} \prec \mathcal{G}$ since $\mathcal{E}^! \in T$. Thus, by (G.6) and Lemma 5.5, there is a $\tilde{\mathcal{E}}^! \in \underline{x}^{\leftarrow}$ or $\tilde{\mathcal{E}}^! \in \underline{x}^{\leftarrow}$ with $\text{family}(\mathcal{E}) = \text{family}(\tilde{\mathcal{E}})$ and $N_{\tilde{\mathcal{E}}} = N_{\mathcal{D}}$. By (G.13), there is $\tilde{\mathcal{F}}$ or $(\tilde{\mathcal{F}}) \in \underline{x}^{\leftarrow}$ with $\text{family}(\mathcal{F}) = \text{family}(\tilde{\mathcal{F}})$ and $N_{\tilde{\mathcal{E}}} = N_{\tilde{\mathcal{F}}}$. Thus, by Lemma 5.6, $\mathcal{F} \neq \tilde{\mathcal{F}}$, contradicting $(\mathcal{F}) \in \underline{x}$. Hence, $\bullet_{\mathcal{G}} \in \underline{x}^{\leftarrow}$.

If $\text{family}(\mathcal{G}) = \text{family}(\mathcal{E})$, we are done. Otherwise, $\mathcal{G} > \mathcal{E}$, and so $\text{family}(\mathcal{G}) = \text{family}(\mathcal{F})$ follows from Lemma 5.5. □

LEMMA 5.10. *If T is \mathcal{G} -good, then no \mathcal{H} is southEast of another.*

Proof. If some \mathcal{H} is SouthEast of another \mathcal{H} , by (G.12), there is a $\bullet_{\mathcal{G}}$ in between the two \mathcal{H} s. If two \mathcal{H} s are box labels of the same row, then, by (G.3), we reach the same conclusion that there is a $\bullet_{\mathcal{G}}$ in between the two \mathcal{H} s. In either case, since this $\bullet_{\mathcal{G}}$ is southeast of the western \mathcal{H} , we have $\mathcal{H} \prec \mathcal{G}$ by (G.9). Since this $\bullet_{\mathcal{G}}$ is northwest of the eastern \mathcal{H} , this eastern \mathcal{H} is marked. This contradicts Lemma 5.7. Finally, suppose that two \mathcal{H} s are edge labels on the bottom of the same row. This contradicts (G.7). □

LEMMA 5.11. *Let T be a \mathcal{G} -good tableau. Suppose that $\text{family}(\mathcal{F}) \leq \text{family}(\mathcal{G})$, $\bullet_{\mathcal{G}} \in \underline{y}$ and $\mathcal{F} \in \underline{z}$ or \underline{z} . Then, \underline{z} is not SouthEast of \underline{y} .*

Proof. Suppose that \underline{z} is SouthEast of \underline{y} . First, assume that $\mathcal{F} < \mathcal{G}$. Consider the box \mathbf{a} that is in \underline{y} 's column and \underline{z} 's row. By Lemma 5.4, either \mathbf{a} contains a marked label (contradicting (G.11)) or $\bullet_{\mathcal{G}} \in \mathbf{a}$ Southeast of \underline{y} (contradicting (G.2)).

Now, assume that $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$. (We do not assume that $\mathcal{F} \leq \mathcal{G}$.) Consider the box \mathbf{b} of T that is in \underline{y} 's row and \underline{z} 's column. By (G.2), \mathbf{b} contains a genetic label. By (G.4), $\text{label}(\mathbf{b}) < \mathcal{F}$. Hence, $\text{label}(\mathbf{b})$ is marked in T . By Lemma 5.5, \mathbf{b} then contains a (possibly virtual) label of the same family as \mathcal{F} and \mathcal{G} . This contradicts (G.4). □

LEMMA 5.12. *Let U be a \mathcal{G}^+ -good tableau. Suppose that $\bullet_{\mathcal{G}^+} \in \underline{x}$ and that either $\mathcal{G} \in \underline{y}$ or $\mathcal{G} \in \underline{y}$. Then, \underline{y} is not NorthWest of \underline{x} .*

Proof. Suppose otherwise. Consider the box \mathbf{b} that is in \mathbf{y} 's column and \mathbf{x} 's row. By (G.2), it contains a genetic label. By (G.4), either $\text{label}(\mathbf{b}) > \mathcal{G}$ or else $\mathcal{G}' \in \mathbf{b}$. If $\mathcal{G}' \in \mathbf{b}$, then \mathbf{b} is southeast of a $\bullet_{\mathcal{G}^+}$ by definition. This contradicts (G.2). If $\mathcal{G} < \text{label}(\mathbf{b})$, we contradict (G.9). \square

LEMMA 5.13. *Let c be a column of a \mathcal{G} -good tableau T . Suppose that $\bullet_{\mathcal{G}} \in c$ and either $\mathcal{G} \in c$ or $\mathcal{G} \in c$. Further suppose that $\mathcal{E}^1 \in \mathbf{y}$, where \mathbf{y} is a box of column c^{\rightarrow} . Then, $\mathcal{G} \in \underline{\mathbf{y}}$.*

Proof. Since \mathcal{E}^1 appears in T , $\mathcal{E} < \mathcal{G}$. Since \mathcal{E} appears East of some \mathcal{G} , by (G.6), this implies $\mathcal{E} < \mathcal{G}$.

Hence, by Lemma 5.5, there is either $\mathcal{G}' \in \underline{\mathbf{y}}$ or $\mathcal{G}' \in \underline{\mathbf{y}}$ with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$. It remains to show $\mathcal{G}' = \mathcal{G}$, for then, by (G.7), $\mathcal{G} \in \underline{\mathbf{y}}$.

Suppose that $\mathcal{G}' \neq \mathcal{G}$. Then, by (G.4), (G.5) and (G.6), $\mathcal{G}' = \mathcal{G}^+$. By Lemma 5.5, $N_{\mathcal{E}} = N_{\mathcal{G}^+}$; thus, $\text{family}(\mathcal{E}^-) = \text{family}(\mathcal{E})$ by (G.8). Moreover, by (G.8), every instance of \mathcal{E}^- must be read before any \mathcal{G} or \mathcal{G} . By (G.4), $\mathcal{E}^- \notin c^{\rightarrow}$. By (G.6), \mathcal{E}^- does not appear East of c^{\rightarrow} . However, by assumption, either $\mathcal{G} \in c$ or $\mathcal{G} \in c$, so \mathcal{E}^- must appear in c .

Consider the box \mathbf{y}^{\leftarrow} . By Lemma 5.4, either $\bullet_{\mathcal{G}} \in \mathbf{y}^{\leftarrow}$ or some $\mathcal{D}^1 \in \mathbf{y}^{\leftarrow}$. The latter is impossible by (G.11), since $\bullet_{\mathcal{G}} \in c$. Hence, $\bullet_{\mathcal{G}} \in \mathbf{y}^{\leftarrow}$.

Now, \mathcal{E}^- cannot appear South of \mathbf{y}^{\leftarrow} in c , for then it would be marked, in violation of (G.11). We have $\mathcal{E}^- \notin \mathbf{y}^{\leftarrow}$, since $\bullet_{\mathcal{G}} \in \mathbf{y}^{\leftarrow}$. By (G.12), \mathcal{E}^- cannot appear North of \mathbf{y}^{\leftarrow} in c . This contradicts that \mathcal{E}^- must appear in c , and therefore the assumption $\mathcal{G}' \neq \mathcal{G}$. \square

6. Snakes of good tableaux

In this section, we give structural results about certain subsets of a good tableau; these will play a critical role in the definition of our generalized *jeu de taquin* (given in Section 7).

6.1. Snakes. Let T be a \mathcal{G} -good tableau. Let $\mathcal{G} = g_k$, and consider the set of boxes in T that contain either $\bullet_{\mathcal{G}}$ or \mathcal{G} . This set decomposes into edge-connected components R that we call *presnakes*. A *short ribbon* is a connected skew shape without a 2×2 subshape and where each row and column contains at most two boxes.

LEMMA 6.1. Each presnake R is a short ribbon. Any row of R with two boxes is $\begin{bmatrix} \bullet_{\mathcal{G}} & \mathcal{G} \end{bmatrix}$. Any column of R with two boxes is $\begin{bmatrix} \bullet_{\mathcal{G}} \\ \mathcal{G} \end{bmatrix}$.

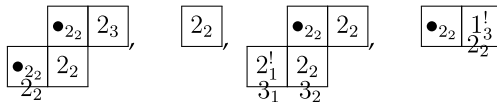
Proof. Since T is \mathcal{G} -good, there is no $\mathcal{G}^!$. Therefore, any column of R has at most one \mathcal{G} by (G.4) and at most one $\bullet_{\mathcal{G}}$ by (G.2). Hence, any column of R has at most two boxes. By (G.9), if $\bullet_{\mathcal{G}}$ and \mathcal{G} are in the same column, the $\bullet_{\mathcal{G}}$ is to the north. The description of rows of R holds by (G.2), (G.3) and (G.9). That R is a skew shape with no 2×2 subshape then follows immediately. \square

A snake S is a presnake R extended by (R.1)–(R.3).

- (R.1) If the box immediately right of the northmost $\bullet_{\mathcal{G}}$ in R contains \mathcal{G}^+ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$, then adjoin this box to R .
- (R.2) If the box immediately left of the southmost \mathcal{G} in R contains a marked label, adjoin this box to R .
- (R.3) If x in the northmost row of R contains $\bullet_{\mathcal{G}}$, $\text{label}(x^{\rightarrow})$ is marked and either \mathcal{G} or $(\mathcal{G}) \in x^{\rightarrow}$, then adjoin x^{\rightarrow} to R .

The entries of S are its box labels and labels appearing on the bottom edges of its boxes.

EXAMPLE 6.2. Below are snakes for $\mathcal{G} = 2_2$:



On the other hand, $\begin{bmatrix} \bullet_{2_2} & 3_1 \\ \bullet_{2_2} & 2_2 \end{bmatrix}$ is *not* a snake, even if $3_1 = 2_2^+$ ((R.1) does not apply). \square

EXAMPLE 6.3 (Snakes can share a row). Thus, $\begin{bmatrix} \bullet_{2_2} & 1_{1,3}^! & 2_2 \\ \bullet_{2_2} & 2_2 & 2_2 \end{bmatrix}$ contains two snakes, as colored. \square

EXAMPLE 6.4 (Snakes can share a column). Thus, $\begin{bmatrix} \bullet_{1_1} & 1_2 \\ \bullet_{1_1} & 1_2 \\ 1_1 \end{bmatrix}$ has two snakes, as colored. \square

LEMMA 6.5. *Every snake S is a short ribbon.*

Proof. A snake S is built by adjoining boxes to a presnake R . By Lemma 6.1, R is a short ribbon. In view of Lemma 6.1, (R.1) and (R.3) only apply if the northmost row of R is a single box with $\bullet_{\mathcal{G}}$. Therefore, adjoining a box to the right maintains shortness. Similarly, (R.2) maintains shortness. \square

LEMMA 6.6 (Disjointness and relative positioning of snakes). *Suppose that S and S' are snakes obtained from distinct presnakes R and R' , respectively. Up to relabeling of the snakes, one of the following holds.*

- (I) *The snake S is entirely SouthWest of the S' (that is, if \mathbf{b}, \mathbf{b}' are respectively boxes of these snakes, then \mathbf{b} is SouthWest of \mathbf{b}').*
- (II) *The snake S consists of a single box containing $\bullet_{\mathcal{G}}$ with neither \mathcal{G} nor $\textcircled{\mathcal{G}}$ on its lower edge; further, this box appears West of and in the same row as the southmost row of S' , and all intervening box labels are marked; see Example 6.3.*
- (III) *The snake S involves an (R.1) extension, adjoining a \mathcal{G}^+ in some box \mathbf{w} , while $S' = \{\bullet_{\mathcal{G}} \in \mathbf{w}^\uparrow\}$ or $S' = \{\bullet_{\mathcal{G}} \in \mathbf{w}^\uparrow, \mathcal{G}^+ \in \mathbf{w}^{\uparrow\rightarrow}\}$; see Example 6.4.*

In particular, S and S' are box disjoint.

Proof. By Lemma 6.1, (G.2) and/or (G.4), R and R' share at most one row and do not share a column. Moreover, one sees that R is southWest of R' (say). By (R.1)–(R.3), S and S' share a row if and only if R and R' do.

Case 1 (R and R' share a row r): The northmost row of R and the southmost row of R' are in row r . We must show that (II) holds and that S, S' are box disjoint.

By (G.2), (G.9) and Lemma 5.10, R has in row r only a $\bullet_{\mathcal{G}} \in \mathbf{x}$, while R' has in r only $\mathcal{G} \in \mathbf{y}$. Since $R \neq R', \mathbf{y} \neq \mathbf{x}^\rightarrow$. By (G.3), $\text{label}(\mathbf{y}^\leftarrow) < \mathcal{G}$, so we have some marked label $\mathcal{F}^\leftarrow \in \mathbf{y}^\leftarrow$. Therefore, R' extends to S' by (R.2).

CLAIM 6.7. *No \mathcal{G} or $\textcircled{\mathcal{G}}$ appears in columns west of \mathbf{y}^\leftarrow .*

Proof. Since $\mathcal{F} < \mathcal{G}$, we are done by (G.4) and (G.6) if $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$. Thus, we assume that $\mathcal{F} < \mathcal{G}$. By Lemma 5.5, there is either $\mathcal{G}' \in \mathbf{y}^\leftarrow$ or $\textcircled{\mathcal{G}'} \in \mathbf{y}^\leftarrow$ such that $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$ and $N_{\mathcal{F}} = N_{\mathcal{G}'}$. By (G.6), $\mathcal{G}' \preceq \mathcal{G}$ because $\mathcal{G} \in \mathbf{y}$. If $\mathcal{G}' = \mathcal{G}$, then since $N_{\mathcal{F}} = N_{\mathcal{G}'(=\mathcal{G})}$, the $\mathcal{G} \in \mathbf{y}$ and $\mathcal{F}^\leftarrow \in \mathbf{y}^\leftarrow$ combine to contradict Lemma 5.6. Thus, $\mathcal{G}' < \mathcal{G}$, and we are done by (G.6) and (G.4). \square

By Claim 6.7, $R = \{\bullet_{\mathcal{G}} \in \underline{x}\}$ without \mathcal{G} or $\mathcal{G} \in \underline{x}$. We observe that R cannot extend to S by (R.1), since (R.1) requires $\mathcal{G}^+ \in \underline{x}^{\rightarrow}$, which contradicts (G.3) in view of $\mathcal{G} \in \underline{y}$. It cannot be extended by (R.2) since $\mathcal{G} \notin \underline{x}$. If R were extended by (R.3), there would be a \mathcal{G} or \mathcal{G} in $\underline{x}^{\rightarrow}$, in violation of Claim 6.7. Thus, $R = S = \{\underline{x}\}$.

By Lemma 5.4(II), all labels strictly between \underline{x} and \underline{y} are marked. Hence, (II) holds. Since $\underline{y}^{\leftarrow} \notin S$, we see, by (R.1)–(R.3), that S and S' are box disjoint.

Case 2 (R and R' do not share a row): We may assume that S and S' share a column, for if they do not, then clearly (I) and box disjointness both hold. Since R and R' do not share a column, S and S' can only share a column if R is extended East by (R.1) or (R.3), or if R' is extended West by (R.2). Let \underline{x} be the northeastmost box of R , and let \underline{y} be the southwestmost box of R' .

Subcase 2.1 (R is extended by (R.1)): Since $\text{label}(\underline{x}^{\rightarrow}) = \mathcal{G}^+$ and $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$, by (G.6), R' cannot contain any \mathcal{G} s and therefore $R' = \{\bullet_{\mathcal{G}} \in \underline{y}\}$. Hence, (R.2) does not extend R' . By assumption, $\underline{x}^{\rightarrow}$ and \underline{y} are in the same column. Hence, by (G.4) and (G.11), $\underline{y} = \underline{x}^{\rightarrow\uparrow}$. By (G.6), R' is not extended by (R.3), since $\mathcal{G}^+ \in \underline{x}^{\rightarrow}$ and (R.3) requires $\mathcal{G} \in \underline{y}^{\rightarrow}$ or $\mathcal{G} \in \underline{y}^{\rightarrow}$. If R' is extended by (R.1), we obtain the second scenario described by (III) (and S, S' are box disjoint). If R' is not extended by any of (R.1)–(R.3), then we have the first scenario described by (III) (and S, S' are box disjoint).

Subcase 2.2 (R is extended by (R.3)): Let c be $\underline{x}^{\rightarrow}$'s column. We have $\mathcal{F}^{\uparrow} \in \underline{x}^{\rightarrow}$ and either $\mathcal{G} \in \underline{x}^{\rightarrow}$ or $\mathcal{G} \in \underline{x}^{\rightarrow}$. Moreover, $N_{\mathcal{F}} = N_{\mathcal{G}}$. Hence, by Lemma 5.6, no \mathcal{G} appears East of c . Thus, $R' = \{\bullet_{\mathcal{G}} \in \underline{y}\}$. By (G.11), $\underline{y} \notin c$. Thus, S and R' do not share a column. Since $\bullet_{\mathcal{G}} \in \underline{y}$, R' is not extended by (R.2). Thus S and S' do not share a column, contradicting our assumption.

Subcase 2.3 (R' is extended by (R.2); R is not extended by either (R.1) or (R.3)): Here, $\mathcal{G} \in \underline{y}$ and $\mathcal{F}^{\uparrow} \in \underline{y}^{\leftarrow}$. By Lemma 5.5, either $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$ or else we have $\mathcal{G}' \in \underline{y}^{\leftarrow}$ or $\mathcal{G}' \in \underline{y}^{\leftarrow}$, such that $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$. Hence, by (G.4) and (G.11), R cannot contain a box in the column of $\underline{y}^{\leftarrow}$. Hence, R and S' do not share a column. Hence, by the assumption of the subcase, S and S' do not share a column. □

6.2. Snake sections. We decompose each snake S into three *snake sections* denoted $\text{head}(S)$, $\text{body}(S)$ and $\text{tail}(S)$ as follows.

DEFINITION-LEMMA 6.8. (I) *If a snake S has at least two rows and its southmost row has two boxes, then $\text{head}(S)$ is the southmost row of S , $\text{tail}(S)$ is the northmost row and $\text{body}(S)$ is the remaining rows.*

- (II) If a snake S has at least two rows and its southmost row has exactly one box, then $\text{head}(S)$ is empty, $\text{tail}(S)$ is the northmost row and $\text{body}(S)$ is the other rows.
- (III) If S has exactly one row, then S is one of the following (edge labels not depicted):

$$\begin{aligned}
 & \text{(i) } S = \boxed{\mathcal{G}} = \text{body}(S); & \text{(ii) } S = \boxed{\bullet_{\mathcal{G}}} = \text{head}(S); \\
 & \text{(iii) } S = \boxed{\bullet_{\mathcal{G}} \mathcal{G}} = \text{head}(S); \\
 & \text{(iv) } R = \boxed{\bullet_{\mathcal{G}} \mathcal{G}^+} = \text{head}(S); & \text{(v) } S = \boxed{\mathcal{F}^1 \mathcal{G}} = \text{head}(S); \\
 & \text{(vi) } S = \boxed{\bullet_{\mathcal{G}} \mathcal{F}^1} = \text{tail}(S) \text{ (with } \mathcal{G} \text{ or } \mathcal{G} \text{ on the lower right edge).}
 \end{aligned}$$

Proof. It is only required to verify that in (III) all possible one-row snakes are shown. This is done by combining Lemma 6.1 and (R.1)–(R.3). □

LEMMA 6.9 (Properties of head, body, tail). (I) If $\text{head}(S) = \{x\}$, then $\bullet_{\mathcal{G}} \in x$.

(II) If $\text{head}(S) = \{x, x^{\rightarrow}\}$, then $\text{head}(S) = \boxed{\mathcal{F}^1 \mathcal{G}}, \boxed{\bullet_{\mathcal{G}} \mathcal{G}}$ or $\boxed{\bullet_{\mathcal{G}} \mathcal{G}^+}$.

(III) $\text{body}(S)$ is a short ribbon consisting only of $\bullet_{\mathcal{G}}s$ and $\mathcal{G}s$ (with no edge label $\mathcal{G}s$ or $\mathcal{G}s$).

(IV) If $\text{tail}(S) = \{x\}$, then $\text{tail}(S) = \boxed{\bullet_{\mathcal{G}}}$, and S has at least two rows.

(V) If $\text{tail}(S) = \{x, x^{\rightarrow}\} = \boxed{\bullet_{\mathcal{G}} \mathcal{G}}$ or $\boxed{\bullet_{\mathcal{G}} \mathcal{G}^+}$, then S has at least two rows, $\mathcal{G} \notin x$ and $\mathcal{G} \notin x$.

(VI) If $\text{tail}(S) = \{x, x^{\rightarrow}\}$ and \mathcal{G} or $\mathcal{G} \in x^{\rightarrow}$, then $\text{tail}(S) = \boxed{\bullet_{\mathcal{G}} \mathcal{F}^1}$ or $\boxed{\bullet_{\mathcal{G}} \mathcal{G}}$.

(VII) If S has at least two rows, then $\mathcal{G} \in x^{\downarrow}$, where x is the westmost box of $\text{tail}(S)$.

Proof. If S has one row, then, by Definition-Lemma 6.8(III), these claims are clear (or irrelevant). Thus, we assume that S has at least two rows.

(I): Under the assumption that S has at least two rows, the claim is vacuous since, by Definition-Lemma 6.8(I,II), we know that $|\text{head}(S)| \neq 1$.

- (II): Either the southmost row of S is $\boxed{\mathcal{F}^{-1} \mathcal{G}}$ if (R.2) was used, or it is $\boxed{\bullet \mathcal{G}}$ if (R.2) was not used; see Lemma 6.1.
- (III): That $\text{body}(S)$ is a short ribbon is clear, since S is a short ribbon by Lemma 6.5. Boxes of $\text{body}(S)$ only contain \mathcal{G} or $\bullet_{\mathcal{G}}$ because (R.1)–(R.3) adjoin boxes only to the northmost or southmost row (and if the southmost row of S has two boxes, then, by definition, that row is not part of $\text{body}(S)$). By (G.12), an edge label \mathcal{G} or \mathcal{G} can only appear in the northmost or southmost row of S . In those cases, the row is not part of $\text{body}(S)$, by Definition-Lemma 6.8(I,II).
- (IV): $\text{tail}(S)$ is the northmost row of S and, since $|\text{tail}(S)| = 1$, it is the northmost row of the presnake of S . Thus, we are done, by Lemma 6.1.
- (V): $\text{tail}(S)$ is the northmost row and, by Lemma 6.1, $\mathcal{G} \in x^\downarrow$ (x^\downarrow is in the presnake of S), so $\mathcal{G}, \mathcal{G} \notin x$ by (G.4).
- (VI): x is in the presnake of S and so, by Lemma 6.1, $\bullet_{\mathcal{G}} \in x$. By (G.2), $\bullet_{\mathcal{G}} \notin x^\rightarrow$. By (G.4), $\text{label}(x^\rightarrow) < \mathcal{G}$, and so $\text{label}(x^\rightarrow)$ is marked, since it is southeast of the $\bullet_{\mathcal{G}} \in x$.
- (VII): x and x^\downarrow are part of the presnake of S . Now, we apply Lemma 6.1. □

7. Genomic jeu de taquin

7.1. Miniswaps. We first define *miniswaps* on snake sections of a \mathcal{G} -good tableau. The output is a formal sum of tableaux. Below, we interpret $\bullet = \bullet_{\mathcal{G}}$ before the miniswap and $\bullet = \bullet_{\mathcal{G}^+}$ after the miniswap. We depict \mathcal{G} whenever it exists. Labels and virtual labels from other genes are not depicted unless they are relevant to the miniswap’s definition. For a box x , we define

$$\beta(x) := 1 - \frac{t_{\text{Man}(x)}}{t_{\text{Man}(x)+1}} \quad \text{and} \quad \hat{\beta}(x) := 1 - \beta(x) = \frac{t_{\text{Man}(x)}}{t_{\text{Man}(x)+1}}.$$

It should be noted that if $x = \theta/\kappa$, then $\hat{\beta}(x) = \text{wt } \theta/\kappa$, as defined in Section 3. If a snake section is empty, then *mswap* acts trivially, so below we assume otherwise.

7.1.1. Miniswaps on $\text{head}(S)$.

Case H1 ($\text{head}(S) = \{x\}$ and $\mathcal{G} \in x$):

$$\text{head}(S) = \boxed{\begin{matrix} \bullet \\ \mathcal{G} \end{matrix}} \mapsto \text{mswap}(\text{head}(S)) = \beta(x) \cdot \boxed{\mathcal{G}} + \gamma \cdot \boxed{\begin{matrix} \mathcal{G} \\ \bullet \end{matrix}}$$

Set $\gamma := 0$ if $\text{row}(x) = \text{family}(\mathcal{G})$ (that is, if $\mathcal{G} \in \bar{x}$ would be too high); otherwise set $\gamma := 1$.

Case H2 ($\text{head}(S) = \{x\}$ and $(\mathcal{G}) \in \underline{x}$):

$$\text{head}(S) = \begin{array}{|c|} \bullet \\ \mathcal{G} \end{array} \mapsto \text{mswap}(\text{head}(S)) = \begin{array}{|c|} \bullet \\ \square \end{array} + \beta(x) \cdot \begin{array}{|c|} \mathcal{G} \end{array}$$

Case H3 ($\text{head}(S) = \{x\}$ and Cases H1/H2 do not apply):

$$\text{head}(S) = \begin{array}{|c|} \bullet \\ \square \end{array} \mapsto \text{mswap}(\text{head}(S)) = \begin{array}{|c|} \bullet \\ \square \end{array}$$

Case H4 ($\text{head}(S) = \{x, x^\rightarrow\}$, $\mathcal{G} \in x^\rightarrow$, and $\mathcal{G} \in \underline{x}$):

$$\text{head}(S) = \begin{array}{|c|} \bullet \mathcal{G} \\ \mathcal{G} \end{array} \mapsto \text{mswap}(\text{head}(S)) = 0$$

Case H5 ($\text{head}(S) = \{x, x^\rightarrow\}$, $\mathcal{G} \in x^\rightarrow$, and $\mathcal{G} \notin \underline{x}$):

Subcase H5.1 ($\mathcal{H} \in \underline{x}^\rightarrow$, $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\text{head}(S) = \begin{array}{|c|} \bullet \mathcal{G} \\ \mathcal{H} \end{array} \mapsto \text{mswap}(\text{head}(S)) = \begin{array}{|c|} \bullet \mathcal{G} \\ \mathcal{H} \end{array}$$

Subcase H5.2 ($(\mathcal{H}) \in \underline{x}^\rightarrow$, $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\text{head}(S) = \begin{array}{|c|} \bullet \mathcal{G} \\ \mathcal{H} \end{array} \mapsto \text{mswap}(\text{head}(S)) = \begin{array}{|c|} \bullet \mathcal{G} \\ \mathcal{H} \end{array} + \hat{\beta}(x) \cdot \begin{array}{|c|} \mathcal{G} \end{array}$$

Subcase H5.3 (Subcases H5.1/H5.2 do not apply):

$$\text{head}(S) = \begin{array}{|c|} \bullet \mathcal{G} \\ \mathcal{G} \end{array} \text{ or } \begin{array}{|c|} \bullet \mathcal{G} \\ \square \end{array} \mapsto \text{mswap}(\text{head}(S)) = \hat{\beta}(x) \cdot \begin{array}{|c|} \mathcal{G} \end{array}$$

Case H6 ($\text{head}(S) = \{x, x^\rightarrow\}$, $\mathcal{G}^+ \in x^\rightarrow$, and $\mathcal{G} \in \underline{x}$):

$$\text{head}(S) = \begin{array}{|c|} \bullet \mathcal{G}^+ \\ \mathcal{G} \end{array} \mapsto \text{mswap}(\text{head}(S)) = \beta(x) \cdot \begin{array}{|c|} \mathcal{G} \mathcal{G}^+ \end{array} + \alpha \cdot \begin{array}{|c|} \mathcal{G} \\ \mathcal{G}^+ \end{array}$$

Set $\alpha := 0$ if the second tableau has two $\bullet_{\mathcal{G}^+}$ s in the same column; otherwise, set $\alpha := \hat{\beta}(x)$.

Case H7 ($\text{head}(S) = \{x, x^\rightarrow\}$, $\mathcal{G}^+ \in x^\rightarrow$, and $(\mathcal{G}) \in \underline{x}$):

$$\text{head}(S) = \begin{array}{|c|} \bullet \mathcal{G}^+ \\ \mathcal{G} \end{array} \mapsto \text{mswap}(\text{head}(S)) = \begin{array}{|c|} \bullet \mathcal{G}^+ \end{array} + \beta(x) \cdot \begin{array}{|c|} \mathcal{G} \mathcal{G}^+ \end{array} + \alpha \cdot \begin{array}{|c|} \mathcal{G} \\ \mathcal{G}^+ \end{array}$$

Set $\alpha := 0$ if the third tableau has two $\bullet_{\mathcal{G}^+}$ s in the same column; otherwise, set $\alpha := \hat{\beta}(x)$.

Case H8 ($\text{head}(S) = \{x, x^\rightarrow\}$, $\mathcal{G}^+ \in x^\rightarrow$, and Cases H6 and H7 do not apply):

$$\text{head}(S) = \begin{array}{|c|c|} \hline \bullet & \mathcal{G}^+ \\ \hline \end{array} \mapsto \text{mswap}(\text{head}(S)) = \begin{array}{|c|c|} \hline \bullet & \mathcal{G}^+ \\ \hline \end{array}$$

Case H9 ($\text{head}(S) = \{x, x^\rightarrow\}$, $\mathcal{F}^! \in x$, and $\mathcal{G} \in x^\rightarrow$):

$$\text{head}(S) = \begin{array}{|c|c|} \hline \mathcal{F}^! & \mathcal{G} \\ \hline \end{array} \mapsto \text{mswap}(\text{head}(S)) = \begin{array}{|c|c|} \hline \mathcal{F}^! & \mathcal{G} \\ \hline \end{array}$$

LEMMA 7.1. *Every nonempty head(S) falls into exactly one of H1–H9.*

Proof. Since $\text{head}(S) \neq \emptyset$, $|\text{head}(S)| \in \{1, 2\}$, by Lemma 6.5. If $\text{head}(S) = \{x\}$, then, by Lemma 6.9(I), $\bullet_{\mathcal{G}} \in x$. Then, x contains exactly one of \mathcal{G} , \mathcal{G} or neither; these are respectively Cases H1, H2 and H3. If $\text{head}(S) = \{x, x^\rightarrow\}$, see Lemma 6.9(II): one possibility is $\mathcal{F}^! \in x$ and $\mathcal{G} \in x^\rightarrow$; this is H9. Otherwise, $\bullet_{\mathcal{G}} \in x$, and x^\rightarrow contains \mathcal{G} or \mathcal{G}^+ . The cases where $\mathcal{G} \in x^\rightarrow$ are covered by H4–H5. The cases where $\mathcal{G}^+ \in x^\rightarrow$ are covered by H6–H8. \square

7.1.2. *Miniswaps on body(S).* Let $\text{body}_{\bullet_{\mathcal{G}}}(S) = \{x \in \text{body}(S) : \bullet_{\mathcal{G}} \in x\}$.

Case B1 ($\text{body}(S) = S$): By Definition-Lemma 6.8, $S = \mathcal{G}$. Define

$$\text{body}(S) = \mathcal{G} \mapsto \text{mswap}(\text{body}(S)) = \mathcal{G}.$$

Case B2 (The southmost row of $\text{body}(S)$ contains two boxes): Replace each \mathcal{G} in $\text{body}(S)$ with $\bullet_{\mathcal{G}^+}$ and each $\bullet_{\mathcal{G}}$ with \mathcal{G} , emitting a weight $\prod_{x \in \text{body}_{\bullet_{\mathcal{G}}}(S)} \hat{\beta}(x)$. That is (see Lemma 6.9(III)),

$$\text{body}(S) = \begin{array}{ccc} & \bullet & \mathcal{G} \\ & \bullet & \mathcal{G} \\ \bullet & \mathcal{G} & \end{array} \mapsto \text{mswap}(\text{body}(S)) = \prod_{x \in \text{body}_{\bullet_{\mathcal{G}}}(S)} \hat{\beta}(x) \cdot \begin{array}{ccc} & \mathcal{G} & \bullet \\ & \mathcal{G} & \bullet \\ \mathcal{G} & \bullet & \end{array}$$

Case B3 (Cases B1/B2 do not apply): Replace each \mathcal{G} in $\text{body}(S)$ with $\bullet_{\mathcal{G}^+}$ and each $\bullet_{\mathcal{G}}$ with \mathcal{G} , emitting $-\prod_{x \in \text{body}_{\bullet_{\mathcal{G}}}(S)} \hat{\beta}(x)$. That is (see Lemma 6.9(III)),

$$\text{body}(S) = \begin{array}{ccc} & \bullet & \mathcal{G} \\ & \bullet & \mathcal{G} \\ \mathcal{G} & & \end{array} \mapsto \text{mswap}(\text{body}(S)) = -\prod_{x \in \text{body}_{\bullet_{\mathcal{G}}}(S)} \hat{\beta}(x) \cdot \begin{array}{ccc} & \mathcal{G} & \bullet \\ & \mathcal{G} & \bullet \\ \bullet & & \end{array}$$

LEMMA 7.2. *Every nonempty body(S) falls into exactly one of B1–B3.*

Proof. If B1 applies, then by Definition-Lemma 6.8, $S = \mathcal{G}$. The lemma follows. \square

7.1.3. Miniswaps on $\text{tail}(S)$.

Case T1 ($\text{tail}(S) = \{x\}$):

$$\text{tail}(S) = \boxed{\bullet} \mapsto \text{mswap}(\text{tail}(S)) = -\hat{\beta}(x) \cdot \boxed{\mathcal{G}}$$

Case T2 ($\text{tail}(S) = \{x, x^{\rightarrow}\}$ and $\mathcal{G} \in x^{\rightarrow}$):

$$\text{tail}(S) = \boxed{\bullet} \boxed{\mathcal{G}} \mapsto \text{mswap}(\text{tail}(S)) = \hat{\beta}(x) \cdot \boxed{\mathcal{G}} \boxed{\bullet}$$

Case T3 ($\text{tail}(S) = \{x, x^{\rightarrow}\}$ and $\mathcal{G}^+ \in x^{\rightarrow}$):

$$\text{tail}(S) = \boxed{\bullet} \boxed{\mathcal{G}^+} \mapsto \text{mswap}(\text{tail}(S)) = -\hat{\beta}(x) \cdot \boxed{\mathcal{G}} \boxed{\mathcal{G}^+} + \alpha \cdot \boxed{\mathcal{G}} \boxed{\bullet_{\mathcal{G}^+}}$$

Set $\alpha := 0$ if the second tableau has two $\bullet_{\mathcal{G}^+}$ s in the same column; otherwise, set $\alpha := \hat{\beta}(x)$.

Case T4 ($\text{tail}(S) = \{x, x^{\rightarrow}\}$, $\mathcal{G} \in x^{\rightarrow}$): Let $Z = \{\ell \in x^{\rightarrow} : \mathcal{F} < \ell < \mathcal{G}\}$.

Subcase T4.1 ($\mathcal{H} \in x^{\rightarrow}$, $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\text{tail}(S) = \boxed{\bullet} \boxed{\mathcal{F}^!}_{z, \mathcal{G}, \mathcal{H}} \mapsto \text{mswap}(\text{tail}(S)) = \boxed{\bullet} \boxed{\mathcal{F}^!}_{z, \mathcal{G}^!, \mathcal{H}}$$

Subcase T4.2 ($\mathcal{H} \in x^{\rightarrow}$, $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\text{tail}(S) = \boxed{\bullet} \boxed{\mathcal{F}^!}_{z, \mathcal{G}, \mathcal{H}} \mapsto \text{mswap}(\text{tail}(S)) = \boxed{\bullet} \boxed{\mathcal{F}^!}_{z, \mathcal{G}^!, \mathcal{H}} + \hat{\beta}(x) \cdot \boxed{\mathcal{G}}^{\mathcal{F}, Z} \boxed{\bullet}$$

Subcase T4.3 (Subcases T4.1/T4.2 do not apply):

$$\text{tail}(S) = \boxed{\bullet} \boxed{\mathcal{F}^!}_{z, \mathcal{G}} \mapsto \text{mswap}(\text{tail}(S)) = \hat{\beta}(x) \cdot \boxed{\mathcal{G}}^{\mathcal{F}, Z} \boxed{\bullet}$$

Case T5 ($\text{tail}(S) = \{x, x^{\rightarrow}\}$, $\mathcal{G} \in x^{\rightarrow}$, $\mathcal{G} \notin x$): Let $Z = \{\ell \in x^{\rightarrow} : \mathcal{F} < \ell < \mathcal{G}\}$.

$$\text{tail}(S) = \boxed{\bullet} \boxed{\mathcal{F}^!}_{z, \mathcal{G}} \mapsto \text{mswap}(\text{tail}(S)) = \hat{\beta}(x) \cdot \boxed{\mathcal{G}}^{\mathcal{F}, Z} \boxed{\bullet}$$

Case T6 ($\text{tail}(S) = \{x, x^{\rightarrow}\}$, $\mathcal{G} \in x^{\rightarrow}$, $\mathcal{G} \in x$):

$$\text{tail}(S) = \boxed{\bullet} \boxed{\mathcal{F}^!}_{\mathcal{G}, \mathcal{G}} \mapsto \text{mswap}(\text{tail}(S)) = 0$$

LEMMA 7.3. Every nonempty $\text{tail}(S)$ falls into exactly one of T1–T6.

Proof. Since $\text{tail}(S) \neq \emptyset$, $|\text{tail}(S)| \in \{1, 2\}$, by Lemma 6.5. If $|\text{tail}(S)| = 1$, then, by Lemma 6.9(IV), $\text{tail}(S) = \boxed{\bullet \mathcal{G}}$; this is covered by T1. Suppose that $\text{tail}(S) = \{x, x^{\rightarrow}\}$. By Lemma 6.1, (R.1)–(R.3) and Definition-Lemma 6.8, $\text{tail}(S) = \boxed{\bullet \mathcal{G} \mathcal{G}}$ (handled by T2), $\text{tail}(S) = \boxed{\bullet \mathcal{G} \mathcal{G}^+}$ (handled by T3) or $\text{tail}(S) = \boxed{\bullet \mathcal{G} \mathcal{F}^1}$, with \mathcal{G} or $\mathcal{G} \in \underline{x^{\rightarrow}}$ (handled by T4, T5 or T6). \square

7.2. Swaps and slides. We define $\text{swap}_{\mathcal{G}}(T)$ and $\text{slide}_{\{x_i\}}(T)$ for a good tableau T . We define $\text{swap}_{\mathcal{G}}$ on a single snake S by applying mswap simultaneously to $\text{head}(S)$, $\text{body}(S)$ and $\text{tail}(S)$ (where the conditions on each mswap refer to the original S).

LEMMA 7.4. On the edges shared by two adjacent snake sections, the modifications to the labels given by the two miniswaps are compatible.

Proof. Suppose that the lower of the two adjacent sections is $\text{head}(S)$. The only miniswap that introduces a label to the northeast edge (that is, \bar{x} if $\text{head}(S) = \{x\}$ or \bar{x}^{\rightarrow} if $\text{head}(S) = \{x, x^{\rightarrow}\}$) is H1. However, in that case, $\text{head}(S) = S$ and the compatibility issue is moot. Since body miniswaps do not affect edge labels, the remaining check is when a tail miniswap involves \underline{x} where x is the left box of $\text{tail}(S)$. This only occurs in T6. In this case, $\text{tail}(S) = S$, so compatibility is again moot. \square

LEMMA 7.5 (Swap commutation). If S_1 and S_2 are distinct snakes in a \mathcal{G} -good tableau T , then, applying $\text{swap}_{\mathcal{G}}$ to S_1 commutes with applying $\text{swap}_{\mathcal{G}}$ to S_2 .

Proof. By definition, the locations of virtual labels in one snake are unaffected by swapping another snake. Hence, if the snakes do not share a horizontal edge, there is no concern. If they do, this is the situation of Lemma 6.6(III). The northmost row r of the lower snake (say S_1) is $\{x, x^{\rightarrow}\}$ with $\mathcal{G}^+ \in x^{\rightarrow}$. Hence, by (G.4), $\mathcal{G}, \mathcal{G} \notin \bar{x}^{\rightarrow}$. By inspection, no miniswap involving r affects \bar{x}^{\rightarrow} . Now, the upper snake S_2 has a single row, which by the previous sentence is either an H3 or H8 head, regardless of whether we have acted on r already. Therefore, $\text{swap}_{\mathcal{G}}$ acts trivially on S_2 whether we act on S_1 or S_2 first. \square

Lemma 7.5 permits us to define the *swap* operation $\text{swap}_{\mathcal{G}}$ on a \mathcal{G} -good tableau as the result of applying $\text{swap}_{\mathcal{G}}$ to all snakes (in arbitrary order). We extend $\text{swap}_{\mathcal{G}}$ to a $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ -linear operator.

An *inner corner* of ν/λ is a maximally southeast box of λ . An *outer corner* of ν/λ is a maximally southeast box of ν/λ .

Let $T \in \text{Bundled}(\nu/\lambda)$, and let $\{x_i\}$ be a subset of the inner corners of ν/λ . Define $T^{(1)}$ to be T , with \bullet_{i_1} placed in each x_i .

LEMMA 7.6. *Each $T^{(1)}$ is 1_1 -good.*

Proof. Condition (G.2) is clear. By Lemma 5.3, T is good; (G.1), (G.3)–(G.8) and (G.12) are unaffected by adding \bullet_{i_1} s to inner corners. Conditions (G.9)–(G.11) and (G.13) hold vacuously. □

The *slide* of T at $\{x_i\}$ is

$$\text{slide}_{\{x_i\}}(T) := \text{swap}_{\mathcal{G}_{\max}} \circ \text{swap}_{\mathcal{G}_{\max}^-} \circ \cdots \circ \text{swap}_{1_1}(T^{(1)}), \tag{7.1}$$

with all $\bullet_{\mathcal{G}_{\max}^+}$ s deleted. If Σ is a formal $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ -linear sum of tableaux, we write $V \in \Sigma$ to mean that V occurs in Σ with nonzero coefficient. The following proposition, proved in Appendix A, shows that (7.1) is well-defined.

PROPOSITION 7.7 (Swaps preserve goodness). *If T is a \mathcal{G} -good tableau, then each $U \in \text{swap}_{\mathcal{G}}(T)$ is \mathcal{G}^+ -good.*

It should be noted that since $U \in \text{swap}_{\mathcal{G}}(T)$ is \mathcal{G}^+ -good, it has virtual labels that must be determined as in any good tableau; part of Proposition 7.7 is showing that the virtual labels depicted on the right side of certain miniswaps are consistent with these.

LEMMA 7.8 (Swaps preserve content). *If T is a \mathcal{G} -good tableau of content μ , then each $U \in \text{swap}_{\mathcal{G}}(T)$ has content μ .*

Proof. No miniswap eliminates genes in a section. We consider each miniswap that introduces a new gene to a section; this gene must be \mathcal{G} . We show that \mathcal{G} appears elsewhere in T . The first case is H2, which produces a \mathcal{G} in its section, where there was only a \mathcal{G} previously. The \mathcal{G} only appears if some \mathcal{G} is west of it in T . The same analysis applies *verbatim* to H7 and T5. The remaining cases are T1 and T3. By Lemma 6.9(IV, V), the snake on which these miniswaps act has at least two rows. Moreover, there is a \mathcal{G} directly below the $\bullet_{\mathcal{G}}$ under consideration. In particular, \mathcal{G} has already appeared in T . □

LEMMA 7.9. *No label is strictly southeast of a $\bullet_{\mathcal{G}_{\max}^+}$ in any $U \in \text{swap}_{\mathcal{G}_{\max}} \circ \text{swap}_{\mathcal{G}_{\max}^-} \circ \cdots \circ \text{swap}_{1_1}(T^{(1)})$. In particular, all $\bullet_{\mathcal{G}_{\max}^+}$ s are at outer corners of ν/λ .*

Proof. By Proposition 7.7, U is \mathcal{G}_{\max}^+ -good. Let x be a box of U , and let $\bullet_{\mathcal{G}_{\max}^+} \in x$. There is no $\bullet_{\mathcal{G}_{\max}^+}$ strictly southeast of x , by (G.2). By definition, there is no label Q in $T^{(1)}$ with $\text{family}(Q) \geq \text{family}(\mathcal{G}_{\max}^+)$. Hence, by Lemma 7.8, there are no such labels in U . Therefore, any genetic label ℓ southeast of x is marked. Clearly, we may assume that ℓ is in x 's row or column. If ℓ is in x 's column, we contradict (G.11). If ℓ is in x 's row, we contradict Lemma 5.5. \square

Clearly, we have the following.

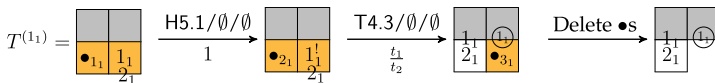
LEMMA 7.10. *If T is a good tableau with no genetic label southeast of a \bullet , then deleting all \bullet s gives a bundled tableau.* \square

COROLLARY 7.11. *Given $\rho \in \lambda^+$ and a tableau $T \in B_{\rho, \mu}^v$, any tableau $U \in \text{slide}_{\rho/\lambda}(T)$ is in either $B_{\lambda, \mu}^v$ or $B_{\lambda, \mu}^\delta$ for some $\delta \in v^-$.*

Proof. By Lemma 5.3, T is a good tableau. By Lemma 7.6, adding \bullet_{1_1} to each box of ρ/λ gives a good tableau $T^{(1)}$. By Proposition 7.7, each swap gives a formal sum of good tableaux. By Lemma 7.9, after all swaps, $\bullet_{\mathcal{G}_{\max}^+}$ s are at outer corners with no labels strictly southeast. By Lemma 7.10, deleting these $\bullet_{\mathcal{G}_{\max}^+}$ s gives a bundled tableau (namely, U). The tableau U has shape ν/λ or δ/λ for $\delta \in v^-$, since there is at most one $\bullet_{\mathcal{G}_{\max}^+}$ deleted in any row or column, by (G.2). Content preservation is shown in Lemma 7.8. \square

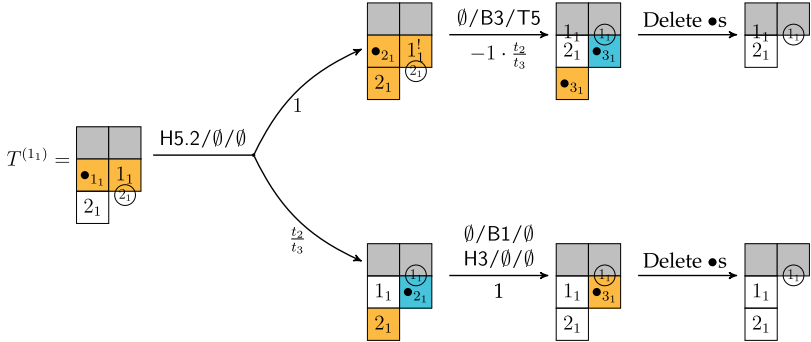
7.3. Examples. We give a number of examples of computing $\text{slide}_{\{x_i\}}(T)$. It is convenient to encode the computation in a diagram. Each nonterminal tableau has its snakes differentiated by color. The notation above each arrow indicates the types of the snakes from southwest to northeast. For example, H5.3/ \emptyset /T2 means that the head of a snake is H5.3, the body is empty and the tail is T2. The notation below arrows indicates the product of the coefficients coming from each miniswap (we will assume for this purpose that the lower-left corner of T coincides with the lower-left corner of $k \times (n - k)$). Each $U \in \text{slide}_{\{x_i\}}(T)$ is a terminal tableau of the diagram. Moreover, $[U]_{\text{slide}_{\{x_i\}}(T)}$ is the sum of the products of the coefficients over all directed paths from T to U .

EXAMPLE 7.12.



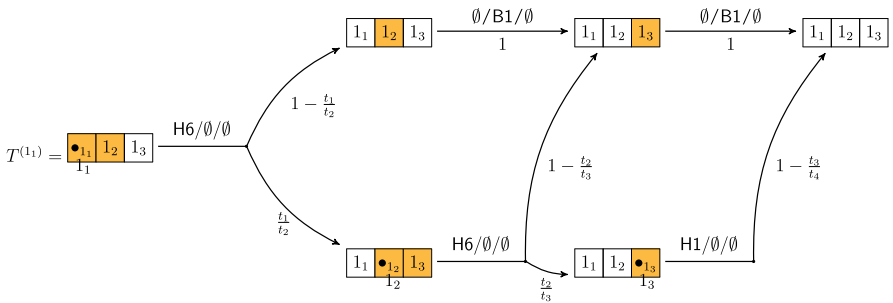
\square

EXAMPLE 7.13.



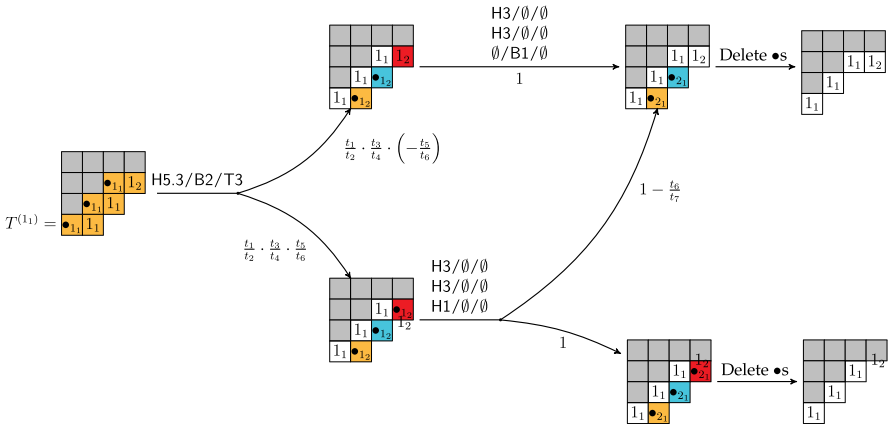
□

EXAMPLE 7.14.



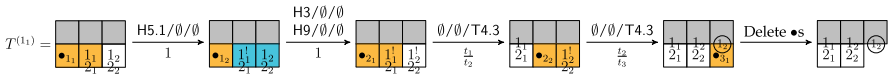
□

EXAMPLE 7.15.



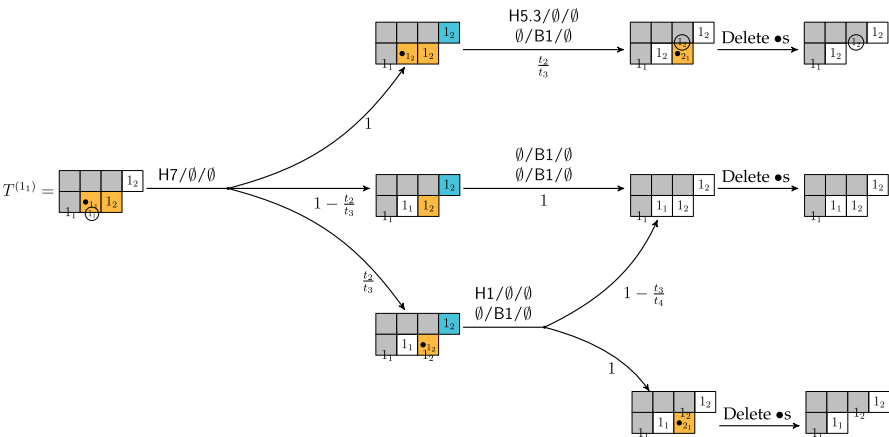
□

EXAMPLE 7.16.



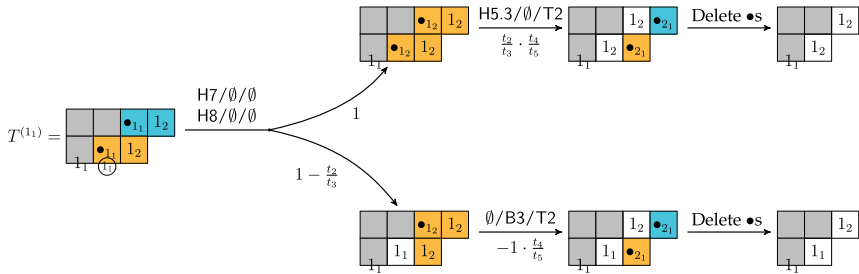
□

EXAMPLE 7.17.



□

EXAMPLE 7.18.

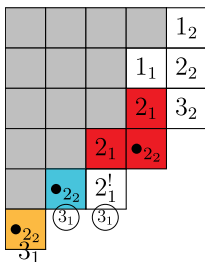


□

8. Ladders

Let U be a \mathcal{G}^+ -good tableau. Consider the boxes of U containing $\bullet_{\mathcal{G}^+}$ or unmarked \mathcal{G} . This set decomposes into maximal edge-connected components, which we call *ladders*.

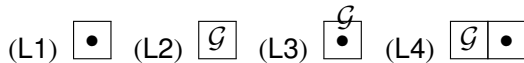
EXAMPLE 8.1.



This 2_2 -good tableau has three ladders; we have given each ladder a separate color. (All virtual labels are depicted.)

□

LEMMA 8.2. A row r of a ladder L is one of the following (edge labels other than \mathcal{G} are not shown and virtual labels are not shown):



Proof. By (G.2), at most one $\bullet_{\mathcal{G}^+}$ occurs in each row. By Lemma 5.10, at most one \mathcal{G} appears in each row. Thus, r has at most two boxes. If it has one box, r is clearly L1, L2 or L3. If r has two boxes, then it has one box label \mathcal{G} and one box label $\bullet_{\mathcal{G}^+}$. Since the \mathcal{G} is not marked, it is West of the $\bullet_{\mathcal{G}^+}$. By (G.4) and (G.7), no edge label \mathcal{G} is possible in this two-box scenario. Thus, L4 is the only two-box possibility. □

LEMMA 8.3. A ladder L is a short ribbon where each column with two boxes is



Proof. In each column, there is at most one $\bullet_{\mathcal{G}^+}$, by (G.2), and at most one \mathcal{G} , by (G.4). If the column consists of $\bullet_{\mathcal{G}^+}$ and \mathcal{G} , then the \mathcal{G} is North of the $\bullet_{\mathcal{G}^+}$, since otherwise the \mathcal{G} is marked. Therefore, the columns are as described.

If L has a 2×2 subsquare, the North box of each column must contain \mathcal{G} , violating (G.3). Each row has at most two boxes, by Lemma 8.2. That L is a skew shape is now immediate from the descriptions of L 's rows and columns. \square

LEMMA 8.4 (Relative positioning of ladders). Suppose that U is \mathcal{G}^+ -good, and that L and M are distinct ladders of U . Then, up to relabeling of the ladders, L is entirely SouthWest of M (that is, if \mathbf{b} and \mathbf{b}' are boxes of L and M , respectively, then \mathbf{b} is SouthWest of \mathbf{b}').

Proof. Suppose the contrary. There are three cases to consider.

Case 1 ($\mathbf{b} \in L$ is NorthWest of $\mathbf{b}' \in M$): By definition, \mathbf{b} and \mathbf{b}' contain either $\bullet_{\mathcal{G}^+}$ or \mathcal{G} . By (G.2) and Lemmas 5.10, 5.11 and 5.12, we see that no combination of these choices is possible.

Case 2 (\mathbf{b} is North and in the same column as \mathbf{b}'): If $\bullet_{\mathcal{G}^+} \in \mathbf{b}$ and $\bullet_{\mathcal{G}^+} \in \mathbf{b}'$, we violate (G.2). If $\bullet_{\mathcal{G}^+} \in \mathbf{b}$ and $\mathcal{G} \in \mathbf{b}'$, then the latter would be marked. Hence, $\mathcal{G} \in \mathbf{b}$. Since $\mathcal{G} \in \mathbf{b}'$ or $\bullet_{\mathcal{G}^+} \in \mathbf{b}'$, we have, by (G.4) and (G.9), that $\mathbf{b}^\downarrow = \mathbf{b}'$, and so \mathbf{b} and \mathbf{b}' are in the same ladder, contradicting $L \neq M$.

Case 3 (\mathbf{b} is West and in the same row as \mathbf{b}'): By (G.2), at least one of \mathbf{b} and \mathbf{b}' contains \mathcal{G} . By Lemma 5.10, at least one of \mathbf{b} and \mathbf{b}' contains $\bullet_{\mathcal{G}^+}$. If $\mathcal{G} \in \mathbf{b}$ and $\bullet_{\mathcal{G}^+} \in \mathbf{b}'$, then, by (G.3) and (G.9), $\mathbf{b}' = \mathbf{b}^\rightarrow$, contradicting $L \neq M$. If $\bullet_{\mathcal{G}^+} \in \mathbf{b}$ and $\mathcal{G} \in \mathbf{b}'$, then the latter is marked. \square

9. Reverse genomic jeu de taquin

Let r be a ladder row in a \mathcal{G}^+ -good tableau U , and let x be the westmost box in r . We define the *reverse miniswap* operation revmswap on r . The cases below are labeled in accordance with the classification of Lemma 8.2. Below, each \bullet on the left of the ' \mapsto ' is a $\bullet_{\mathcal{G}^+}$, while on the right it is a $\bullet_{\mathcal{G}}$.

Case L1:

Subcase L1.1 ($\mathcal{G} \in x^\uparrow$):

$$r = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \mapsto \text{revmswap}(r) = \begin{array}{|c|} \hline \mathcal{G} \\ \hline \end{array}$$

Subcase L1.2 ($\mathcal{G} \notin x^\uparrow$):

$$r = \boxed{\bullet} \mapsto \text{revmswap}(r) = \boxed{\bullet}$$

Case L2:

Subcase L2.1 ($\bullet_{\mathcal{G}^+} \in x^\downarrow$ or $\mathcal{G}^! \in x^\downarrow$):

$$r = \boxed{\mathcal{G}} \mapsto \text{revmswap}(r) = \boxed{\bullet}$$

Subcase L2.2 ($\bullet_{\mathcal{G}^+} \notin x^\downarrow$, $\mathcal{G}^! \notin x^\downarrow$, x contains the westmost \mathcal{G}):

$$r = \boxed{\mathcal{G}} \mapsto \text{revmswap}(r) = \boxed{\mathcal{G}} + \boxed{\bullet_{\mathcal{G}}}$$

Subcase L2.3 ($\bullet_{\mathcal{G}^+} \notin x^\downarrow$, $\mathcal{G}^! \notin x^\downarrow$, x does *not* contain the westmost \mathcal{G}):

$$r = \boxed{\mathcal{G}} \mapsto \text{revmswap}(r) = \boxed{\mathcal{G}} + \boxed{\bullet_{\mathcal{G}}}$$

Case L3:

$$r = \boxed{\begin{array}{c} \mathcal{G} \\ \bullet \end{array}} \mapsto \text{revmswap}(r) = \boxed{\mathcal{G}}$$

Case L4:

Subcase L4.1 ($\mathcal{G}^+ \in \underline{x}$ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$, and either $\bullet_{\mathcal{G}^+} \in x^\downarrow$ or $\mathcal{G}^! \in x^\downarrow$):

$$r = \boxed{\mathcal{G}} \boxed{\bullet_{\mathcal{G}^+}} \mapsto \text{revmswap}(r) = \boxed{\bullet} \boxed{\mathcal{G}^+}$$

Subcase L4.2 ($\mathcal{G}^+ \in \underline{x}$ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$, $\bullet_{\mathcal{G}^+} \notin x^\downarrow$, $\mathcal{G}^! \notin x^\downarrow$ and x contains the westmost \mathcal{G}):

$$r = \boxed{\mathcal{G}} \boxed{\bullet_{\mathcal{G}^+}} \mapsto \text{revmswap}(r) = \boxed{\bullet} \boxed{\mathcal{G}^+}$$

Subcase L4.3 ($\mathcal{G}^+ \in \underline{x}$ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$, $\bullet_{\mathcal{G}^+} \notin x^\downarrow$, $\mathcal{G}^! \notin x^\downarrow$ and x does *not* contain the westmost \mathcal{G}):

$$r = \boxed{\mathcal{G}} \boxed{\bullet_{\mathcal{G}^+}} \mapsto \text{revmswap}(r) = \boxed{\bullet_{\mathcal{G}}} \boxed{\mathcal{G}^+}$$

Subcase L4.4 (there is *no* $\mathcal{G}^+ \in \underline{x}$ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$, and x contains the westmost \mathcal{G}): Let A be the labels in \bar{x} , $Z = \{\mathcal{E} \in A : N_{\mathcal{G}} = N_{\mathcal{E}}\}$, $Z^\sharp = Z \cup \{\mathcal{G}\}$, $\mathcal{F} = \min Z^\sharp$, $A'' = Z^\sharp \setminus \{\mathcal{F}\}$ and $A' = A \setminus Z$.

$$r = \boxed{\begin{array}{c} A \\ \mathcal{G} \end{array}} \boxed{\bullet} \mapsto \text{revmswap}(r) = \boxed{\begin{array}{c} A' \\ \bullet \end{array}} \boxed{\begin{array}{c} \mathcal{F} \\ A'' \end{array}}$$

Subcase L4.5 (there is no $\mathcal{G}^+ \in \underline{x}^{\rightarrow}$ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$, and \underline{x} does not contain the westmost \mathcal{G}): Let $A, Z, Z^\sharp, \mathcal{F}$ and A' be as in L4.4; also, let $A''' = Z \setminus \{\mathcal{F}\}$.

$$r = \begin{array}{|c|c|} \hline A \\ \hline \mathcal{G} & \bullet \\ \hline \end{array} \mapsto \text{revmswap}(r) = \begin{array}{|c|c|} \hline A' \\ \hline \bullet & \mathcal{F} \\ \hline \end{array} \begin{array}{c} \\ \\ \\ \textcircled{A'''} \end{array},$$

where the virtual $\textcircled{\mathcal{G}}$ on the right does not appear if $\mathcal{F} = \mathcal{G}$.

LEMMA 9.1. *Every ladder row falls into exactly one of the above cases.*

Proof. This is tautological, given Lemma 8.2. □

LEMMA 9.2. *No revmswap affects an edge that is shared by two rows of the same ladder L.*

Proof. No revmswap affects the upper (virtual) edge labels of the right box of a ladder row. Hence, it suffices to analyze those cases that affect the lower (virtual) edge labels of the left box of a ladder row. These are L2.2, L2.3, L3, L4.2 and L4.3. In each case, there can be no ladder row of L below, by Lemma 8.3. Hence, that edge is not shared. □

Thus, it makes sense to define $\text{revmswap}_{\mathcal{G}^+}$ on a ladder L , by applying revmswap to each row of L simultaneously (where the conditions on each revmswap refer to the original ladder L).

LEMMA 9.3. *If L_1 and L_2 are distinct ladders in a \mathcal{G}^+ -good tableau U , then, applying $\text{revmswap}_{\mathcal{G}^+}$ to L_1 commutes with applying $\text{revmswap}_{\mathcal{G}^+}$ to L_2 .*

Proof. This follows, since, by definition, L_1 and L_2 do not share any edges. □

Lemma 9.3 permits us to define the reverse swap $\text{revmswap}_{\mathcal{G}^+}$ on a \mathcal{G}^+ -good tableau by applying $\text{revmswap}_{\mathcal{G}^+}$ to all ladders (in arbitrary order). We extend this to a $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ -linear operator.

LEMMA 9.4 (Reverse swaps preserve content). *If U is \mathcal{G}^+ -good and of content μ , then each $T \in \text{revmswap}_{\mathcal{G}^+}(U)$ has content μ .*

Proof. Let \mathcal{H} be a gene in U . We must show that $\mathcal{H} \in T$. Let ℓ be the westmost instance of \mathcal{H} in U . If ℓ is not part of a ladder, \mathcal{H} appears in the same location in T and we are done. Thus, suppose that ℓ is in a ladder row r . Consider the

reverse miniswap applied to r . If it is anything but L2.1 or L4.1, then there is an \mathcal{H} in that row of T . If it is L2.1 or L4.1, let x be the box containing ℓ . By definition, U has $\bullet_{\mathcal{H}^+} \in x^\downarrow$ or $\mathcal{H}^\downarrow \in x^\downarrow$. In the former case, the miniswap applied at x^\downarrow is L1.1, so \mathcal{H} appears in x^\downarrow in T . In the latter case, x^\downarrow is not in a ladder, so \mathcal{H} appears in x^\downarrow in T .

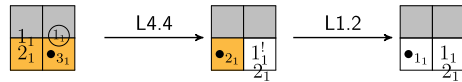
Conversely, suppose that \mathcal{H} is not a gene in U . We must show that it does not appear in T . If it appears in T , it must be created by some miniswap. Clearly, no miniswap other than L1.1 could possibly introduce a new gene. However, if we apply L1.1 at some box x of U , introducing $\mathcal{H} \in x$ in T , then, U has $\mathcal{H} \in x^\uparrow$, by definition, so \mathcal{H} was indeed a gene of U . □

We prove the following proposition in Appendix B.

PROPOSITION 9.5 (Reverse swaps preserve goodness). *If U is \mathcal{G}^+ -good, each $T \in \text{revswap}_{\mathcal{G}^+}(U)$ is \mathcal{G} -good.*

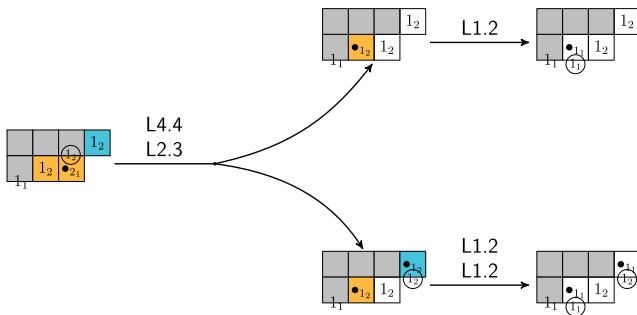
We now provide some examples of computing repeated applications of revswap . In the diagrams below, each nonterminal tableau has its ladder rows differentiated by color. We indicate on the arrows the type(s) of the ladder row(s) from southwest to northeast. (It should be noted that this ordering is consistent with the one used in the examples of Section 7.3.)

EXAMPLE 9.6.



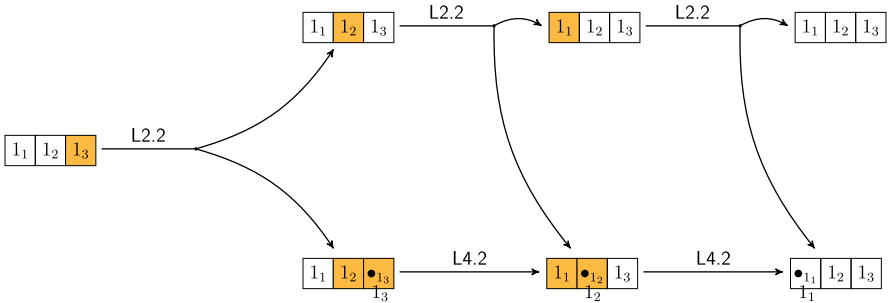
□

EXAMPLE 9.7.



□

EXAMPLE 9.8.



□

LEMMA 9.9. Let T be a \mathcal{G} -good tableau, and let $U \in \text{swap}_{\mathcal{G}}(T)$.

- (I) If $\text{label}_U(\mathbf{x}) = \mathcal{G}$, then $\text{label}_T(\mathbf{x}) \in \{\bullet_{\mathcal{G}}, \mathcal{G}\}$.
- (II) If $\text{label}_U(\mathbf{x}) = \bullet_{\mathcal{G}^+}$, then $\text{label}_T(\mathbf{x}) \in \{\bullet_{\mathcal{G}}, \mathcal{G}, \mathcal{F}^!, \mathcal{G}^+\}$.
- (III) If $\text{label}_U(\mathbf{x}) = \mathcal{G}^!$, then $\text{label}_T(\mathbf{x}) = \mathcal{G}$.

Proof. This is proved by inspection of the miniswaps. □

LEMMA 9.10. Let U be a \mathcal{G}^+ -good tableau, and let $T \in \text{revswap}_{\mathcal{G}^+}(U)$.

- (I) If $\text{label}_U(\mathbf{x}) = \mathcal{G}^!$, then $\text{label}_T(\mathbf{x}) = \mathcal{G}$.
- (II) If $\text{label}_U(\mathbf{x}) = \mathcal{G}$, then $\text{label}_T(\mathbf{x}) \in \{\mathcal{G}, \bullet_{\mathcal{G}}\}$.
- (III) If $\text{label}_U(\mathbf{x}) = \bullet_{\mathcal{G}^+}$, then $\text{label}_T(\mathbf{x}) \in \{\bullet_{\mathcal{G}}, \mathcal{G}, \mathcal{G}^+, \mathcal{F}^!\}$. If, moreover, $\text{label}_T(\mathbf{x}) = \mathcal{G}^+$, then $\text{label}_T(\mathbf{x}^{\leftarrow}) = \bullet_{\mathcal{G}}$, while if, moreover, $\text{label}_T(\mathbf{x}) = \mathcal{F}^!$, then $N_{\mathcal{F}} = N_{\mathcal{G}}$, $\text{label}_T(\mathbf{x}^{\leftarrow}) = \bullet_{\mathcal{G}}$ and either $\mathcal{G} \in \underline{\mathbf{x}}$ or $\mathcal{G} \in \underline{\mathbf{x}}$.

Proof. This is proved by inspection of the reverse miniswaps. □

For a good tableau T of shape ν/λ , we define a T -patch of ν/λ as one of the following:

- (Pat.1) a row of a snake of T (including both upper and lower edges of the row);
- (Pat.2) a box not in a snake (the box excludes the edges);
- (Pat.3) a horizontal edge not bounding a box of a snake.

Clearly, the set $\{P\}$ of T -patches covers ν/λ . Given a tableau W of shape ν/λ , let $W|_P$ be the tableau obtained by restricting W to P .

PROPOSITION 9.11. *Let T and U be good. Then, $U \in \text{swap}_{\mathcal{G}}(T)$ if and only if $T \in \text{revswap}_{\mathcal{G}^+}(U)$.*

Proof. (\Rightarrow) Suppose that $U \in \text{swap}_{\mathcal{G}}(T)$. We show that $T \in \text{revswap}_{\mathcal{G}^+}(U)$.

CLAIM 9.12. *Every ladder row r of U is contained in a distinct T -patch.*

Proof. Distinctness is clear. We now argue containment. If r has one box, containment is trivial. Otherwise, r has two boxes, and we are in case L4 of the ladder row classification of Lemma 8.2. Therefore, in U , each box of r contains $\bullet_{\mathcal{G}^+}$ or \mathcal{G} . One considers all possibilities, under Lemma 9.9, for the entries in T of the boxes of r . Since T is good, these boxes of T either form a row of a snake section or are $\boxed{\mathcal{G} \ \mathcal{G}^+}$. We are done by (Pat.1) in the former case. The latter case cannot occur, since, by inspection of the miniswaps, this cannot swap to L4. \square

By the definitions, we notice that $\text{revswap}_{\mathcal{G}^+}(U) \neq 0$. Moreover, we have the following.

CLAIM 9.13. *For each T -patch P , there exists $W \in \text{revswap}_{\mathcal{G}^+}(U)$ such that $W|_P = T|_P$ (ignoring virtual labels).*

Proof. If P is type (Pat.2), then, by definition, $T|_P = U|_P$, since P is not part of a snake. In particular, $U|_P$ does not contain \mathcal{G} or $\bullet_{\mathcal{G}^+}$. Therefore, $U|_P$ is not part of a ladder of U . Hence, for any $W \in \text{revswap}_{\mathcal{G}^+}(U)$, $W|_P = U|_P = T|_P$, as desired.

If P is type (Pat.3), then $T|_P = U|_P$, since P is not part of a snake. Moreover, by definition, no box y bounded by the edge P is part of a snake in T . Therefore, $\bullet_{\mathcal{G}}, \mathcal{G} \notin y$ in T . Hence, $\bullet_{\mathcal{G}^+}, \mathcal{G} \notin y$ in U . Therefore, P does not bound a box of a ladder of U . Thus, for any $W \in \text{revswap}_{\mathcal{G}^+}(U)$, $W|_P = U|_P = T|_P$.

Finally, if P is type (Pat.1), by inspection of the miniswaps, combined with Claim 9.12, $U|_P$ contains at most one ladder row r , and possibly a nonladder box y . Since $\text{revswap}_{\mathcal{G}^+}$ does not affect y , it suffices to indicate the reverse miniswap on r to give our desired $W|_P = T|_P$. We refer to the list of outputs described in Section 7.

H1: Use L2.2 or L3, respectively, on the two mswap outputs.

H2: Use L1.2 or L2.3, respectively, on the two mswap outputs.

H3: Use L1.2. By T 's (G.2) and (G.9) and Lemma 9.9(I) applied to T , we have $\mathcal{G} \notin x^\uparrow$ in U .

H4: This case does not arise, since here U does not exist.

H5.1: Use L1.2.

H5.2: For the first output, use L1.2. For the second output, use L4.4 or L4.5. We must show in the latter cases that $Z = \emptyset$. Otherwise, if $\mathcal{E} \in Z$, then $\mathcal{E} \in \bar{x}$ in T . Since $N_{\mathcal{E}} = N_{\mathcal{G}}$ in both T and U , this contradicts Lemma 5.6 for T .

H5.3: Use L4.4 or L4.5. The argument that these apply is the same as for H5.2.

H6: Use L2.2 for the first output and L4.2 for the second. By Lemma 9.9(II) and T 's (G.2) and (G.4), $\bullet_{\mathcal{G}^+} \notin x^\downarrow$; by Lemma 9.9(I) and T 's (G.2) or (G.4), $\mathcal{G}^! \notin x^\downarrow$; that the $\mathcal{G} \in x$ is westmost follows from T 's (G.7) and Claim A.3 applied to T .

H7: Use L1.2 for the first output. By Lemma 9.9(I) and T 's (G.2) and (G.9), U has $\mathcal{G} \notin x^\uparrow$. Use L2.3 for the second output and L4.3 for the third. By Lemma 9.9(II) and T 's (G.2) or (G.4), $\bullet_{\mathcal{G}^+} \notin x^\downarrow$; by Lemma 9.9(I) and T 's (G.2) or (G.4), $\mathcal{G}^! \notin x^\downarrow$; that the $\mathcal{G} \in x$ is not westmost follows from T 's $(\mathcal{G}) \in \underline{x}$.

H8: Use L1.2. By T 's (G.2) and (G.9) and Lemma 9.9(I), U has $\mathcal{G} \notin x^\uparrow$.

H9: Here, r does not exist.

B1: Use L2.2 or L2.3. By Lemma 9.9(II) and T 's (G.2) or (G.4), $\bullet_{\mathcal{G}^+} \notin x^\downarrow$; by Lemma 9.9(III) and T 's (G.4), $\mathcal{G}^! \notin x^\downarrow$.

B2: Use L4.4 or L4.5; applicability is as for H5.2.

B3: If we are not in the bottom row, we may use L4.4 or L4.5 as for B2. Otherwise, use L1.1.

T1: Use L2.1. By Lemma 6.9(IV, VII), T has $\mathcal{G} \in x^\downarrow$, so the hypothesis holds by inspection of the miniswaps.

T2: Use L4.4 or L4.5; applicability is as for H5.2.

T3: Use L2.1 or L4.1; applicability is as for T1.

T4.1: Use L1.2. By T 's (G.2) and (G.12) and Lemma 9.9(I), U has $\mathcal{G} \notin x^\uparrow$ and $\mathcal{G} \notin \bar{x}$.

T4.2: Use L1.2 on the first output; applicability is as for T4.1. Use L4.4 on the second output; applicability is as for H5.2.

T4.3: Use L4.4; applicability is as for H5.2.

T5: Use L4.5; applicability is as for H5.2.

T6: This case does not arise, since here U does not exist. □

By definition, $\text{revswap}_{\mathcal{G}^+}(U)$ is obtained by acting on ladder rows of U independently. By Claim 9.12, it follows that $\text{revswap}_{\mathcal{G}^+}(U)$ is also obtained by acting on the T -patches of U independently. Thus (\Rightarrow) holds by Claim 9.13.

(\Leftarrow) Suppose that $T \in \text{revswap}_{\mathcal{G}^+}(U)$. We show that U is in $\text{swap}_{\mathcal{G}}(T)$.

It should be recalled that $\text{swap}_{\mathcal{G}}(T)$ is a formal sum, given by independently replacing each snake section in each prescribed way. Trivially, by (Pat.1), each

snake section is a union of T -patches. Moreover, if a snake section σ consists of more than one T -patch, then σ is a body with at least two rows, and hence either **B2** or **B3**. Therefore, $\text{mswap}(\sigma)$ has a unique output in this case. Since $\text{swap}_{\mathcal{G}}$ acts trivially on the T -patches of types (Pat.2) and (Pat.3), by Lemma 7.5, it follows that $\text{swap}_{\mathcal{G}}(T)$ is also given by acting independently on the T -patches of T . It remains to show that locally at P , we may swap $T|_P$ to obtain $U|_P$.

To make these local verifications, we use the following.

- CLAIM 9.14. (I) Every ladder row of U sits in a distinct T -patch of type (Pat.1).
 (II) Every T -patch P of type (Pat.1) not coming from an **H9** snake section contains a ladder row of U .

Proof. (I): By Lemma 9.10, every ladder row of U is contained in a T -patch of type (Pat.1). Consider a T -patch P of type (Pat.1); P consists of at most two boxes. If P does not consist of two boxes, clearly at most one ladder row of U can be contained in it. If P consists of two boxes, they are joined by a vertical edge. Since distinct ladder rows do not share a vertical edge, it follows that distinct ladder rows of U are contained in distinct T -patches.

(II): This is proved by inspection of the reverse miniswaps. □

If P is type (Pat.2) or (Pat.3), then, by Claim 9.14, P does not intersect any ladder row of U . Thus, $T|_P = U|_P$. By definition, P is not part of any snake in T . Hence, for any $V \in \text{swap}_{\mathcal{G}}(T)$, $V|_P = T|_P = U|_P$, as desired.

Finally, suppose that P is a patch of type (Pat.1). If it comes from an **H9** snake section, then $V \in \text{swap}_{\mathcal{G}}(T)$ and $V|_P = T|_P = U|_P$. Otherwise, by Claim 9.14, P contains a unique ladder row in U . We consider each ladder row type in turn and indicate the miniswaps on $T|_P$ that give our desired $V|_P = U|_P$. We refer to the list of outputs described at the beginning of Section 9. The following case analysis completes the proof of (\Rightarrow).

L1.1: Use **B3**. Since $\text{label}_U(x^\uparrow) = \mathcal{G}$, we apply at x^\uparrow either L2.1, L4.1, L4.4 or L4.5. In each case, $\text{label}_T(x^\uparrow) = \bullet_{\mathcal{G}^+}$. Hence, x and x^\uparrow are part of the southmost two rows of a snake of T . We claim that x^\leftarrow is not part of this snake. It should be noted that, by assumption, x^\leftarrow is not part of any ladder of U . Thus, $\text{label}_U(x^\leftarrow) = \text{label}_T(x^\leftarrow)$ and $\text{label}_T(x^\leftarrow) \notin \{\bullet_{\mathcal{G}}, \mathcal{G}\}$. If x^\leftarrow is part of x 's snake in T , then $\text{label}_T(x^\leftarrow) = \mathcal{F}^! \prec \mathcal{G}$ and southeast of some $\bullet_{\mathcal{G}}$. Hence, in U , x^\leftarrow is southeast of some $\bullet_{\mathcal{G}^+}$; this contradicts U 's (G.2) in view of U 's $\bullet_{\mathcal{G}} \in x$. Thus, x is the unique box of the southmost row of its snake and, by Definition-Lemma 6.8, it is the southmost row of a **B3** snake section.

L1.2: Use H2, H3, H7, H8, T4.1 or T4.2. Since $\text{label}_T(x^\downarrow) = \mathcal{G}$, $\text{label}_U(x^\downarrow) \in \{\bullet_{\mathcal{G}^+}, \mathcal{G}\}$. Hence, by Lemma 8.3, x^\downarrow is not in x 's snake in T . Since $\text{label}_T(x) = \bullet_{\mathcal{G}}$, x^\uparrow is not in x 's snake in T . Hence, x is in a one-row snake. Since L1.2 applies, $\text{label}_U(x^\rightarrow) \neq \mathcal{G}$, so $\text{label}_T(x^\rightarrow) \neq \mathcal{G}$. Thus, x 's snake in T is type (ii), (iv) or (vi) in Definition-Lemma 6.8(III). Type (ii) uses H2 or H3; type (iv) uses H7 or H8; type (vi) uses T4.1 or T4.2.

L2.1: Use T1 or T3. By assumption, $\text{label}_U(x^\downarrow) \in \{\bullet_{\mathcal{G}^+}, \mathcal{G}^\dagger\}$. Hence, by inspection of the reverse miniswaps, $\text{label}_T(x^\downarrow) = \mathcal{G}$. Since $\text{label}_T(x) = \bullet_{\mathcal{G}}$, x^\uparrow is not in x 's snake. Hence, by Definition-Lemma 6.8(I,II), x is in its snake's tail. By T 's (G.3), $\text{label}_T(x^\rightarrow) \succ \mathcal{G}$, so $\text{label}_U(x^\rightarrow) \neq \mathcal{F}^\dagger$. Thus, either T1 or T3 applies.

L2.2: Use B1 for the first output. By assumption and U 's (G.9), U has no $\bullet_{\mathcal{G}^+}$ adjacent to x . Moreover, by U 's (G.4), no box adjacent to x is in any ladder. Hence, T has no $\bullet_{\mathcal{G}}$ adjacent to x . If $\mathcal{F}^\dagger \in x^\leftarrow$ in T , then (possibly marked) $\mathcal{F} \in x^\leftarrow$ in U . If $\text{label}_U(x^\leftarrow) = \mathcal{F}^\dagger$, then we contradict unmarked $\mathcal{G} \in x$ in U . If $\text{label}_U(x^\leftarrow)$ is unmarked, then U has no $\bullet_{\mathcal{G}^+}$ northwest of x^\leftarrow . By U 's (G.3) and (G.4), U has no \mathcal{G} northwest of x^\leftarrow . However, since $\mathcal{F}^\dagger \in x^\leftarrow$ in T , T has a $\bullet_{\mathcal{G}^+}$ northwest of x^\leftarrow . Hence, by Lemma A.3, U has a $\bullet_{\mathcal{G}^+}$ or \mathcal{G} northwest of x^\leftarrow , which is a contradiction.

Use H1 or H6 for the second output. Since x^\rightarrow is not in any ladder of U , $\text{label}_U(x^\rightarrow) = \text{label}_T(x^\rightarrow)$. Moreover, by U 's (G.3), $\text{label}_U(x^\rightarrow) \succ \mathcal{G}$, so $\text{label}_T(x^\rightarrow) \succ \mathcal{G}$. If $\text{label}_T(x^\rightarrow) = \mathcal{G}^+$, H6 applies. Otherwise, H1 applies.

L2.3: Use B1 for the first output; applicability is as for the first output of L2.2. Use H2 or H7 for the second output. Since x^\rightarrow is not in any ladder of U , $\text{label}_U(x^\rightarrow) = \text{label}_T(x^\rightarrow)$. Moreover by U 's (G.3), $\text{label}_U(x^\rightarrow) \succ \mathcal{G}$, so $\text{label}_T(x^\rightarrow) \succ \mathcal{G}$. If $\text{label}_T(x^\rightarrow) = \mathcal{G}^+$, H7 applies. Otherwise, H2 applies.

L3: Use H1. By U 's (G.12), $\text{label}_U(x^\rightarrow) \notin \{\mathcal{G}, \mathcal{G}^+\}$. Moreover, by U 's (G.13) and (G.12), $\text{label}_U(x^\rightarrow)$ is not marked, so $\text{label}_U(x^\rightarrow) \geq \mathcal{G}^+$. Thus, $\text{label}_U(x^\rightarrow) \succ \mathcal{G}^+$. Since x^\rightarrow is not in any ladder of U , $\text{label}_T(x^\rightarrow) \succ \mathcal{G}^+$.

L4.1: Use T3. By inspection of the reverse miniswaps, T has $\mathcal{G} \in x^\downarrow$. Hence, x 's snake in T has at least two rows. Hence, x is part of its snake's tail.

L4.2: Use H6.

L4.3: Use H7.

L4.4: If $Z \neq \emptyset$, use T4.2 or T4.3. Otherwise, use H5.3, B2, B3 or T2. If $Z \neq \emptyset$, some T4 applies. If it is T4.1, T has $\mathcal{H} \in x^\rightarrow$ with $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$. Hence, U also has $\mathcal{H} \in x^\rightarrow$, contradicting Lemma 5.6 for U . If $Z = \emptyset$, there is nothing to check.

L4.5: If $Z \neq \emptyset$, use T5. Otherwise, use H5.3, B2, B3 or T2. \square

The following proposition characterizes good tableaux in terms of forward swapping.

PROPOSITION 9.15. *A tableau U is \mathcal{G} -good if and only if $U \in \text{swap}_{\mathcal{G}^-} \circ \cdots \circ \text{swap}_{\mathbb{1}_2} \circ \text{swap}_{\mathbb{1}_1}(T^{(1)})$ for some bundled tableau T and choice of inner corners of T to initially place $\bullet_{\mathbb{1}_1}$ s in.*

Proof. (\Rightarrow) Given a \mathcal{G} -good tableau U , let $T^{(1)}$ be any tableau appearing in $\text{revswap}_{\mathbb{1}_2} \circ \cdots \circ \text{revswap}_{\mathcal{G}^-} \circ \text{revswap}_{\mathcal{G}}(U)$. By Proposition 9.5, $T^{(1)}$ is a $\mathbb{1}_1$ -good tableau. By $T^{(1)}$'s (G.2) and (G.9), the $\bullet_{\mathbb{1}_1}$ s of $T^{(1)}$ are at inner corners and there is no genetic label northwest of a $\bullet_{\mathbb{1}_1}$. Let T be obtained by removing the $\bullet_{\mathbb{1}_1}$ s of $T^{(1)}$. Then, it is clear that T is a bundled tableau. Now, $U \in \text{swap}_{\mathcal{G}^-} \circ \cdots \circ \text{swap}_{\mathbb{1}_2} \circ \text{swap}_{\mathbb{1}_1}(T^{(1)})$ holds by Propositions 7.7 and 9.11.

(\Leftarrow) The proof is immediate from Lemma 7.6 and Proposition 7.7. \square

10. The reversal tree

10.1. Walkways. An i -walkway W in an $(i+1)_1$ -good tableau T is an edge-connected component of the collection of boxes x in T such that

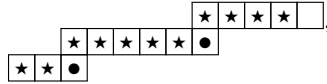
(W.1) $\bullet_{(i+1)_1} \in x$; or

(W.2) $i_k \in x$, and x is not southeast of a $\bullet_{(i+1)_1}$ (equivalently, $i_k \in x$ is not marked).

LEMMA 10.1 (Structure of an i -walkway). *Let W be an i -walkway.*

- (I) *Each column c of W has at most two boxes; if c has two boxes, the southern box contains $\bullet_{(i+1)_1}$.*
- (II) *W has no 2×2 subsquare.*
- (III) *W is an edge-connected skew shape.*
- (IV) *The $\bullet_{(i+1)_1}$ s are at outer corners of W .*
- (V) *The box and upper edge labels of family i form a \prec -interval in the set of genes.*

Therefore, each i -walkway looks like



where each \star is a genetic label and the blank box contains either $\bullet_{(i+1)_1}$ or a genetic label.

Proof of Lemma 10.1. (I): By (G.2), at most one box of c comes from (W.1). By (G.4), at most one box of c comes from (W.2). Thus, the first assertion of (I) holds. The second assertion holds by (W.2).

(II): Suppose that W contains a 2×2 subsquare. Then, the two southern boxes of the subsquare contain $\bullet_{(i+1)_1}$ s by (I), contradicting (G.2).

(III): By definition, W is edge-connected. In view of (II), it remains to show that there are no two boxes y, z of W with y NorthWest of z . Suppose otherwise. By (G.2), at least one of y and z contains a genetic label. If $\bullet_{(i+1)_1} \in y$ and $i_k \in z$, we violate (W.2). If $\bullet_{(i+1)_1} \in z$ and $i_k \in y$, consider the box b in y 's column and z 's row. By (G.2), b contains a genetic label. By (G.4), $\text{label}(b) > i_k$. Since $\bullet_{(i+1)_1} \in z$, this contradicts (G.9). Finally, if $i_k \in y$ and $i_h \in z$, then we contradict (G.12).

(IV): The proof is immediate from (W.2) and (G.2).

(V): By the edge-connectedness of W , we know that W occupies consecutive columns. Thus, we are done by (G.4)–(G.6). □

10.2. Walkway reversal. Let $U \in B_{\lambda, \mu}^\alpha$ for some $\alpha \in \{v\} \cup v^-$. Obtain $U^{(0)}$ from U by placing $\bullet_{(\ell(\mu)+1)_1}$ in each box of v/α . The root of the reversal tree \mathfrak{T}_U is $U^{(0)}$. The children $\{U^{(1)}\}$ of $U^{(0)}$ are the tableaux in the formal sum $\text{revswap}_{\ell(\mu)_1^+} \circ \dots \circ \text{revswap}_{(\ell(\mu)+1)_1}(U^{(0)})$. By Proposition 9.5, each $U^{(1)}$ is $\ell(\mu)_1$ -good. We define the children $\{U^{(2)}\}$ of a $U^{(1)}$ by reverse swapping successively through labels of family $\ell(\mu) - 1$, and so on. Similarly, all tableaux thus obtained are also good. (A tableau may have a copy of itself as a child; this occurs only if $U^{(0)}$ has no $\bullet_{(\ell(\mu)+1)_1}$ s.) After $\ell(\mu) - i$ steps, a descendant $U' = U^{(\ell(\mu)-i)}$ is an $(i + 1)_1$ -good tableau.

LEMMA 10.2. *Let U' be an $(i + 1)_1$ -good tableau. If ℓ is a box or edge label that is not in an i -walkway, then ℓ appears in the same location in every $T \in \text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}(U')$.*

Proof. The case analysis is as follows.

Case 1 ($\ell \in \mathbf{x}$ is a box label in U'):

Subcase 1.1 ($\text{family}(\ell) \neq i$): During the reversal process $\text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}$, the label ℓ is never part of any ladder consisting of \mathcal{H} and $\bullet_{\mathcal{H}^+}$, where $\mathcal{H} \in \{i_1, \dots, i_{\mu_i}\}$. Thus, $\text{revswap}_{\mathcal{H}^+}$ does not move ℓ .

Subcase 1.2 ($\text{family}(\ell) = i$): Since \mathbf{x} is not part of an i -walkway, by (W.2), it is southeast of a $\bullet_{(i+1)_1}$ in U' . By inspection of the reverse miniswaps, this remains true for each tableau V appearing in the reversal process $\text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}$. The box \mathbf{x} is never part of a ladder during this process, as when we apply $\text{revswap}_{\mathcal{H}^+}$, where \mathcal{H} is ℓ 's gene, $\bullet_{\mathcal{H}^+}$ is northwest of \mathbf{x} and so $\ell^1 \in \mathbf{x}$. The case then follows.

Case 2 (ℓ is an edge label in U'): Let \mathbf{x} and \mathbf{x}^\downarrow be the boxes adjacent to the edge.

Subcase 2.1 (\mathbf{x} and \mathbf{x}^\downarrow do not contain a label of family i in U'): As above, \mathbf{x} and \mathbf{x}^\downarrow are not part of a ladder consisting of \mathcal{H} and $\bullet_{\mathcal{H}^+}$, where $\mathcal{H} \in \{i_1, \dots, i_{\mu_i}\}$. Hence, neither is the ℓ in question, and so this ℓ remains fixed throughout the reversal process.

Subcase 2.2 (\mathbf{x} or \mathbf{x}^\downarrow contains a label \mathcal{H} of family i in U'): By (G.4), at most one of \mathbf{x} or \mathbf{x}^\downarrow contains such a label. Without loss of generality, suppose that it is \mathbf{x} (the argument in the other case is the same). Since $\ell \in \mathbf{x}$ is not part of an i -walkway, neither is \mathbf{x} . By the arguments of **Subcase 1.2**, \mathbf{x} is never part of a ladder, since $\mathcal{H}^1 \in \mathbf{x}$. Thus, \mathbf{x} is unchanged. □

Consider an i -walkway W of U' . By Lemma 10.1(V), the genes of family i in W form an interval; let it be (w_1, \dots, w_n) in increasing \prec -order.

LEMMA 10.3 (Characterization of one-row walkway reversals). *Let W be a one-row i -walkway in an $(i + 1)_1$ -good tableau U' . Let \mathbf{a} and \mathbf{z} be the westmost and eastmost boxes of W , respectively. Consider the region \mathcal{R} occupied by W .*

- (I) *Suppose that U' has $\bullet_{(i+1)_1} \in \mathbf{z}$ and no label of family i in $\bar{\mathbf{z}}$. Then, there exists a filling R of \mathcal{R} with $\bullet_{i_1} \in \mathbf{a}$ and $w_1 \notin \underline{\mathbf{a}}$ such that for any $V \in \text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.*
- (II) *Suppose that U' has $\bullet_{(i+1)_1} \in \mathbf{z}$ and a label of family i in $\bar{\mathbf{z}}$. Then, there exists a filling R of \mathcal{R} with $\bullet_{i_1} \in \mathbf{a}$ and either $w_1 \in \underline{\mathbf{a}}$ or $(w_1) \in \underline{\mathbf{a}}$ such that for any $V \in \text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.*
- (III) *Suppose that U' has a label of family i in \mathbf{z} . Then, there exist two fillings R, R' of \mathcal{R} such that*

- (i) R has $w_1 \in \mathbf{a}$;
- (ii) R' has $\bullet_{i_1} \in \mathbf{a}$ and either $w_1 \in \underline{\mathbf{a}}$ or $(w_1) \in \underline{\mathbf{a}}$;
- (iii) R and R' are otherwise identical; and
- (iv) for any $V \in \text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} \in \{R, R'\}$.

Proof. We argue (I)–(III) separately, by induction on the number of boxes of W . The base cases (where W consists of a single box, $\mathbf{a} = \mathbf{z}$) are clear by Lemma 8.2 and inspection of the reverse miniswaps. Assume that W has at least two boxes and let \overline{W} be W with \mathbf{a} removed.

(I): By induction, \overline{W} reverses uniquely to some \overline{R} , which has a $\bullet_{w_2} \in \mathbf{a}^{\rightarrow}$ and $w_2 \notin \underline{\mathbf{a}}^{\rightarrow}$. (By a technical modification of the hypotheses, we may apply the inductive hypothesis to this partial walkway here and below.) This extends uniquely by L4.4 or L4.5 (followed by some number of applications of L1.2) to an R with the claimed properties.

(II): The unique reversal \overline{R} of \overline{W} has a $\bullet_{w_2} \in \mathbf{a}^{\rightarrow}$ and $w_2 \in \underline{\mathbf{a}}^{\rightarrow}$. (By (V.2), $(w_2) \notin \underline{\mathbf{a}}^{\rightarrow}$.) We obtain the desired unique reversal R by applying L4.2 or L4.3 to $\{\mathbf{a}, \mathbf{a}^{\rightarrow}\}$ in $\overline{R} \cup \{\mathbf{a}\}$.

(III): There are precisely two reversals of \overline{W} : \overline{R} and \overline{R}' . The former reversal has $w_2 \in \mathbf{a}^{\rightarrow}$, while the latter has $\bullet_{w_2} \in \mathbf{a}^{\rightarrow}$ and $w_2 \in \underline{\mathbf{a}}^{\rightarrow}$. (By (V.2), $(w_2) \notin \underline{\mathbf{a}}^{\rightarrow}$.) Applying L4.2 or L4.3 (as appropriate) to $\{\mathbf{a}, \mathbf{a}^{\rightarrow}\}$ in $\overline{R}' \cup \{\mathbf{a}\}$ returns R' as described. Applying L2.2 or L2.3 (as appropriate) to \mathbf{a} in $\overline{R} \cup \{\mathbf{a}\}$ returns precisely R and R' . (We apply L4.2 to $\overline{R}' \cup \{\mathbf{a}\}$ exactly when we apply L2.2 to $\overline{R} \cup \{\mathbf{a}\}$.) □

LEMMA 10.4 (Characterization of multirow walkway reversals). *Let W be an i -walkway with at least two rows in an $(i + 1)_1$ -good tableau U' . Let \mathbf{a} and \mathbf{z} be the westmost and eastmost boxes, respectively, in its southmost row. Thus, $\bullet_{(i+1)_1} \in \mathbf{z}$. Let \mathcal{R} be the region occupied by W .*

- (I) *Suppose that $\mathbf{a} = \mathbf{z}$. Then, there exists a filling R of \mathcal{R} with $w_1 \in \mathbf{a}$ such that for any $V \in \text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.*
- (II) *Suppose that $\mathbf{a} \neq \mathbf{z}$ and $\text{label}_W(\mathbf{z}^{\leftarrow}) = \text{label}_W(\mathbf{z}^{\uparrow})$. Then, there exists a filling R of \mathcal{R} with $\bullet_{i_1} \in \mathbf{a}$ and no label of family i on $\underline{\mathbf{a}}$ such that for any $V \in \text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.*
- (III) *Suppose that $\mathbf{a} \neq \mathbf{z}$ and $\text{label}_W(\mathbf{z}^{\leftarrow}) \neq \text{label}_W(\mathbf{z}^{\uparrow})$. Then, there exist two fillings R, R' of \mathcal{R} such that*

- (i) R has $w_1 \in \mathfrak{a}$;
- (ii) R' has $\bullet_{i_1} \in \mathfrak{a}$ and either $w_1 \in \underline{\mathfrak{a}}$ or $(w_1) \in \underline{\mathfrak{a}}$;
- (iii) R and R' are otherwise identical; and
- (iv) for any $V \in \text{revswap}_{i_1^+} \circ \dots \circ \text{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} \in \{R, R'\}$.

Proof. (I): Let \overline{W} be W with the two boxes in the westmost column of W removed. If $\overline{W} = \emptyset$, then $W = \{\bullet_{(i+1)_1} \in \mathfrak{z}, w_1 \in \mathfrak{z}^{\uparrow}\}$; here, we obtain the desired result by use of L1.1 and L2.1. Hence, assume that $\overline{W} \neq \emptyset$. Clearly,

$$\text{label}_W(\mathfrak{z}^{\uparrow\rightarrow}) \in \{w_2, \bullet_{(i+1)_1}\}. \tag{10.1}$$

Depending on whether \overline{W} has multiple rows, by induction or by Lemma 10.3, there are at most two reversals of \overline{W} .

Case 1 (\overline{W} has a unique reversal \overline{R}): By (10.1) and induction/Lemma 10.3, we have two possible scenarios.

Subcase 1.1 (\overline{R} has $\bullet_{w_2} \in \mathfrak{z}^{\uparrow\rightarrow}$ and no labels of family i appear on $\underline{\mathfrak{z}^{\uparrow\rightarrow}}$): Here, we extend to a unique reversal of W by applying L4.4 or L4.5 at \mathfrak{z}^{\uparrow} and L1.1 at \mathfrak{z} . This results in $w_1 \in \mathfrak{z} = \mathfrak{a}$.

Subcase 1.2 (\overline{R} has $w_2 \in \mathfrak{z}^{\uparrow\rightarrow}$): We extend to a unique reversal of W by applying L2.1 at \mathfrak{z}^{\uparrow} and L1.1 at \mathfrak{z} . This results in $w_1 \in \mathfrak{z} = \mathfrak{a}$, as desired.

Case 2 (\overline{W} has two reversals \overline{R} and \overline{R}'): By (10.1) and induction/Lemma 10.3, \overline{R} and \overline{R}' differ only in $\mathfrak{z}^{\uparrow\rightarrow}$: \overline{R} has $w_2 \in \mathfrak{z}^{\uparrow\rightarrow}$ whereas \overline{R}' has $\bullet_{w_2} \in \mathfrak{z}^{\uparrow\rightarrow}$ and $w_2 \in \underline{\mathfrak{z}^{\uparrow\rightarrow}}$. By L2.1 and L1.1 in the \overline{R} case and by L4.1 and L1.1 in the \overline{R}' case, both extend to the same reversal R of W ; here, R has $w_1 \in \mathfrak{z} = \mathfrak{a}$, as claimed.

(II): We have some cases.

Case 1 (The southmost row of W has exactly two boxes $\{\mathfrak{a} = \mathfrak{z}^{\leftarrow}, \mathfrak{z}\}$): Let \overline{W} be W with $\{\mathfrak{a}, \mathfrak{z}, \mathfrak{z}^{\uparrow}\}$ removed. If \overline{W} is empty, the result is clear, so we may assume otherwise. Thus, (10.1) still holds. Depending on whether \overline{W} has multiple rows or not, either by induction or by Lemma 10.3, it follows that there are at most two reversals of \overline{W} .

Subcase 1.1 (\overline{W} has a unique reversal \overline{R}): By (10.1) and induction/Lemma 10.3, two scenarios are possible.

Subcase 1.1.1 (\overline{R} has $\bullet_{w_2} \in \mathfrak{z}^{\uparrow\rightarrow}$ and no label of family i on $\underline{\mathfrak{z}^{\uparrow\rightarrow}}$): We extend to a unique reversal R of W by applying L4.5 at $\{\mathfrak{z}^{\uparrow}, \mathfrak{z}^{\uparrow\rightarrow}\}$ and either L4.4 or L4.5 (as required) at $\{\mathfrak{a}, \mathfrak{z}\}$; R has $\bullet_{w_1} \in \mathfrak{a}$ and no label of family i on $\underline{\mathfrak{a}}$.

Subcase 1.1.2 ($w_2 \in z^{\uparrow}$): We extend to a unique reversal R of W by applying L2.1 at z^{\uparrow} and either L4.4 or L4.5 (as required) at z . This again results in $\bullet_{w_1} \in \underline{a}$ and no label of family i on \underline{a} .

Subcase 1.2 (\overline{W} has two reversals \overline{R} and \overline{R}'): By (10.1) and induction/Lemma 10.3, \overline{R} and \overline{R}' differ only in z^{\uparrow} : \overline{R} has a $w_2 \in z^{\uparrow}$ whereas \overline{R}' has a $\bullet_{w_2} \in z^{\uparrow}$ and $w_2 \in \underline{z}$. By L2.1 and L4.4 or L4.5 in the \overline{R} case and by L4.1 and L4.4 or L4.5 in the \overline{R}' case, both extend to the same reversal R of W . Thus, R has $\bullet_{w_1} \in z = \underline{a}$.

In each of the subcases above, we are done after applying a sequence of L1.2s at \underline{a} .

Case 2 (The southmost row of W contains at least three boxes): Let \overline{W} be W with \underline{a} removed. By induction, \overline{W} has a unique reversal \overline{R} with \bullet_{w_2} in $\underline{a}^{\rightarrow}$ and no label of family i on $\underline{a}^{\rightarrow}$. Now, we uniquely extend \overline{R} to a reversal R of W by applying L4.4 or L4.5 at $\{\underline{a}, \underline{a}^{\rightarrow}\}$; R has $\bullet_{w_1} \in \underline{a}$ and no label of family i on \underline{a} , and the result follows after applying a sequence of L1.2s at \underline{a} .

(III): Let \overline{W} be W with the southmost row and z^{\uparrow} removed. We recall that $\text{label}_W(z) = \bullet_{(i+1)_1}$, and suppose that W has $w_{q-1} \in z^{\leftarrow}$ and $w_q \in z^{\uparrow}$. If \overline{W} is empty, we are done by applying L2.1 at z^{\uparrow} and L1.1 at z , followed by application of Lemma 10.3(III) to the southmost row. Thus, we assume that \overline{W} is not empty. By induction or Lemma 10.3, there are at most two reversals of \overline{W} .

Case 1 (\overline{W} has a unique reversal \overline{R}): Observe that exactly one of the following two cases holds.

Subcase 1.1 (\overline{R} has $\bullet_{w_{q+1}} \in z^{\uparrow}$ and no label of family i on z^{\uparrow}): Apply L4.4 at z^{\uparrow} and L1.1 at z .

Subcase 1.2 (\overline{R} has $w_{q+1} \in z^{\uparrow}$): Apply L2.1 at z^{\uparrow} and L1.1 at z .

Case 2 (\overline{W} has two reversals \overline{R} and \overline{R}'): By induction/Lemma 10.3, \overline{R} and \overline{R}' differ only in z^{\uparrow} : \overline{R} has $w_{q+1} \in z^{\uparrow}$ whereas \overline{R}' has a $\bullet_{w_{q+1}} \in z^{\uparrow}$ and $w_{q+1} \in \underline{z}$. Apply L2.1 and L1.1 in the \overline{R} case. Apply L4.1 and L1.1 in the \overline{R}' case.

In each of the cases above, the indicated reverse miniswaps leave us with the southmost row having $w_1 \in \underline{a}$ and $w_q \in z$. We complete the reversal using Lemma 10.3(III), yielding the desired conclusion. \square

PROPOSITION 10.5. *The children of a node U' in \mathfrak{T}_U are obtained by replacing each walkway W with R or R, R' (as defined in Lemmas 10.3 and 10.4) independently in all possible ways.*

Proof. That nothing changes outside the walkways is given by Lemma 10.2. Independence follows from walkways being edge-disjoint. □

PROPOSITION 10.6. \mathfrak{T}_U is a tree.

Proof. Let U' and U'' be distinct i_1 -good nodes of \mathfrak{T}_U . By induction and Lemmas 10.3 and 10.4, U' and U'' differ in the placement of a label of family strictly larger than i . This label is unaffected by later reverse swaps, so U' and U'' cannot have the same child. □

PROPOSITION 10.7 (Characterization of reversal tree leaves). (I) *Let L be a leaf of \mathfrak{T}_U . Then, if we ignore the \bullet_{i_1} s, either $L = U$ or $L \in \Lambda^+$ and has shape ν/ρ for some $\rho \in \lambda^+$. Moreover, $[U]_{\text{slide}_{\rho/\lambda}}(L) \neq 0$.*

(II) *If $M \in \Lambda^+$ has shape ν/ρ and $[U]_{\text{slide}_{\rho/\lambda}}(M) \neq 0$, then M appears as a leaf of \mathfrak{T}_U .* □

Proof. (I): By Proposition 9.5, L is 1_1 -good. By (G.9), there are no labels northwest of a \bullet_{i_1} . By (G.2), \bullet_{i_1} s appear in distinct rows and columns. This proves the second sentence. The third sentence then follows from Proposition 9.11.

(II): This is immediate from Proposition 9.11. □

11. The recurrence coefficients

Given $U \in B_{\lambda, \mu}^\alpha$, where $\alpha \in \{\nu\} \cup \nu^-$, let $\text{leaf}(\mathfrak{T}_U)$ be the collection of leaves of the tree \mathfrak{T}_U defined in Section 10.

Let W be an i -walkway of shape $\bar{\nu}/\bar{\lambda}$ with $\bullet_{(i+1)_1}$ s in $\bar{\nu}/\bar{\alpha}$. Let S be a reversal of W , as defined by Lemmas 10.3 and 10.4. Let \mathbf{a} be the southwestmost box of W , let \mathbf{b} be the northeastmost box of W , and let \mathbf{z} be the eastmost box of W 's southmost row. By Lemma 10.1(V), the labels of family i of S form an interval (w_1, \dots, w_n) with respect to \prec . Let $\bar{\alpha}_*$ denote $\bar{\alpha}$ with its southmost row deleted, and set $\bar{\lambda}_* := \bar{\lambda} \cap \bar{\alpha}_*$. Let $\Delta(S, W) := (\#\bullet_{i_1} \text{ s in } S) - (\#\bullet_{(i+1)_1} \text{ s in } W)$. For a tableau T , let \tilde{T} denote T excluding boxes containing w_1 and outer corners containing $\bullet_{w_1^+}$.

CLAIM 11.1. (I.i) *If S has $w_1 \notin \mathbf{a}^\rightarrow$ and w_1 or $(w_1) \in \mathbf{a}$, while W has either at least two rows or $w_n \in \mathbf{b}$, then $[W]_{\text{slide}_{\bar{\nu}/\bar{\lambda}}}(S) = (-1)^{\Delta(S, W)-1} (1 - \text{wt } \bar{\alpha}/(\bar{\alpha}_* \cup \bar{\lambda})) \text{wt } \bar{\alpha}_*/\bar{\lambda}_*$.*

(I.ii) If S has $w_1 \notin \mathbf{a}^\rightarrow$ and w_1 or $(w_1) \in \underline{\mathbf{a}}$, while W has exactly one row and $w_n \in \overline{\mathbf{b}}$, then $[W]_{\text{slide}_{\overline{\rho}/\overline{\lambda}}}(S) = (-1)^{\Delta(S,W)} \text{wt } \overline{\alpha}/\overline{\lambda}$.

(II) If S has $\bullet_{i_1} \in \mathbf{a}$, $w_1 \in \mathbf{a}^\rightarrow$ and $w_1 \notin \underline{\mathbf{a}}$, then $[W]_{\text{slide}_{\overline{\rho}/\overline{\lambda}}}(S) = (-1)^{\Delta(S,W)} \text{wt } \overline{\alpha}/\overline{\lambda}$.

(III) If S has $w_1 \in \mathbf{a}$, then $[W]_{\text{slide}_{\overline{\rho}/\overline{\lambda}}}(S) = (-1)^{\Delta(S,W)} \text{wt } \overline{\alpha}_*/\overline{\lambda}_*$.

Proof. We simultaneously induct on the number of genes of family i in S . (We gloss over some technical reindexing in the arguments below.) We check the base case of one gene directly from the swapping rules of Section 7. Now, let us assume that S has at least two genes of family i , and the claims hold for situations with fewer genes of family i .

In the illustrative examples below that accompany the general analysis, we use, for simplicity, $1, 2, \dots$ to represent w_1, w_2, \dots , respectively. Moreover, for simplicity, our examples assume that \mathbf{a} is the southwest corner of $k \times (n - k)$; that is, $\beta(\mathbf{a}) = 1 - t_1/t_2$.

Case (I.i).1 ($\mathbf{a}^\rightarrow \neq z$): Consider $S = \begin{array}{|c|c|c|c|} \hline & & \bullet & 4 \ 5 \\ \hline \bullet & 2 & 3 & \\ \hline \end{array}$ and $W = \begin{array}{|c|c|c|} \hline & 3 & 4 \ 5 \\ \hline 1 & 2 & \bullet \\ \hline \end{array}$. Then,

$$\text{swap}_1(S) = (1 - \frac{t_1}{t_2}) \begin{array}{|c|c|c|c|} \hline & & \bullet & 4 \ 5 \\ \hline 1 & 2 & 3 & \\ \hline \end{array} + \frac{t_1}{t_2} \begin{array}{|c|c|c|c|} \hline & & \bullet & 4 \ 5 \\ \hline 1 & \bullet & 3 & \\ \hline \end{array} := (1 - \frac{t_1}{t_2})S' + \frac{t_1}{t_2}S''.$$

Inductively by (III), $[W]_{\text{slide}}(\tilde{S}') = t_4/t_7$. Inductively by (I.i), $[W]_{\text{slide}}(\tilde{S}'') = (1 - t_2/t_3)(t_4/t_7)$. Hence, $[W]_{\text{slide}}(S) = (1 - t_1/t_2)(t_4/t_7) + (t_1/t_2)(1 - t_2/t_3)(t_4/t_7) = (1 - t_1/t_3)(t_4/t_7)$, as desired. In general,

$$\begin{aligned} [W]_{\text{slide}}(S) &= (1 - \hat{\beta}(\mathbf{a}))(-1)^{\Delta(S,W)-1} \text{wt } \overline{\alpha}_*/\overline{\lambda}_* \\ &\quad + \hat{\beta}(\mathbf{a})(-1)^{\Delta(S,W)-1} \left(1 - \frac{\text{wt } \overline{\alpha}/(\overline{\alpha}_* \cup \overline{\lambda})}{\hat{\beta}(\mathbf{a})} \right) \text{wt } \overline{\alpha}_*/\overline{\lambda}_* \\ &= (-1)^{\Delta(S,W)-1} (1 - \text{wt } \overline{\alpha}/(\overline{\alpha}_* \cup \overline{\lambda})) \text{wt } \overline{\alpha}_*/\overline{\lambda}_*. \end{aligned}$$

Case (I.i).2 ($\mathbf{a}^\rightarrow = z$): Let $S = \begin{array}{|c|c|c|} \hline & & \bullet \\ \hline \bullet & 2 & 3 \\ \hline \end{array}$ and $W = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & \bullet & \bullet \\ \hline \end{array}$. Then,

$$\text{swap}_1(S) = \left(1 - \frac{t_1}{t_2} \right) \begin{array}{|c|c|c|} \hline & & \bullet \\ \hline 1 & 2 & 3 \\ \hline \end{array} := \left(1 - \frac{t_1}{t_2} \right) S'.$$

By (III), $[W]_{\text{slide}}(\tilde{S}') = (t_3/t_5)(t_6/t_7)$. Hence,

$$[W]_{\text{slide}}(S) = \left(1 - \frac{t_1}{t_2}\right) \frac{t_3}{t_5} \frac{t_6}{t_7},$$

as desired. In general,

$$\begin{aligned} [W]_{\text{slide}}(S) &= (1 - \hat{\beta}(\mathbf{a}))(-1)^{\Delta(S,W)-1} \text{wt } \bar{\alpha}_*/\bar{\lambda}_* \\ &= (-1)^{\Delta(S,W)-1} (1 - \text{wt } \bar{\alpha}/(\bar{\alpha}_* \cup \bar{\lambda})) \text{wt } \bar{\alpha}_*/\bar{\lambda}_*. \end{aligned}$$

Case (I.ii): Let $S = \begin{array}{|c|c|c|} \bullet & 2 & 3 \\ \hline \end{array}$ and $W = \begin{array}{|c|c|c|} 1 & 2 & \bullet \\ \hline \end{array}$. Then,

$$\text{swap}_1(S) = \left(1 - \frac{t_1}{t_2}\right) \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline \end{array} + \frac{t_1}{t_2} \begin{array}{|c|c|c|} 1 & \bullet & 3 \\ \hline 2 & & \end{array} := \left(1 - \frac{t_1}{t_2}\right) S' + \frac{t_1}{t_2} S''.$$

By Lemma 10.3, $[W]_{\text{slide}}(\tilde{S}') = 0$. By (I.ii), $[W]_{\text{slide}}(\tilde{S}'') = t_2/t_3$. Hence, $[W]_{\text{slide}}(S) = (t_1/t_2)(t_2/t_3) = t_1/t_3$, as desired. In general,

$$[W]_{\text{slide}}(S) = \hat{\beta}(\mathbf{a})(-1)^{\Delta(S,W)} \frac{1}{\hat{\beta}(\mathbf{a})} \text{wt } \bar{\alpha}/\bar{\lambda} = (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}/\bar{\lambda}.$$

Case (II).1 ($\mathbf{a}^\rightarrow \neq \mathbf{z}$): Let $S = \begin{array}{|c|c|c|c|} \bullet & 4 & 5 & \\ \hline \bullet & 1 & 2 & 3 \\ \hline \end{array}$ and $W = \begin{array}{|c|c|c|c|} 1 & 2 & 3 & \bullet \\ \hline \bullet & 4 & 5 & \\ \hline \end{array}$. Then, $\text{swap}_1(S) = (t_1/t_2) \begin{array}{|c|c|c|c|} \bullet & 4 & 5 & \\ \hline 1 & \bullet & 2 & 3 \\ \hline \end{array} := (t_1/t_2)S'$. By (II), $[W]_{\text{slide}}(\tilde{S}') = (t_2/t_4)(t_5/t_8)$. Hence, $[W]_{\text{slide}}(S) = t_1/t_2 \cdot (t_2/t_4)(t_5/t_8) = (t_1/t_4)(t_5/t_8)$, as desired. In general,

$$[W]_{\text{slide}}(S) = \hat{\beta}(\mathbf{a}) \cdot (-1)^{\Delta(S,W)} \frac{1}{\hat{\beta}(\mathbf{a})} \text{wt } \bar{\alpha}/\bar{\lambda} = (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}/\bar{\lambda}.$$

Case (II).2 ($\mathbf{a}^\rightarrow = \mathbf{z}$ and the northmost $w_1 \in S$ is *not* immediately below \bullet_{i_1}): Let $S = \begin{array}{|c|c|c|c|} \bullet & 4 & 5 & \\ \hline \bullet & 1 & 2 & 3 \\ \hline \bullet & 1 & & \end{array}$ and $W = \begin{array}{|c|c|c|c|} 1 & 2 & 3 & \bullet \\ \hline \bullet & 4 & 5 & \\ \hline 1 & \bullet & & \end{array}$. Then, $\text{swap}_1(S) = \begin{array}{|c|c|c|c|} \bullet & 4 & 5 & \\ \hline 1 & \bullet & 2 & 3 \\ \hline \bullet & 1 & & \end{array} := (t_1/t_2)(t_3/t_4)S'$. By (II), $[W]_{\text{slide}}(\tilde{S}') = -(t_4/t_6)(t_7/t_{10})$. Hence, $[W]_{\text{slide}}(S) = (t_1/t_2)(t_3/t_4) \cdot (-(t_4/t_6)(t_7/t_{10})) = -(t_1/t_2)(t_3/t_6)(t_7/t_{10})$, as desired. In general,

$$\begin{aligned} [W]_{\text{slide}}(S) &= \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot (-1)^{\Delta(S,W)} \prod_{y:\text{label}_W(y)>1} \hat{\beta}(y) \\ &= (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}/\bar{\lambda}. \end{aligned}$$

Case (II).3 ($\mathbf{a}^\rightarrow = \mathbf{z}$ and the northmost $w_1 \in S$ is immediately below \bullet_{i_1}):

Let $S = \begin{array}{cccc} & & \bullet & 5 & 6 \\ & \bullet & 2 & 3 & 4 \\ \bullet & 1 & & & \end{array}$ and $W = \begin{array}{cccc} & & & 4 & 5 & 6 \\ & 1 & 2 & 3 & \bullet \\ 1 & \bullet & & & \end{array}$. Then,

$$\text{swap}_1(S) = -\frac{t_1 t_3}{t_2 t_4} \begin{array}{cccc} & & \bullet & 5 & 6 \\ & 1 & 2 & 3 & 4 \\ 1 & \bullet & & & \end{array} + \frac{t_1 t_3}{t_2 t_4} \begin{array}{cccc} & & \bullet & 5 & 6 \\ & 1 & \bullet & 3 & 4 \\ 1 & \bullet & & & \end{array} := -\frac{t_1 t_3}{t_2 t_4} S' + \frac{t_1 t_3}{t_2 t_4} S''$$

By (III), $[W]_{\text{slide}}(\tilde{S}') = t_7/t_{10}$. By (I.i), $[W]_{\text{slide}}(\tilde{S}'') = (1 - t_4/t_6)(t_7/t_{10})$. Hence, $[W]_{\text{slide}}(S) = -(t_1/t_2)(t_3/t_4)(t_7/t_{10}) + (t_1/t_2)(t_3/t_4)(1 - t_4/t_6)(t_7/t_{10}) = -(t_1/t_2)(t_3/t_6)(t_7/t_{10})$, as desired. Depending on whether (I.i) or (I.ii) applies inductively, we have in general, respectively,

$$\begin{aligned} [W]_{\text{slide}}(S) &= - \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot Y(-1)^{\Delta(S,W)-1} \\ &\quad + \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot (1 - Z)Y(-1)^{\Delta(S,W)-1} \\ &= (-1)^{\Delta(S,W)} YZ \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) = (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}/\bar{\lambda} \end{aligned}$$

or

$$\begin{aligned} [W]_{\text{slide}}(S) &= \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot Z(-1)^{\Delta(S,W)} \\ &= (-1)^{\Delta(S,W)} Z \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) = (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}/\bar{\lambda}, \end{aligned}$$

where Y is the weight of the boxes of W that contain genetic labels and are North of all w_1 s, and Z is the weight of the boxes of W that contain genetic labels greater than w_1 and are not North of all w_1 s.

Case (III).1 ($a \neq z$): Let $S = \begin{array}{cccc} & & \bullet & 5 & 6 \\ & \bullet & 2 & 3 & 4 \\ 1 & 2 & & & \end{array}$ and $W = \begin{array}{cccc} & & & 4 & 5 & 6 \\ & 2 & 3 & 4 & \bullet \\ 1 & \bullet & & & \end{array}$. Then,

$$\text{swap}_1(S) = \begin{array}{cccc} & & \bullet & 5 & 6 \\ & \bullet & 2 & 3 & 4 \\ 1 & 2 & & & \end{array} := S'. \text{ By (III), } [W]_{\text{slide}}(\tilde{S}') = (t_3/t_6)(t_7/t_{10}).$$

Hence, $[W]_{\text{slide}}(S) = (t_3/t_6)(t_7/t_{10})$, as desired. In general, $[W]_{\text{slide}}(S) = (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}_*/\bar{\lambda}_*$.

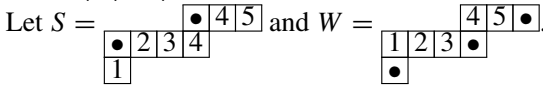
Case (III).2 ($a = z$ and the northmost $w_1 \in S$ is *not* immediately below \bullet_{i_1}): Let $S = \begin{array}{cccc} & & \bullet & 4 & 5 \\ & \bullet & 1 & 2 & 3 \\ 1 & & & & \end{array}$ and $W = \begin{array}{cccc} & & & 3 & 4 & 5 \\ & 1 & 2 & 3 & \bullet \\ & \bullet & & & \end{array}$. Then, $\text{swap}_1(S) =$

$$-\frac{(t_2/t_3)}{\bullet} \begin{array}{cccc} & & \bullet & 4 & 5 \\ & \bullet & 1 & 2 & 3 \\ 1 & \bullet & 2 & 3 & \end{array} := -(t_2/t_3)S'. \text{ By (II), } [W]_{\text{slide}}(\tilde{S}') = -(t_3/t_5)(t_6/t_9).$$

Hence, $[W]_{\text{slide}}(S) = -t_2/t_3 \cdot (-t_3/t_5)(t_6/t_9) = (t_2/t_5)(t_6/t_9)$, as desired. In general,

$$\begin{aligned}
 [W]_{\text{slide}}(S) &= - \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot (-1)^{\Delta(S,W)-1} \prod_{y:\text{label}_W(y)>1} \hat{\beta}(y) \\
 &= (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}_*/\bar{\lambda}_*.
 \end{aligned}$$

Case (III).3 ($a = z$ and the northmost $w_1 \in S$ is immediately below \bullet_{i_1}):



Then,

$$\text{swap}_1(S) = \frac{t_2}{t_3} \begin{array}{ccccccc} & & & & \bullet & 4 & 5 \\ \bullet & 1 & 2 & 3 & 4 & & \\ \bullet & & & & & & \end{array} - \frac{t_2}{t_3} \begin{array}{ccccccc} & & & & \bullet & 4 & 5 \\ \bullet & 1 & \bullet & 2 & 3 & 4 & \\ \bullet & & & & & & \end{array} := \frac{t_2}{t_3} S' - \frac{t_2}{t_3} S''.$$

By (III), $[W]_{\text{slide}}(\tilde{S}') = -t_6/t_8$. By (I.i), $[W]_{\text{slide}}(\tilde{S}'') = -(1 - t_3/t_5)(t_6/t_8)$. Hence, $[W]_{\text{slide}}(S) = t_2/t_3 \cdot (-t_6/t_8) - t_2/t_3 \cdot (-(1 - t_3/t_5)(t_6/t_8)) = -(t_2/t_5)(t_6/t_8)$, as desired. Depending on whether (I.i) or (I.ii) applies inductively, we have in general, respectively,

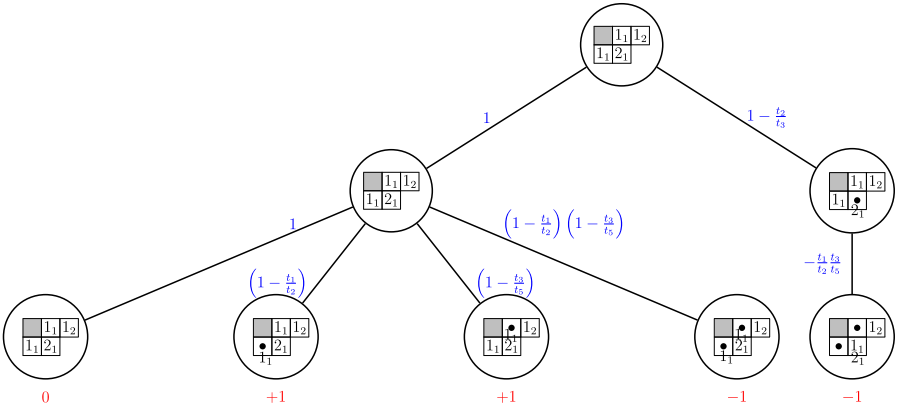
$$\begin{aligned}
 [W]_{\text{slide}}(S) &= \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot (-1)^{\Delta(S,W)} Y \\
 &\quad - \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot (-1)^{\Delta(S,W)} (1 - Z) Y \\
 &= (-1)^{\Delta(S,W)} Y Z \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) = (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}_*/\bar{\lambda}_*
 \end{aligned}$$

or

$$\begin{aligned}
 [W]_{\text{slide}}(S) &= - \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) \cdot (-1)^{\Delta(S,W)-1} Z \\
 &= (-1)^{\Delta(S,W)} Z \prod_{x:\text{label}_W(x)=1} \hat{\beta}(x) = (-1)^{\Delta(S,W)} \text{wt } \bar{\alpha}_*/\bar{\lambda}_*,
 \end{aligned}$$

where Y is the weight of the boxes of W containing genetic labels and that are North of all w_1 s, and Z is the weight of the boxes of W containing genetic labels greater than w_1 and that are not North of all w_1 s. □

EXAMPLE 11.2. Let $\lambda = (1)$, $\nu = (3, 2)$ and $\mu = (2, 1)$. Consider $U = \begin{array}{|c|c|} \hline 1_1 & 1_2 \\ \hline 1_1 & 2_1 \\ \hline \end{array} \in \Lambda$. Below, we give the reversal tree \mathfrak{T}_U .



Each edge is labeled (in blue) by $[U']_{\text{swap}_{i_{\mu_i}} \circ \dots \circ \text{swap}_{i_1}}(V')$, where U' is the parent of the i_1 -good tableau V' . This label agrees with the application of Claim 11.1 to each i -walkway of V' . Below each leaf (in red) is the coefficient in Λ^+ (that is, $(-1)^{|\rho/\lambda|+1}$ if nonzero). □

LEMMA 11.3. Suppose that U' is an $(i + 1)_1$ -good node of \mathfrak{T}_U . Let Γ be the boxes of U containing labels of family i . Then, $\sum_{V'} (-1)^{1+\#\bullet s} \text{in } V' [U']_{\text{swap}_{i_{\mu_i}} \circ \dots \circ \text{swap}_{i_1}}(V') = (-1)^{1+\#\bullet s} \text{in } U' \text{ wt } \Gamma$, where the sum is over all children V' of U' in \mathfrak{T}_U .

Proof. Consider boxes of U' containing unmarked labels of family i or $\bullet_{(i+1)_1}$. By (W.1) and (W.2), these boxes decompose into an edge-disjoint union of i -walkways W_1, W_2, \dots, W_t . Let Γ_j be the boxes of W_j in U containing labels of family i ; thus, $\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \dots \sqcup \Gamma_t$. (It should be noted that the locations of the boxes of Γ in U are the same as the locations of the boxes in U' containing labels of family i .) Let R_j and R'_j (if it exists) be the reversal(s) defined by Lemmas 10.3 and 10.4 with respect to the walkway W_j . As computed by Claim 11.1, let a_j be the coefficient of W_j obtained by sliding R_j . Let b_j be the coefficient of W_j obtained by sliding R'_j if it exists; set $b_j := 0$ if R'_j does not exist. We now assert that

$$(-1)^{\#\bullet s \text{ in } R_j} a_j + (-1)^{\#\bullet s \text{ in } R'_j} b_j = (-1)^{\#\bullet s \text{ in } W_j} \text{ wt } \Gamma_j. \tag{11.1}$$

Suppose that there is a unique reversal (that is, $b_j = 0$). This occurs under Lemmas 10.3(I,II) and 10.4(I,II). In these four cases, R_j is, respectively, the

S from (II), (I.ii), (III) and (II) of Claim 11.1. Hence, in each of these cases, (11.1) is immediate from the corresponding case of Claim 11.1. (It should be noted that for Lemma 10.4(I), the southmost row of W_j has a single box and $\bar{\alpha}_*/\bar{\lambda}_* = \bar{\alpha}/\bar{\lambda} = \Gamma_j$.) Suppose that there are two reversals. This occurs under Lemmas 10.3(III) and 10.4(III), which show that R_j is the S from Claim 11.1(III) and R'_j is the S from Claim 11.1(I.i). Hence, (11.1) also follows in these cases, by adding the two opposite coefficients given by Claim 11.1.

Since, by Proposition 10.5, all V' are obtained by independent replacements of W_j by R_j and R'_j (if it exists),

$$\begin{aligned} & \sum_{V'} (-1)^{1+\#\bullet s \text{ in } V'} [U'] \text{swap}_{i_{\mu_i}} \circ \cdots \circ \text{swap}_{i_1}(V') \\ &= - \prod_{j=1}^t ((-1)^{\#\bullet s \text{ in } R_j} a_j + (-1)^{\#\bullet s \text{ in } R'_j} b_j) \\ &= - \prod_{j=1}^t (-1)^{\#\bullet s \text{ in } W_j} \text{wt } \Gamma_j \\ &= (-1)^{1+\#\bullet s \text{ in } U'} \text{wt } \Gamma. \end{aligned} \quad \square$$

LEMMA 11.4. *Let U' be an $(i + 1)_1$ -good node of \mathfrak{T}_U . Let $\Gamma^{(i)}$ be the set of boxes $\{\mathbf{x} \in \alpha/\lambda : \text{family}(\text{label}_{1_U}(\mathbf{x})) \leq i\}$. Then,*

$$\begin{aligned} & \sum_T (-1)^{1+\#\bullet s \text{ in } T} [U'] \text{swap}_{i_{\mu_i}} \circ \text{swap}_{i_{\mu_i}^-} \circ \cdots \circ \text{swap}_{1_1^+} \circ \text{swap}_{1_1}(T) \\ &= \text{wt}(\Gamma^{(i)}) (-1)^{1+\#\bullet s \text{ in } U'}, \end{aligned}$$

where the sum is over all $T \in \text{leaf}(\mathfrak{T}_U)$ that are descendants of U' .

Proof. We induct on $i \geq 0$. In the base case $i = 0$, $U' = T$ for $T \in \text{leaf}(\mathfrak{T}_U)$ and the left-hand side equals $(-1)^{1+\#\bullet s \text{ in } T}$. This equals the right-hand side since $\Gamma^{(0)} = \emptyset$, so $\text{wt } \Gamma^{(0)} = 1$.

Now, let $i > 0$. We have $\sum_T (-1)^{1+\#\bullet s \text{ in } T} [U'] \text{swap}_{i_{\mu_i}} \circ \text{swap}_{i_{\mu_i}^-} \circ \cdots \circ \text{swap}_{1_1^+} \circ \text{swap}_{1_1}(T)$

$$\begin{aligned} &= \sum_{V' \text{ a child of } U'} \sum_{T \in \text{leaf}(\mathfrak{T}_{U'}) \text{ below } V'} (-1)^{1+\#\bullet s \text{ in } T} [U'] \text{swap}_{i_{\mu_i}} \circ \cdots \circ \text{swap}_{i_1} \circ \\ & \quad \text{swap}_{(i-1)_{\mu_{i-1}}} \circ \cdots \circ \text{swap}_{1_1}(T) \\ &= \sum_{V' \text{ a child of } U'} \sum_{T \in \text{leaf}(\mathfrak{T}_{U'}) \text{ below } V'} (-1)^{1+\#\bullet s \text{ in } T} [U'] \text{swap}_{i_{\mu_i}} \circ \cdots \circ \text{swap}_{i_1}(V') \cdot \\ & \quad [V'] \text{swap}_{(i-1)_{\mu_{i-1}}} \circ \cdots \circ \text{swap}_{1_1}(T). \end{aligned}$$

The previous equality is since \mathfrak{T}_U is a tree (Proposition 10.6) and V' is the unique child of U' that is an ancestor of T . The previous summation equals

$$\begin{aligned} & \sum_{V' \text{ a child of } U'} [U']\text{swap}_{i_{\mu_i}} \circ \dots \circ \text{swap}_{i_1}(V') \\ & \times \sum_{T \in \text{leaf}(\mathfrak{T}_{U'}) \text{ below } V'} (-1)^{1 + \#\bullet \text{ in } T} [V']\text{swap}_{(i-1)_{\mu_{i-1}}} \circ \dots \circ \text{swap}_{i_1}(T) \\ & = \sum_{V' \text{ a child of } U'} [U']\text{swap}_{i_{\mu_i}} \circ \dots \circ \text{swap}_{i_1}(V') \cdot \text{wt}(\Gamma^{(i-1)})(-1)^{1 + \#\bullet \text{ in } V'} \\ & \hspace{15em} \text{(by induction)} \\ & = \text{wt}(\Gamma^{(i-1)}) \sum_{V' \text{ a child of } U'} (-1)^{1 + \#\bullet \text{ in } V'} [U']\text{swap}_{i_{\mu_i}} \circ \dots \circ \text{swap}_{i_1}(V') \\ & = \text{wt}(\Gamma^{(i-1)}) \cdot (-1)^{1 + \#\bullet \text{ in } U'} \text{wt}(\Gamma) \quad \text{(by Lemma 11.3)} \\ & = (-1)^{1 + \#\bullet \text{ in } U'} \text{wt}(\Gamma^{(i)}), \end{aligned}$$

since by definition $\text{wt}(\Gamma^{(i)}) = \text{wt}(\Gamma) \cdot \text{wt}(\Gamma^{(i-1)})$. □

PROPOSITION 11.5. For $U \in B_{\lambda, \mu}^\alpha$,

$$\sum_{T \in \text{leaf}(\mathfrak{T}_U)} (-1)^{|\rho(T)/\lambda|+1} [U]\text{slide}_{\rho(T)/\lambda}(T) = \text{wt}(\alpha/\lambda)(-1)^{|\nu/\alpha|+1}, \quad (11.2)$$

where $\rho(T) \in \{\lambda\} \cup \lambda^+$ is the ‘inner shape’ of T ; that is, T has shape $\nu/\rho(T)$.

Proof. Take $U' = U$ in Lemma 11.4. □

Now, assume that $U \in B_{\lambda, \mu}^\nu$. The root of \mathfrak{T}_U contains no $\bullet_{(\ell(\mu)+1)_1}$ s. One leaf of \mathfrak{T}_U is U itself. This is the unique leaf not in Λ^+ . Let $\text{leaf}^*(\mathfrak{T}_U)$ be the collection of all other leaves.

PROPOSITION 11.6. For $U \in B_{\lambda, \mu}^\nu$,

$$\sum_{T \in \text{leaf}^*(\mathfrak{T}_U)} (-1)^{|\rho(T)/\lambda|+1} [U]\text{slide}_{\rho(T)/\lambda}(T) = 1 - \text{wt}(\nu/\lambda), \quad (11.3)$$

where $\rho(T) \in \lambda^+$ is the ‘inner shape’ of T ; that is, T has shape $\nu/\rho(T)$.

Proof. This is immediate from Proposition 11.5, since $\nu = \alpha$ and the contribution from the excluded leaf is 1. □

EXAMPLE 11.7. In Example 11.2, summing the weights below the left child of U gives $1 - (t_1/t_2)(t_3/t_5)$, in agreement with Lemma 11.3. Proposition 11.6 asserts in this case that

$$1 - \text{wt}(v/\lambda) = 1 - \frac{t_1}{t_5} = \left(1 - \frac{t_1}{t_2}\right) + \left(1 - \frac{t_3}{t_5}\right) - \left(1 - \frac{t_1}{t_2}\right)\left(1 - \frac{t_3}{t_5}\right) + \frac{t_1}{t_2} \frac{t_3}{t_5} \cdot \left(1 - \frac{t_2}{t_3}\right),$$

as the reader may verify. □

We recall that $\Lambda^+ = \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} \sum_{T \in B_{\rho, \mu}^v} T$. For $T \in B_{\rho, \mu}^v$, we write $T^{(1_1)}$ (see Section 7.2) for T , with \bullet_{1_1} in each box of ρ/λ .

Now, we set

$$P_{\mathcal{G}} := \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|-1} \sum_{T \in B_{\rho, \mu}^v} \text{swap}_{\mathcal{G}^-} \circ \text{swap}_{(\mathcal{G}^-)^-} \circ \dots \circ \text{swap}_{P_{1_1}}(T^{(1_1)}). \tag{11.4}$$

In particular, P_{1_1} is Λ^+ , where each T is replaced by $T^{(1_1)}$. By Lemma 7.6 and Proposition 7.7, each $P_{\mathcal{G}}$ is a formal sum of \mathcal{G} -good tableaux.

The main conclusion of this section is the following.

PROPOSITION 11.8. $P_{\mathcal{G}_{\max}^+}$ with all $\bullet_{\mathcal{G}_{\max}^+}$ s removed equals $\Lambda + \Lambda^-$.

Proof. By Corollary 7.11, each tableau appearing in $P_{\mathcal{G}_{\max}^+}$ (with $\bullet_{\mathcal{G}_{\max}^+}$ s removed) is a tableau in $\Lambda + \Lambda^-$. On the other hand, given any U appearing in $\Lambda + \Lambda^-$, we constructed the tree \mathfrak{T}_U in Section 10. By Proposition 10.7, the leaves of \mathfrak{T}_U are exactly those tableaux $T \in \Lambda^+$ such that $U \in \text{slide}_{\rho/\lambda}(T)$. It remains to show that $[U]P_{\mathcal{G}_{\max}^+} = 1 - \text{wt}(v/\lambda)$ if $U \in \Lambda$, and $[U]P_{\mathcal{G}_{\max}^+} = (-1)^{|v/\delta|+1} \text{wt}(\delta/\lambda)$ if $U \in \Lambda^-$ and the shape of U is δ/λ . These are precisely the statements of Propositions 11.6 and 11.5, respectively. □

12. Weight preservation

12.1. Fine tableaux and their weights. A tableau is *fine* if it is good or can be obtained from a good tableau by swapping some subset of its snakes; that is, it appears in the formal sum of tableaux resulting from this partial swap.

Let T be fine, and fix $x \in T$. Suppose that $\ell \in \underline{x}$. Define $\text{edgefactor}(\ell)$ as in Section 1.4; see (1.1). The *edge weight* $\text{edgewt}(T) := \prod_{\ell} \text{edgefactor}(\ell)$, where the product is over all (nonvirtual) edge labels of T .

Suppose that T is obtained by swapping some of the snakes of the good tableau S , and U is obtained from T by swapping the remaining snakes. We define the positions in T of a *virtual label* (\mathcal{H}) as follows. Consider a box x in column c .

If c intersects a snake in S that has been swapped in T , and that snake is not the upper snake described in Lemma 6.6(III), then $\mathcal{H} \in \underline{x}$ (in T) if and only if $\mathcal{H} \in \underline{x}$ (in U). Otherwise, $\mathcal{H} \in \underline{x}$ (in T) if and only if $\mathcal{H} \in \underline{x}$ (in S). We observe that if T is indeed good, this definition is clearly consistent with the definition of virtual labels in a good tableau.

Suppose that $\mathcal{H} \in \underline{x}$. If $\text{label}_T(x)$ is marked and each $\mathcal{F} \in \underline{x}$ with $\mathcal{F} \prec \mathcal{H}$ is marked, then

$$\begin{aligned} \text{virtualfactor}_{\underline{x} \in T}(\mathcal{H}) &:= -\text{edgfactor}_{\underline{x} \in T}(\mathcal{H}) \\ &= \frac{t_{\text{Man}(x)}}{t_{r+N_{\mathcal{H}+1}-\text{family}(\mathcal{H})+\text{Man}(x)}} - 1. \end{aligned} \tag{12.1}$$

Otherwise,

$$\begin{aligned} \text{virtualfactor}_{\underline{x} \in T}(\mathcal{H}) &:= 1 - \text{edgfactor}_{\underline{x} \in T}(\mathcal{H}) \\ &= \frac{t_{\text{Man}(x)}}{t_{r+N_{\mathcal{H}+1}-\text{family}(\mathcal{H})+\text{Man}(x)}}. \end{aligned} \tag{12.2}$$

The *virtual weight* $\text{virtualwt}(T)$ is $\prod_{\mathcal{L}} \text{virtualfactor}(\mathcal{L})$, where the product is over all instances of virtual labels.

We call $x \in T$ *productive* if any of the following holds:

- (P.1) $\text{label}_T(x) < \text{label}_T(x^\rightarrow)$ or $x^\rightarrow \notin T$;
- (P.2) $\bullet_{i_{k+1}} \in x$, $i_k \in x^\leftarrow$, $i_{k+1} \in \underline{x}$, and either $\text{family}(\text{label}(x^\rightarrow)) \neq i$ or $x^\rightarrow \notin T$;
- (P.3) $\mathcal{H} \in x$, $\bullet_g \in x^\rightarrow$, and $\underline{x^\rightarrow}$ does not contain a label of the same family as \mathcal{H} ;
or
- (P.4) $i_k \in x$, $i_{k+1} \in x^\rightarrow$ and $\bullet_{i_{k+1}} \in x^{\rightarrow\uparrow}$, with x not SouthEast of a $\bullet_{i_{k+1}}$.

We define $\text{boxfactor}(x)$ and *box weight* $\text{boxwt}(T) = \prod_x \text{boxfactor}(x)$ as in Section 1.4, specifically (1.2), with the addendum that $\bullet_{\mathcal{H}} \in x$ is evaluated like $\mathcal{H} \in x$.

EXAMPLE 12.1. \bullet The right two boxes of $\boxed{1_1 1_2 2_1}$ are productive, by (P.1). The left box is not productive.

- \bullet The left box of $\boxed{1_1 \bullet_{1_2} 1_2}$ is not productive. The right box is productive, by (P.2).

- The first and third boxes of $\begin{array}{|c|c|c|} \hline 1_1 & \bullet_{1_2} & 2_1 \\ \hline \end{array}$ are productive, by (P.3) and (P.1), respectively. The middle box is not productive; a box with \bullet_{1_2} is productive only if (P.2) holds.
- The right box of the second row in both $\begin{array}{|c|c|} \hline \bullet_{1_2} & \\ \hline 1_1 & 1_2 \\ \hline \end{array}$ and $\begin{array}{|c|c|} \hline \bullet_{1_1} & \\ \hline 1_1 & 1_2 \\ \hline \end{array}$ is productive, by (P.1). The left box in the second row is productive only in the first case, by (P.4). \square

Finally, the *weight* is

$$\text{wt}(T) := (-1)^{d(T)} \text{edgewt}(T) \cdot \text{virtualwt}(T) \cdot \text{boxwt}(T),$$

where $d(T) = \sum_{\mathcal{G}} (|\mathcal{G}| - 1)$, the sum is over genes \mathcal{G} , and $|\mathcal{G}|$ is the (multiset) cardinality of \mathcal{G} (not including virtual labels). We view wt as a $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ -linear operator of formal sums of tableaux.

By Lemma 5.3, bundled tableaux are good and hence also fine. Hence, for a bundled tableau B , we have two *a priori* distinct notions of $\text{wt } B$. The following lemma justifies our failure to distinguish these notationally.

LEMMA 12.2. *For B a bundled tableau, $\text{wt } B$ as a fine tableau equals $\text{wt } B$ as a bundled tableau.*

Proof. By definition, the two notions of $\text{edgewt}(B)$ coincide, as do the two notions of $d(B)$. Since B has no \bullet s, only (P.1) is available to effect productivity. Hence, the two notions of productive boxes coincide, and thus, by definition, so too do the two notions of $\text{boxwt}(B)$. As remarked above, the locations of virtual labels are the same, whether we think of B as bundled or fine. By Lemma 2.4, $\text{wt } B$ as a bundled tableau is

$$(-1)^{d(B)} \text{edgewt}(B) \text{boxwt}(B) \prod_{\textcircled{\ell}} (1 - \text{edgefactor}(\ell)),$$

where the product is over all instances of virtual labels, and $\text{edgefactor}(\ell)$ means the factor that would be given by ℓ in $\textcircled{\ell}$'s place. Since B is bundled, it has no marked labels. Hence, $\text{virtualwt}(B)$ is calculated using only (12.2), not (12.1). Thus, $\text{virtualwt}(B) = \prod_{\textcircled{\ell}} (1 - \text{edgefactor}(\ell))$, and the lemma follows. \square

12.2. Main claim about weight preservation.

PROPOSITION 12.3. (I) $\text{wt } P_{1_1} = \text{wt } \Lambda^+$.

(II) For every \mathcal{G} , $\text{wt } P_{\mathcal{G}} = \text{wt } P_{1_1}$.

(III) $\text{wt } P_{\mathcal{G}_{\max}^+} = \text{wt } \Lambda + \text{wt } \Lambda^-$.

Proof. We will first prove the easier statements (I) and (III).

(I): Suppose that $T \in B_{\rho, \mu}^v$ for some $\rho \in \lambda^+$. It is enough to show that $\text{wt } T = \text{wt } T^{(1)}$. Certainly, $\text{edgewt}(T) = \text{edgewt}(T^{(1)})$ and $d(T) = d(T^{(1)})$. The addition of \bullet_{1_i} s preserves the virtual labels' locations, so that $\text{virtualwt}(T) = \text{virtualwt}(T^{(1)})$.

A productive box in T is also productive in $T^{(1)}$ and has the same boxfactor . Suppose that x is a productive box of $T^{(1)}$ that is not productive in T . It satisfies one of (P.1)–(P.4). If x satisfies (P.1), it is productive in T . If it satisfies (P.2), then $\bullet_{1_1} \in x$, and x^{\leftarrow} contains a label, contradicting $x^{\leftarrow} \in \rho$. If it satisfies (P.3), then $x^{\rightarrow} \in \rho$, contradicting that x contains a label. Finally, if x satisfies (P.4), then $\bullet_{i_{k+1}} \in x^{\rightarrow\uparrow}$ and $i_k \in x$. However, every \bullet in $T^{(1)}$ is \bullet_{1_1} . Hence, $i_{k+1} = 1_1$, which is impossible since 1_0 is not a label in our alphabet. Thus, the productive boxes of T and $T^{(1)}$ are the same, and have the same respective boxfactors . Therefore, $\text{wt } T = \text{wt } T^{(1)}$.

(III): Suppose that $U \in P_{\mathcal{G}_{\max}^+}$ and let \tilde{U} be given by deleting each $\bullet_{\mathcal{G}_{\max}^+}$. Proposition 11.8 states that $P_{\mathcal{G}_{\max}^+}$ with all $\bullet_{\mathcal{G}_{\max}^+}$ s removed equals $\Lambda + \Lambda^-$. Thus, it suffices to show that $\text{wt } U = \text{wt } \tilde{U}$. Clearly, $\text{edgewt}(U) = \text{edgewt}(\tilde{U})$ and $d(U) = d(\tilde{U})$. One checks that the virtual labels of U and the virtual labels of \tilde{U} appear in the same places. Hence, $\text{virtualwt}(U) = \text{virtualwt}(\tilde{U})$.

Suppose that x is productive in U . Then, it satisfies one of (P.1)–(P.4). If x satisfies (P.1) in U , then it satisfies (P.1) in \tilde{U} . Now, x cannot satisfy (P.2) in U , since if it did, $\bullet_{\mathcal{G}_{\max}^+} \in x$, and x contains a label, contradicting Lemma 7.9. If x satisfies (P.3) in U , then it satisfies (P.1) in \tilde{U} . If x satisfies (P.4) in U , then $\bullet_{\mathcal{G}_{\max}^+} \in x^{\rightarrow\uparrow}$ but is not an outer corner, again contradicting Lemma 7.9. Thus, if x is productive in U , it is productive in \tilde{U} . Conversely, if x is productive in \tilde{U} , it satisfies (P.1), since there are no $\bullet_{\mathcal{G}_{\max}^+}$ s in \tilde{U} . Hence, x satisfies (P.1) or (P.3) in U . Thus, the productive boxes of U and \tilde{U} are the same. These boxes have the same boxfactors . Thus, $\text{boxwt}(U) = \text{boxwt}(\tilde{U})$.

(II): We induct on \mathcal{G} with respect to \prec . The base case $\mathcal{G} = 1_1$ is trivial. The inductive hypothesis is that $\text{wt } P_{\mathcal{G}} = \text{wt } P_{1_1}$. Our inductive step is to show that $\text{wt } P_{\mathcal{G}^+} = \text{wt } P_{\mathcal{G}}$.

We consider the set

$$\text{Snakes}_{\mathcal{G}} = \{S \text{ is a snake in } T : [T]P_{\mathcal{G}} \neq 0\}.$$

We emphasize that each $S \in \text{Snakes}_{\mathcal{G}}$ refers to a particular instance of a snake in a specific tableau $T \in P_{\mathcal{G}}$. In particular, $\text{Snakes}_{\mathcal{G}}$ is not a multiset.

For $\mathcal{B} \subseteq \text{Snakes}_{\mathcal{G}}$, we define $\text{swapset}_{\mathcal{B}}(T)$ to be the formal sum of fine tableaux obtained by swapping each snake of \mathcal{B} that appears in T (done in any order, as permitted by Lemma 7.5).

We will construct m subsets \mathcal{B}_i such that (D.1) and (D.2) below hold.

(D.1) We have a disjoint union $\text{Snakes}_G = \bigsqcup_{1 \leq i \leq m} \mathcal{B}_i$.

(D.2) For every $1 \leq i \leq m$ and $J \subseteq \{1, \dots, \hat{i}, \dots, m\}$, let $\mathcal{B}_J := \bigcup_{j \in J} \mathcal{B}_j$. Then,

$$\begin{aligned} & \sum_{T \in \Gamma_i} [T] P_G \cdot \text{wt}(\text{swapset}_{\mathcal{B}_J}(T)) \\ &= \sum_{T \in \Gamma_i} [T] P_G \cdot \text{wt}(\text{swapset}_{\mathcal{B}_i} \circ \text{swapset}_{\mathcal{B}_J}(T)), \end{aligned} \quad (12.3)$$

where $\Gamma_i := \{T \in P_G : T \text{ contains a snake from } \mathcal{B}_i\}$.

CLAIM 12.4. *The existence of $\{\mathcal{B}_i\}$ satisfying (D.1) and (D.2) implies $\text{wt}(P_{G^+}) = \text{wt}(P_G)$.*

Proof. By Lemma 7.5, snakes may be swapped in any order, so we choose an arbitrary ordering of the blocks \mathcal{B}_i . By (D.1), $P_{G^+} := \text{swap}_G(P_G) = \text{swapset}_{\mathcal{B}_m} \circ \dots \circ \text{swapset}_{\mathcal{B}_1}(P_G)$. Thus,

$$\begin{aligned} \text{wt}(P_{G^+}) &= \text{wt}(\text{swap}_G(P_G)) \\ &= \text{wt}(\text{swapset}_{\mathcal{B}_m} \circ \dots \circ \text{swapset}_{\mathcal{B}_1}(P_G)) \\ &= \text{wt}(\text{swapset}_{\mathcal{B}_{m-1}} \circ \dots \circ \text{swapset}_{\mathcal{B}_1}(P_G)). \end{aligned}$$

Here, we have just used (12.3) from (D.2) together with linearity of wt and $\text{swapset}_{\mathcal{B}_i}$ and the triviality $\text{swapset}_{\mathcal{B}_J}(T) = \text{swapset}_{\mathcal{B}_i} \circ \text{swapset}_{\mathcal{B}_J}(T)$ for $T \notin \Gamma_i$. Repeating this argument $m - 1$ further times, we obtain the desired equality with $\text{wt}(P_G)$. \square

In order to provide the desired decomposition, as explained at the beginning of Section 12.3, we need to first construct certain ‘pairing’ maps. These are given in Section 12.3. Given these, the description of the decomposition satisfying (D.1) and (D.2) is relatively straightforward and is found in Appendix C. \square

12.3. Pairing maps. The basic point of this subsection is as follows. The proof of Proposition 12.3(II) would amount to only straightforward computational checks *if* it were true that for any snake $S \in \text{Snakes}_G$ (in the notation of Section 12.2) and fine tableau T containing S , we had $\text{wt}(T) = \text{wt}(\text{swapset}_{\{S\}}(T))$. However, this is not true in general (see the example below). What we do here is to identify precisely those situations where this fails and to establish pairings of said situations. Now, in Appendix C, where we give the desired decomposition $\text{Snakes}_G = \bigsqcup_{1 \leq i \leq m} \mathcal{B}_i$, ‘most’

\mathcal{B}_i s are actually singletons. In those cases, the key check of (12.3) is indeed straightforward. However, those \mathcal{B}_i that are not singletons exactly correspond to the paired cases classified here. Moreover, the checks of (12.3) in these more complicated situations are also straightforward given the lemmas developed in this subsection. In summary, we regard this subsection as containing our main ingredients in the proof of Proposition 12.3(II), whereas Appendix C contains the mechanical verifications needed.

To give a paradigmatic example, let $k = 1, n = 3$ and $(\lambda, \mu, \nu) = (\emptyset, (1), (2))$; thus, $\lambda^+ = \{(1)\}$. Here,

$$P_{1_1} = \begin{array}{|c|c|} \hline \bullet_{1_1} & 1_1 \\ \hline \hline \end{array} + \begin{array}{|c|c|} \hline \bullet_{1_1} & 1_1 \\ \hline \hline \end{array} \quad \text{and} \quad P_{1_2} = 0 + \frac{t_1}{t_2} \begin{array}{|c|c|} \hline 1_1 & \bullet_{1_2} \\ \hline \hline \end{array},$$

where we have used the miniswaps H4 and H5.3, respectively. It is not true that $\text{wt}(\begin{array}{|c|c|} \hline \bullet_{1_1} & 1_1 \\ \hline \hline \end{array}) = \text{wt}(0)$. Nor is it true that $\text{wt}(\begin{array}{|c|c|} \hline \bullet_{1_1} & 1_1 \\ \hline \hline \end{array}) = (t_1/t_2)\text{wt}(\begin{array}{|c|c|} \hline 1_1 & \bullet_{1_2} \\ \hline \hline \end{array})$. Thus, Proposition 12.3(II) does not hold in this case for the ‘naive reason’. Instead, our third pairing below $(\mathcal{C}_3, \mathcal{C}'_3)$, applied to this case, pairs the two tableaux appearing in P_{1_1} . Moreover, Proposition 12.17 asserts that the coefficients of these two paired tableaux are the same in P_{1_1} (in this case, the coefficients being 1). Finally, in Appendix C, at the needed generality, the routine check that $\text{wt}(P_{1_1}) = \text{wt}(P_{1_2})$ is discussed twice. The first time is Case 4 for the H4 tableau of P_{1_1} ; the check itself is Claim C.3. The second time is for its pairing, the H5.3 tableau of P_{1_1} , which in fact refers back to the Case 4 analysis. All other pairings discussed below are handled similarly.

We now begin the description of the pairing maps.

Let $G_{\lambda, \mu}^{\nu}(\mathcal{G})$ be the set of \mathcal{G} -good tableaux of shape ν/λ and content μ . For $\mathcal{Q} < \mathcal{G}$ and $T \in G_{\lambda, \mu}^{\nu}(\mathcal{G})$, let $\mathfrak{R}_{\mathcal{Q}}(T) := \{V \in \text{revswap}_{\mathcal{Q}^+} \circ \dots \circ \text{revswap}_{\mathcal{G}}(T)\}$.

LEMMA 12.5. For any genes $\mathcal{Q} < \mathcal{G}$ and any tableau $T \in G_{\lambda, \mu}^{\nu}(\mathcal{G})$,

$$\mathfrak{R}_{\mathcal{Q}}(T) = \{W \in G_{\lambda, \mu}^{\nu}(\mathcal{Q}) : T \in \text{swap}_{\mathcal{G}} \circ \dots \circ \text{swap}_{\mathcal{Q}}(W)\}.$$

Proof. This is immediate from Proposition 9.11, noting that, by Lemmas 7.8 and 9.4 and Propositions 7.7 and 9.5, both forward and reverse swaps preserve goodness and content. □

For $\ell \geq k$, let $\mathcal{S}_1(\ell)$ be the subset of tableaux in $G_{\lambda, \mu}^{\nu}(i_k)$ with a box x such that $\bullet_{i_k} \in x, \bullet_{i_k} \in x^{\rightarrow \uparrow}, i_{\ell+1} \in x^{\rightarrow}$ and $(i_{\ell}) \in \underline{x}$; that is, locally the

tableau is $\mathcal{C}_1(\ell) = \begin{array}{|c|c|} \hline & \bullet_{i_k} \\ \hline \bullet_{i_k} & i_{\ell+1} \\ \hline \end{array}$ (with possibly additional edge labels), where x is the

southwestmost depicted box. Let $\mathcal{S}'_1(\ell)$ be the subset of tableaux in $G_{\lambda, \mu}^v(i_k)$ with a box x such that $i_\ell \in x$, $i_{\ell+1} \in x^\rightarrow$, $\bullet_{i_k} \in x^{\rightarrow\uparrow}$, i_ℓ appears outside of x and no

\bullet_{i_k} appears West of x in x 's row. Locally, the tableau is $\mathcal{C}'_1(\ell) = \begin{array}{|c|c|} \hline & \bullet_{i_k} \\ \hline i_\ell & i_{\ell+1} \\ \hline \end{array}$ (with possibly additional edge labels). Actually, in our application in Appendix C, we will only need the case $k = \ell$ of what follows. However, in order to establish this case, we use an inductive argument that involves $\ell \geq k$; see specifically (12.6).

LEMMA 12.6. *If $T \in \mathcal{S}_1(\ell)$ (respectively, $\mathcal{S}'_1(\ell)$), there is a unique $\mathcal{C}_1(\ell)$ (respectively, $\mathcal{C}'_1(\ell)$) that it contains.*

Proof. Let x be the lower-left box of any fixed choice of $\mathcal{C}_1(\ell)$ in T . Since $(i_\ell) \in \underline{x}$, the $i_{\ell+1} \in x^\rightarrow$ is westmost in T , by (G.6). Hence, this configuration is unique. The argument for the other claim is the same, except that we replace ' $(i_\ell) \in \underline{x}$ ' with ' $i_\ell \in x$ '. □

For $T \in \mathcal{S}_1(\ell)$, let $\phi_{1,\ell}(T)$ be the same tableau with the unique $\mathcal{C}_1(\ell)$ replaced by $\mathcal{C}'_1(\ell)$. (By this we mean that we delete the labels specified in $\mathcal{C}_1(\ell)$ and add the labels specified in $\mathcal{C}'_1(\ell)$; any additional edge labels in T are unchanged.)

LEMMA 12.7. *The map $\phi_{1,\ell} : \mathcal{S}_1(\ell) \rightarrow \mathcal{S}'_1(\ell)$ is a bijection.*

Proof. Since ℓ is fixed, we may, for simplicity, drop ℓ from the notation of these sets and maps throughout this argument.

Let $\phi_1^{-1} : \mathcal{S}'_1 \rightarrow \mathcal{S}_1$ be the putative inverse of ϕ_1 , defined by replacing \mathcal{C}'_1 in a $T \in \mathcal{S}'_1$ by \mathcal{C}_1 . We are done once we show that ϕ_1 and ϕ_1^{-1} are well-defined since the maps are clearly injective and are mutually inverse.

Let x be the southwestmost box in the unique (by Lemma 12.6) \mathcal{C}_1 in T .

(The map ϕ_1 is well-defined): Let $T \in \mathcal{S}_1$. We only need that $\phi_1(T)$ is good. Conditions (G.1) and (G.2) hold trivially in $\phi_1(T)$. Condition (G.3) holds if x^\leftarrow is empty. Suppose that $\mathcal{F} \in x^\leftarrow$. By T 's (G.9), $\mathcal{F} < i_k$. Hence, $\mathcal{F} < i_\ell$, and (G.3) holds in $\phi_1(T)$. The $(i_\ell) \in \underline{x}$ in T shows that $\phi_1(T)$ satisfies (G.4), (G.6) and (G.8). Conditions (G.5), (G.7), (G.9), (G.11) hold trivially. Since $i_{\ell+1} \in x^\rightarrow$ is not marked, by Lemma 5.4(II), there is no marked label in T in x 's row, so (G.10) and (G.13) hold for $\phi_1(T)$. For (G.12), suppose that T has labels ℓ, ℓ' that violate (G.12) in $\phi_1(T)$. Since ℓ must be northWest of x , by T 's (G.9), $\ell < i_k$. Since ℓ' must be southeast of x , by T 's (G.3), (G.4) and (G.11), $\ell' > i_{\ell+1}$. Hence, $\text{family}(\ell) = \text{family}(\ell') = i$. If ℓ is North of x , then, by (G.4), the box of x 's row directly below ℓ contains a label that violates T 's (G.9). By T 's (G.4), ℓ' is not South of x^\rightarrow . Hence, ℓ and ℓ' are box labels in the row of x , and no violation of $\phi_1(T)$'s (G.12) occurs.

(The map ϕ_1^{-1} is well-defined): Let $T' \in \mathcal{S}'_1$. We must show that (G.7) and (G.13) hold in $\phi_1^{-1}(T')$ and that (G.1)–(G.6) and (G.8)–(G.12) hold even if the virtual label is replaced by a nonvirtual one (see (V.1)–(V.3)). Conditions (G.1), (G.3)–(G.10), (G.12) and (G.13) are trivial to verify. To verify (G.2) for $\phi_1^{-1}(T')$, it suffices to show that T' has no \bullet_{i_k} South of x in the same column, or West of x in the same row. Condition (G.9) for T' rules out the possibility of \bullet_{i_k} South of x in the same column of T' . By definition, there is no \bullet_{i_k} West of x in the same row. To see (G.11) for $\phi_1^{-1}(T')$, we check that there is no marked label \mathcal{F}^\dagger in the column of x . Such a label cannot appear North of x in T' , by Lemma 5.4 and (G.2), considering the $\bullet_{i_k} \in x \rightarrow^\dagger$. By (G.4), it cannot appear South of x in T' either. \square

PROPOSITION 12.8. For each $T \in \mathcal{S}_1(\ell)$, $[T]P_{i_k} = -[\phi_{1,\ell}(T)]P_{i_k}$.

Proof. Let $T^\dagger := \phi_{1,\ell}(T)$.

Special case $k = 1$: Let \tilde{T} be the tableau obtained from T by deleting:

- all labels of family i and greater;
- all marked labels; and
- all boxes containing a deleted box label.

It should be noticed that any label SouthEast of a deleted label or a \bullet_{i_1} will have been deleted.

We also reindex the genes so that the subscripts of each family form an initial segment of $\mathbb{Z}_{>0}$. (This reindexing is only possibly needed if T contains a marked label.) We leave \bullet_{i_1} s in place. In the same way, we produce \tilde{T}^\dagger from T^\dagger . By definition of $\phi_{1,\ell}$, \tilde{T} has one more \bullet_{i_1} than \tilde{T}^\dagger , and otherwise the two tableaux are exactly the same (the family i labels of $\mathcal{C}_1(\ell)$ and $\mathcal{C}'_1(\ell)$ having been deleted).

Ignoring \bullet_{i_1} s, \tilde{T} and \tilde{T}^\dagger are of some common skew shape θ/λ . If we include the \bullet_{i_1} s, their respective total shapes are some ω/λ and ω^\dagger/λ , where $\omega, \omega^\dagger \in \theta^+$.

CLAIM 12.9. $\tilde{T} \in G_{\lambda, \tilde{\mu}}^\omega(i_1)$ and $\tilde{T}^\dagger \in G_{\lambda, \tilde{\mu}}^{\omega^\dagger}(i_1)$, where $\tilde{\mu}$ is a partition (for example, if T has no marked labels, then $\tilde{\mu} := (\mu_1, \mu_2, \dots, \mu_{i-1})$). Thus, \tilde{T} and \tilde{T}^\dagger (with \bullet_{i_1} s removed) are in $B_{\lambda, \tilde{\mu}}^\theta$.

Proof. We prove the claim for \tilde{T} ; the proof for \tilde{T}^\dagger is essentially the same.

Clearly, (G.1)–(G.7), (G.9) and (G.12) for \tilde{T} are inherited from the assumption that T is good. Conditions (G.10), (G.11) and (G.13) are vacuous for \tilde{T} . It remains to show that (G.8) holds for \tilde{T} (which moreover implies that $\tilde{\mu}$ is a partition).

Suppose that \tilde{T} fails (G.8). Then, there is a least q such that \tilde{T} has a ballotness violation between families q and $q + 1$. That is, in some genotype G of \tilde{T} , there are more labels of family $q + 1$ than of family q in some initial segment of $\text{word}(G)$. Since we have deleted all labels of family i and greater, $q < i - 1$. By failure of (G.8), either there exist labels q_r and $(q + 1)_s$ of \tilde{T} with $N_{q_r} = N_{(q+1)_s}$ such that $(q + 1)_s$ appears before q_r in $\text{word}(G)$, or else there is a label $(q + 1)_s$ of \tilde{T} with $N_{(q+1)_s} > N_{q_v}$ for all v . Let $q_{r'}$ (if q_r exists) and $(q + 1)_{s'}$ be the corresponding labels of T . We assert, in the former case, that $N_{q_{r'}} \leq N_{(q+1)_{s'}}$ in T . In the latter case, we assert that $N_{(q+1)_{s'}} > N_{q_{v'}}$ in T for all v' . Either of these inequalities contradicts T 's (G.8).

To see these assertions, suppose that q_h is a gene of T that is entirely deleted in the construction of \tilde{T} (that is, every instance of q_h in T is marked). Consider an instance of q_h in T in \mathbf{x} or $\underline{\mathbf{x}}$. Since this q_h is marked and $q < i - 1$, by Lemma 5.5, we know that T has some nonvirtual and marked $(q + 1)_z^!$ $\in \underline{\mathbf{x}}$ with $N_{q_h} = N_{(q+1)_z}$. By T 's (G.7), there is no $(q + 1)_z$ West of \mathbf{x} in T . By T 's Lemma 5.6, there is no $(q + 1)_z$ East of \mathbf{x} in T . Hence, the $(q + 1)_z^!$ $\in \underline{\mathbf{x}}$ is the only $(q + 1)_z$ in T . Since it is marked, the gene $(q + 1)_z$ is entirely deleted in \tilde{T} . By this argument, if $q_{\hat{h}}$ is any other gene of T that is entirely deleted in the construction of \tilde{T} , there is a distinct $(q + 1)_{\hat{z}}$ with $N_{q_{\hat{h}}} = N_{(q+1)_{\hat{z}}}$ that is also entirely deleted in \tilde{T} . Hence, $N_{q_{r'}} \geq N_{(q+1)_{s'}}$ or $N_{(q+1)_{s'}} > N_{q_{v'}}$ in T for all v' , as asserted.

The last sentence of the claim follows from the first by Lemma 7.10, since no genetic label is southeast of a \bullet_{i_1} . □

In view of Claim 12.9, it makes sense to speak of $\mathfrak{T}_{\tilde{T}}$ and of $\mathfrak{T}_{\tilde{T}^\dagger}$. By Proposition 11.5,

$$\sum_{L \in \text{leaf}(\mathfrak{T}_{\tilde{T}})} (-1)^{|\rho(L)/\lambda|+1} [\tilde{T}] \text{slide}_{\rho(L)/\lambda}(L) = (-1)^{1+\#\text{of } \bullet\text{'s in } \tilde{T}} \cdot \text{wt}(\theta/\lambda). \tag{12.4}$$

Similarly,

$$\sum_{L \in \text{leaf}(\mathfrak{T}_{\tilde{T}^\dagger})} (-1)^{|\rho(L)/\lambda|+1} [\phi_{1,\ell}(\tilde{T})] \text{slide}_{\rho(L)/\lambda}(L) = (-1)^{1+\#\text{of } \bullet\text{'s in } \tilde{T}^\dagger} \cdot \text{wt}(\theta/\lambda). \tag{12.5}$$

In particular, these quantities differ by a factor of -1 .

By inspection of the reverse miniswaps, revswap_{a_q} for $1 \leq a \leq i - 1$ does not affect any labels of family i or greater, or any labels that are marked in T . Hence, one sees that $\text{revswap}_{1_2} \circ \dots \circ \text{revswap}_{(i-1)_{\mu_{i-1}}} \circ \text{revswap}_{i_1}(T)$ (respectively, T^\dagger) is the same as $\text{revswap}_{1_2} \circ \dots \circ \text{revswap}_{(i-1)_{\mu_{i-1}}} \circ \text{revswap}_{i_1}(\tilde{T})$ (respectively, \tilde{T}^\dagger) followed by adding back the labels of

$T \setminus \tilde{T}$ (respectively, $T^\dagger \setminus \tilde{T}^\dagger$). Therefore, by our comparison of (12.4) and (12.5) above,

$$[T]P_{i_1} = (-1)^{1+\#\text{of } \bullet\text{'s in } \tilde{T}} \cdot \text{wt}(\theta/\lambda) = -[T^\dagger]P_{i_1},$$

as desired.

Reduction to the $k = 1$ case: In the calculation of $\text{revswap}_{i_2} \circ \text{revswap}_{i_3} \circ \dots \circ \text{revswap}_{i_k}(T)$ and $\text{revswap}_{i_2} \circ \text{revswap}_{i_3} \circ \dots \circ \text{revswap}_{i_k}(T^\dagger)$, it is straightforward by inspection that each reverse miniswap involving either \bullet of $\mathcal{C}_1(\ell)$ or the \bullet of $\mathcal{C}'_1(\ell)$ is L1.2. Therefore, there exists an instance of $\mathcal{C}_1(\ell)$ in each $W \in \mathfrak{R}_{i_1}(T)$ and an instance of $\mathcal{C}'_1(\ell)$ in each $W' \in \mathfrak{R}_{i_1}(T^\dagger)$. By Lemma 12.6, these instances are unique. Extending ϕ_1 linearly, since T and T^\dagger are the same outside of the regions $\mathcal{C}_1(\ell)$ and $\mathcal{C}'_1(\ell)$, it is easy to see inductively that for all $2 \leq q \leq k$,

$$\phi_{1,\ell}(\text{revswap}_{i_q} \circ \dots \circ \text{revswap}_{i_k}(T)) = \text{revswap}_{i_q} \circ \dots \circ \text{revswap}_{i_k}(T^\dagger). \tag{12.6}$$

In particular, $\phi_{1,\ell}$ bijects $\mathfrak{R}_{i_1}(T)$ with $\mathfrak{R}_{i_1}(T^\dagger)$.

Let $V \in \mathfrak{R}_{i_1}(T)$. By the $k = 1$ case above, $[V]P_{i_1} = -[\phi_{1,\ell}(V)]P_{i_1}$. Moreover, when we apply $\text{swap}_{i_k}^- \circ \dots \circ \text{swap}_{i_1}$ to V and $\phi_{1,\ell}(V)$, each miniswap involving a \bullet of $\mathcal{C}_1(\ell)$ or $\mathcal{C}'_1(\ell)$ is H3. Hence, $[T] \text{swap}_{i_k}^- \circ \dots \circ \text{swap}_{i_1}(V) = [T^\dagger] \text{swap}_{i_k}^- \circ \dots \circ \text{swap}_{i_1}(\phi_{1,\ell}(V))$. Thus, by Lemma 12.5, $[T]P_{i_k} = -[T^\dagger]P_{i_k}$. □

Let \mathcal{S}_2 be the subset of tableaux in $G_{\lambda,\mu}^v(i_k)$ with a box x such that $\bullet_{i_k} \in x$,

$\bullet_{i_k} \in x^{\rightarrow\uparrow}$, $i_{k+1} \in x^{\rightarrow}$ and $i_k \in \underline{x}$; that is, locally, the tableau is $\mathcal{C}_2 = \begin{array}{|c|c|} \hline & \bullet_{i_k} \\ \hline \bullet_{i_k} & i_{k+1} \\ \hline \end{array}$ (with possibly additional edge labels). Let \mathcal{S}'_2 be the subset of tableaux in $G_{\lambda,\mu}^v(i_k)$ with a box x such that $i_k \in x$, $i_{k+1} \in x^{\rightarrow}$, $\bullet_{i_k} \in x^{\rightarrow\uparrow}$, no i_k appears outside of x and

no \bullet_{i_k} appears West of x in x 's row. Locally, the tableau is $\mathcal{C}'_2 = \begin{array}{|c|c|} \hline & \bullet_{i_k} \\ \hline i_k & i_{k+1} \\ \hline \end{array}$ (with possibly additional edge labels).

LEMMA 12.10. *If $T \in \mathcal{S}_2$ (respectively, \mathcal{S}'_2), there is a unique \mathcal{C}_2 (respectively, \mathcal{C}'_2) that it contains.*

Proof. Let x be the southwestmost box of a \mathcal{C}_2 in T . By (G.7), the $i_k \in \underline{x}$ is the westmost i_k in T ; hence, this configuration is unique. The claim about \mathcal{C}'_2 is clear since the i_k is unique. □

For $T \in \mathcal{S}_2$, let $\phi_2(T)$ be T with the unique \mathcal{C}_2 replaced by \mathcal{C}'_2 .

LEMMA 12.11. *The map $\phi_2 : \mathcal{S}_2 \rightarrow \mathcal{S}'_2$ is a bijection.*

Proof. This may be proved almost exactly as Lemma 12.7. □

PROPOSITION 12.12. *For each $T \in \mathcal{S}_2$, $[T]P_{i_k} = -[\phi_2(T)]P_{i_k}$.*

Proof. Let $T^\dagger := \phi_2(T)$. Let x be the southwestmost box of \mathcal{C}_2 in T . Then, x is also the southwestmost box of \mathcal{C}'_1 in T^\dagger .

Special case $k = 1$: The proof is *verbatim* the argument for the $k = 1$ case of Proposition 12.8.

Reduction to the $k = 1$ case: Suppose that $k > 1$. Let \mathcal{Z} be the set of boxes in an i_k -good tableau that either (1) contain \bullet_{i_k} or (2) contain a label \mathcal{F} with $i_1 \preceq \mathcal{F} \preceq i_{k-1}$ and are not southeast of a \bullet_{i_k} . Call an edge-connected component of \mathcal{Z} an i_k -walkway. We will now apply the development of i -walkways, from Sections 10 and 11, in a slightly modified form to the i_k -walkways. To be more precise, Lemmas 10.1, 10.2, 10.3 and 10.4 are true after replacing ‘ $(i + 1)_1$ ’ with ‘ i_k ’, and ‘ i -walkway’ with ‘ i_k -walkway’. In addition, Claim 11.1 holds *verbatim*. The proofs are trivial modifications of those given.

Let W be the i_k -walkway of T containing x (W includes all edges of boxes in W). Let W^\dagger be the analogous i_k -walkway of T^\dagger . It should be noted that W and W^\dagger have the same skew shape.

CLAIM 12.13. *Let \mathbb{S} , \mathbb{S}' and \mathbb{T} be, respectively, the sets of reversals of W , W^\dagger and W^c (the complement of W) under $\text{revswap}_{i_2} \circ \dots \circ \text{revswap}_{i_k}$. Then,*

- (I) $\mathfrak{R}_{i_1}(T) = \{V \in G_{\lambda, \mu}^v(i_1) : V|_W \in \mathbb{S}, V|_{W^c} \in \mathbb{T}\};$
- (II) $\mathfrak{R}_{i_1}(T^\dagger) = \{V' \in G_{\lambda, \mu}^v(i_1) : V'|_{W^\dagger} \in \mathbb{S}', V'|_{(W^\dagger)^c} \in \mathbb{T}\}.$

Proof. We prove only (I), as the proof of (II) is similar (using $T|_{W^c} = T^\dagger|_{W^c}$). We fix $2 \leq h \leq k$ and let L be a ladder of $A \in \mathfrak{R}_{i_h}(T)$. The ladder L contains only \bullet_{i_h} and unmarked i_{h-1} . Each of the boxes x of L is in \mathcal{Z} : This is clear if $h = k$, and follows for smaller h by induction. Thus, $L \subseteq \mathcal{Z}$. Therefore, since L is edge-connected, it sits inside an edge-connected component of \mathcal{Z} . Thus, since W is one such component, reverse swapping acts independently on W and W^c . □

Case 1 (W (and hence W^\dagger) has a single row): By construction, x is the eastmost box of W and W^\dagger . By Lemma 10.3(II), for every $V \in \mathfrak{R}_{i_1}(T)$, $V|_W = R'$. By Lemma 10.3(III), for every $V' \in \mathfrak{R}_{i_1}(T^\dagger)$, $V'|_{W^\dagger} \in \{R, R'\}$, where this R' is the same as in the previous sentence.

Since R' is the unique reversal of W and is a reversal of W^\dagger , we have $\mathfrak{R}_{i_1}(T) \subseteq \mathfrak{R}_{i_1}(T^\dagger)$, by Claim 12.13. Let $\iota : \mathfrak{R}_{i_1}(T) \rightarrow \mathfrak{R}_{i_1}(T^\dagger)$ be the inclusion map. Let $f : \mathfrak{R}_{i_1}(T) \rightarrow \mathfrak{R}_{i_1}(T^\dagger)$ be the map given by replacing the R' occupying the region W with R . Again appealing to Claim 12.13, we see that these maps are well-defined, injective and $\mathfrak{R}_{i_1}(T^\dagger) = \text{im } \iota \sqcup \text{im } f$.

By Claim 11.1(III), forward swapping R produces W^\dagger with coefficient 1. By Claim 11.1 (part (I.i) or (I.ii), as appropriate), forward swapping R' produces $\beta W + (1 - \beta)W^\dagger$ for some β . Moreover, when applying $\text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}$ to $V \in \mathfrak{R}_{i_1}(T)$ or $V' \in \mathfrak{R}_{i_1}(T^\dagger)$, every snake lies entirely inside some edge-connected component of \mathcal{Z} . Further, W is one of these components. Thus, for each $V \in \mathfrak{R}_{i_1}$, $[T] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(V)$ factors as a contribution from the region W times a contribution from $\mathcal{Z} \setminus W$. That is, for the same α ,

$$\begin{aligned} \sum_{V \in \mathfrak{R}_{i_1}(T)} [T] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(V) &= \alpha\beta, \\ \sum_{V' \in \mathfrak{R}_{i_1}(T^\dagger)} [T^\dagger] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(V') &= \alpha, \\ \sum_{V' \in \mathfrak{R}_{i_1}(T^\dagger)} [T^\dagger] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(V') &= \alpha(1 - \beta). \end{aligned}$$

Therefore,

$$[T]P_{i_k} = \sum_{V \in \mathfrak{R}_{i_1}(T)} [V]P_{i_1} \cdot [T] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(V) = [V]P_{i_1}\alpha\beta,$$

while

$$\begin{aligned} [T^\dagger]P_{i_k} &= \sum_{V' \in \mathfrak{R}_{i_1}(T^\dagger)} [V']P_{i_1} \cdot [T^\dagger] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(V') \\ &= \sum_{V \in \mathfrak{R}_{i_1}(T)} [\iota(V)]P_{i_1} \cdot [T^\dagger] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(\iota(V)) \\ &\quad + \sum_{V \in \mathfrak{R}_{i_1}(T)} [f(V)]P_{i_1} \cdot [T^\dagger] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(f(V)) \\ &= \sum_{V \in \mathfrak{R}_{i_1}(T)} [V]P_{i_1} \cdot [T^\dagger] \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(V) \\ &\quad - \sum_{V \in \mathfrak{R}_{i_1}(T)} [V]P_{i_1} \text{swap}_{i_{k-1}} \circ \dots \circ \text{swap}_{i_1}(f(V)) \\ &= [V]P_{i_k}(\alpha(1 - \beta) - \alpha). \end{aligned}$$

Now, $[T]P_{i_k} = -[T^\dagger]P_{i_k}$ follows.

Case 2 (W (and hence W^\dagger) has at least two rows): There are three cases to consider, corresponding to the case of Lemma 10.4.

In Cases (I) and (II) of Lemma 10.4, W and W^\dagger have a unique reversal R . By Claim 11.1(III) or Claim 11.1(II), respectively, forward swapping R produces $\beta W - \beta W^\dagger$ for some β . In Case (III) of Lemma 10.4, W and W^\dagger share the same pair of reversals R, R' . By Claim 11.1(III) and (I.i), forward swapping R produces $\beta W - \beta W^\dagger$ for some β , while forward swapping R' produces $\beta' W - \beta' W^\dagger$ for some β' . Using these facts, one may argue similarly to Case 1 to deduce $[T]P_{i_k} = -[T^\dagger]P_{i_k}$. \square

Let \mathcal{S}_3 be the subset of tableaux in $G_{\lambda, \mu}^v(i_k)$ with a box x such that $\bullet_{i_k} \in x$, $i_k \in x^\rightarrow$ and $i_k \in \underline{x}$; that is, locally, the tableau is $\mathcal{C}_3 = \begin{array}{|c|c|} \bullet_{i_k} & i_k \\ \hline & \end{array}$ (with possibly additional edge labels). Let \mathcal{S}'_3 be the subset of tableaux in $G_{\lambda, \mu}^v(i_k)$ with a box x such that $\bullet_{i_k} \in x$, $i_k \in x^\rightarrow$, no i_k appears West of x^\rightarrow , $i_{k-1} \notin x^\leftarrow$ and $(i+1)_h \notin \underline{x}^\rightarrow$, where $N_{i_k} = N_{(i+1)_h}$. Locally, the tableau is $\mathcal{C}'_3 = \begin{array}{|c|c|} \bullet_{i_k} & i_k \\ \hline & \end{array}$ (with possibly additional edge labels).

LEMMA 12.14. *If $T \in \mathcal{S}_3$ (respectively, \mathcal{S}'_3), there is a unique \mathcal{C}_3 (respectively, \mathcal{C}'_3) that it contains.*

Proof. If \mathcal{C}_3 occurs in a good tableau, it is unique since the edge i_k is westmost in its gene, by (G.7). Similarly, \mathcal{C}'_3 is unique since the $i_k \in x^\rightarrow$ is westmost, by assumption. \square

We define $\phi_3(T)$ to be T with the unique \mathcal{C}_3 replaced by \mathcal{C}'_3 .

LEMMA 12.15. *The map $\phi_3 : \mathcal{S}_3 \rightarrow \mathcal{S}'_3$ is a bijection.*

Proof. We define the (putative) inverse ϕ_3^{-1} by replacing \mathcal{C}'_3 with \mathcal{C}_3 . Once we have established that ϕ_3 and ϕ_3^{-1} are well-defined, we are done, since ϕ_3 and ϕ_3^{-1} are clearly mutually inverse.

Let $T \in \mathcal{S}_3$. Trivially, each (G.n) holds for $\phi_3(T)$. By T 's (G.12), $i_{k-1} \notin x^\leftarrow$. If $(i+1)_h \in \underline{x}^\rightarrow$ in $\phi_3(T)$ with $N_{i_k} = N_{(i+1)_h}$, then T would violate Lemma 5.6. By T 's (G.4) and (G.7), the $i_k \in x^\rightarrow$ is westmost in $\phi_3(T)$.

Now, let $T \in \mathcal{S}'_3$. We check the goodness conditions for $\phi_3^{-1}(T)$.

CLAIM 12.16. *No label of family i appears in x 's column in T .*

Proof. By T 's (G.12), there are no labels of family i North of x and in its column. By T 's (G.11), a label ℓ South of x and in its column is not marked; that is, $\ell \geq i_k$.

Since we have assumed that the $i_k \in \mathbf{x}^\rightarrow$ is westmost, $\ell \neq i_k$. By T 's (G.6), $\ell \neq i_l$ for $l > k$. Hence, $i_k < \ell$. □

Conditions (G.4) and (G.5): By T 's (G.9), all labels North of \mathbf{x} and in its column are of family at most i . By T 's (G.11), all labels South of \mathbf{x} and in its column are of family at least i . Hence, by Claim 12.16, $\phi_3^{-1}(T)$'s (G.4) and (G.5) follow.

Condition (G.8): If there is a genotype G of $\phi_3^{-1}(T)$ that is not ballot, then it uses the $i_k \in \underline{\mathbf{x}}$. Furthermore, since T is ballot, some $(i + 1)_h$ with $N_{i_k} = N_{(i+1)_h}$ appears in $\text{word}(G)$ before the $i_k \in \underline{\mathbf{x}}$. By Lemma 5.6 applied to T , this $(i + 1)_h$ can only be South of \mathbf{x}^\rightarrow and in \mathbf{x}^\rightarrow 's column, or North of \mathbf{x} and in \mathbf{x} 's column. By T 's (G.9), it cannot be North of \mathbf{x} and in its column. Suppose that it appears South of \mathbf{x}^\rightarrow and in its column. By assumption, $(i + 1)_h \notin \mathbf{x}^\rightarrow$. Hence, suppose that it appears south of \mathbf{x}^\rightarrow , and consider $\text{label}(\mathbf{x}^\downarrow)$. By (G.11), $\text{family}(\text{label}(\mathbf{x}^\downarrow)) \geq i$. By Claim 12.16, $\text{family}(\text{label}(\mathbf{x}^\downarrow)) \neq i$. By T 's (G.3) and (G.4), $\text{label}(\mathbf{x}^\downarrow) < (i + 1)_h$. Hence, $\text{family}(\text{label}(\mathbf{x}^\downarrow)) = i + 1$. However, by Lemma 5.10, $\text{label}(\mathbf{x}^\downarrow) \neq (i + 1)_h$. Hence, by T 's (G.6), $(i + 1)_{h-1} \in \mathbf{x}^\downarrow$. This creates a (G.8) violation in T , as this label is read before any i_{k-1} .

Condition (G.12): Since T is good, if $\phi_3^{-1}(T)$ violates (G.12), the violation involves the $i_k \in \underline{\mathbf{x}}$. Since, by assumption, $i_{k-1} \notin \mathbf{x}^\leftarrow$, the last sentence of (G.12) does not apply. Suppose that i_j is SouthEast of $\underline{\mathbf{x}}$; then, it is also SouthEast of $i_k \in \mathbf{x}^\rightarrow$, which will lead to a violation of T 's (G.12). Suppose that i_j is NorthWest of $\underline{\mathbf{x}}$; then, to avoid a violation of T 's (G.12) with the $i_k \in \mathbf{x}^\rightarrow$, i_j must be either in \mathbf{x} 's row or in an upper edge of that row. Since we have $\bullet_{i_k} \in \mathbf{x}$, this avoids violating $\phi_3^{-1}(T)$'s (G.12).

All of the remaining (G.n)-conditions are trivial to verify. □

PROPOSITION 12.17. For $T \in \mathcal{S}_3$, $[T]P_{i_k} = [\phi_3(T)]P_{i_k}$.

Proof. Let $T^\dagger := \phi_3(T)$. By inspection of the reverse miniswaps, and downward induction on \mathcal{Q} , there is a bijection $f_{\mathcal{Q}} : \mathfrak{R}_{\mathcal{Q}}(T) \rightarrow \mathfrak{R}_{\mathcal{Q}}(T^\dagger)$ given by deletion of the $i_k \in \underline{\mathbf{x}}$. If $L \in \mathfrak{R}_{1_1}(T)$, then L and $f_{1_1}(L)$ have the same number of \bullet_{1_1} 's. Hence, $[L]P_{1_1} = [f_{1_1}(L)]P_{1_1}$; see (11.4).

We extend $f_{\mathcal{Q}}$ linearly. By inspection of the miniswaps,

$$f_{i_k}(\text{swap}_{i_k^-} \circ \dots \circ \text{swap}_{1_1}(L)) = \text{swap}_{i_k^-} \circ \dots \circ \text{swap}_{1_1}(f_{1_1}(L)).$$

Hence, by Lemma 12.5, $[T]P_{i_k} = [T^\dagger]P_{i_k}$. □

Let \mathcal{S}_4 be the subset of tableaux in $G_{\lambda, \mu}^v(i_k)$ with a box x such that $\bullet_{i_k} \in x$, $\mathcal{F}^1 \in x^{\rightarrow}$, $i_k \in \underline{x}$ and $(i_k) \in \underline{x}^{\rightarrow}$; that is, locally, the tableau is $\mathcal{C}_4 = \begin{array}{|c|c|} \hline \bullet_{i_k} & \mathcal{F}^1 \\ \hline i_k & (i_k) \\ \hline \end{array}$ (with possibly additional edge labels). Let \mathcal{S}'_4 be the subset of tableaux in $G_{\lambda, \mu}^v(i_k)$ with a box x such that $\bullet_{i_k} \in x$, $\mathcal{F}^1 \in x^{\rightarrow}$, $i_k \in \underline{x}^{\rightarrow}$, $(i+1)_h \notin \underline{x}^{\rightarrow}$ if $N_{(i+1)_h} = N_{i_k}$, and $i_{k-1} \notin x^{\leftarrow}$. Locally, the tableau is $\mathcal{C}'_4 = \begin{array}{|c|c|} \hline \bullet_{i_k} & \mathcal{F}^1 \\ \hline i_k & \\ \hline \end{array}$ (with possibly additional edge labels).

LEMMA 12.18. *If $T \in \mathcal{S}_4$ (respectively, \mathcal{S}'_4), there is a unique \mathcal{C}_4 (respectively, \mathcal{C}'_4) that it contains.*

Proof. This follows since, by (G.7), T contains at most one edge label i_k . □

We set $\phi_4 : \mathcal{S}_4 \rightarrow \mathcal{S}'_4$ by replacing \mathcal{C}_4 with \mathcal{C}'_4 .

LEMMA 12.19. *The map $\phi_4 : \mathcal{S}_4 \rightarrow \mathcal{S}'_4$ is a bijection.*

Proof. We define a putative inverse $\phi_4^{-1} : \mathcal{S}'_4 \rightarrow \mathcal{S}_4$ by replacing \mathcal{C}'_4 with \mathcal{C}_4 . Clearly, ϕ_4 and ϕ_4^{-1} are mutually inverse. It remains to check well-definedness. Indeed, it is trivial to check that each goodness condition holds for $\phi_4(T)$. By Lemma 5.6 for T , there is not $(i+1)_h \in \underline{x}^{\rightarrow}$ with $N_{(i+1)_h} = N_{i_k}$. By T 's (G.12), $i_{k-1} \notin x^{\leftarrow}$. Thus, ϕ_4 is well-defined.

CLAIM 12.20. *No label of family i appears in x 's column in T .*

Proof. By T 's (G.12), i_ℓ cannot appear North of x and in its column. If i_ℓ is South of x and in its column, then, by T 's (G.6) and (G.7), $\ell < k$, so this i_ℓ is marked, contradicting T 's (G.11). □

Now let $T \in \mathcal{S}'_4$. We check the goodness conditions for $\phi_4^{-1}(T)$.

Conditions (G.4) and (G.5): By T 's (G.9), every label North of x and in its column has family at most i . By T 's (G.11), every label South of x and in its column has family at least i . Moreover, by Claim 12.20, no label of family i appears in x 's column in T . Hence, (G.4) and (G.5) hold in $\phi_4^{-1}(T)$.

Condition (G.8): Suppose that $\phi_4^{-1}(T)$ has a nonballot genotype G . By T 's (G.8), G must use the $i_k \in \underline{x}$. Also by T 's (G.8), some $(i+1)_h$ with $N_{(i+1)_h} = N_{i_k}$ appears in $\text{word}(G)$ before this $i_k \in \underline{x}$. By T 's (G.9) and (G.8), this $(i+1)_h$ appears South of x^{\rightarrow} and in x^{\rightarrow} 's column. By T 's (G.4) and the first hypothesis on \mathcal{S}'_4 , in fact $(i+1)_h \in x^{\rightarrow \downarrow}$. By T 's (G.3), $\text{family}(\text{label}(x^{\downarrow})) \leq i+1$. By (G.11) and the $\bullet_{i_k} \in x$, $\text{family}(\text{label}(x^{\downarrow})) \geq i$. By Claim 12.20, no label of

family i appears in x 's column in T . Thus, $\text{family}(\text{label}(x^\downarrow)) = i + 1$. Then, by T 's (G.3) and (G.6), $(i + 1)_{h-1} \in x^\downarrow$. Hence, by Claim 12.20, this contradicts Lemma 5.6 for T .

Condition (G.12): If there is an i_ℓ SouthEast of the $i_k \in x$ in $\phi_4^{-1}(T)$, then we either violate T 's (G.2), (G.4) or (G.12). Now, suppose that there is an i_ℓ NorthWest of $i_k \in x$ in $\phi_4^{-1}(T)$. By T 's (G.12), this i_ℓ is West and either in x 's row or on the upper edge of that row. If $i_\ell \in x^\leftarrow$, then $\ell = k - 1$ by T 's (G.6). However, then we contradict the last hypothesis on S'_4 . Therefore, the i_ℓ and i_k satisfy (G.12).

The remaining goodness conditions are trivial to verify. □

PROPOSITION 12.21. For each $T \in \mathcal{S}_4$, $[T]P_{i_k} = [\phi_4(T)]P_{i_k}$.

Proof. Let $T^\dagger = \phi_4(T)$. Let $f_Q : \mathfrak{R}_Q(T) \rightarrow \mathfrak{R}_Q(T^\dagger)$ be defined by deleting the $i_k \in x$ and replacing the $(i_k) \in x^\rightarrow$ by i_k . Now, the proof proceeds exactly as that for Proposition 12.17. □

13. Proof of the conjectural K_T rule from [ThYo13]

We briefly recap the conjectural rule for $K_{\lambda, \mu}^\nu$ from [ThYo13, Section 8]. This rule directly extends the rule of [ThYo09b] for the corresponding coefficients in nonequivariant K -theory. An *equivariant increasing tableau* is an edge-labeled filling of ν/λ using the labels $1, 2, \dots, |\mu|$ such that each label is strictly smaller than any label below it in its column and each *box* label is strictly smaller than the box label immediately to its right. Any subset of the boxes of ν/λ may be marked by a \star , except that, if i and $i + 1$ are box labels in the same row, then the box containing i may *not* have a \star . Let $\text{EqInc}(\nu/\lambda, |\mu|)$ denote the set of all such equivariant increasing tableaux.

An *alternating ribbon* R is a filling of a short ribbon by two symbols such that adjacent boxes are filled differently; all edges except the southwestmost edge are empty; if this edge is filled, it is filled with the other symbol than that in the box above it. Let $\text{switch}(R)$ be the alternating ribbon of the same shape where each box is instead filled with the other symbol. If the southwestmost edge was filled by one of these symbols, that symbol is deleted. If R consists of a single box with only one symbol used, then switch does nothing to it. We define switch to act on an edge-disjoint union of alternating ribbons, by acting on each independently.

EXAMPLE 13.1. Let $R = \begin{array}{|c|c|} \hline \heartsuit & \spadesuit \\ \hline \heartsuit & \spadesuit \\ \hline \end{array}$. Then, $\text{switch}(R) = \begin{array}{|c|c|} \hline \spadesuit & \heartsuit \\ \hline \spadesuit & \heartsuit \\ \hline \end{array}$. □

Given $T \in \text{EqInc}(v/\lambda, |\mu|)$ and an inner corner $x \in \lambda$, label x with \bullet and erase all \star s. Call this tableau V_1 . Consider the alternating ribbons $\{R_1\}$ made of \bullet and 1. The tableau V_2 is obtained by applying `switch` to each R_1 . Now, let $\{R_2\}$ be the collection of ribbons consisting of \bullet and 2, and produce V_3 by applying `switch` to each R_2 . Repeat until the \bullet s have been switched past all of the numerical labels in T ; the final placement of numerical labels gives $\text{KEqjdt}_x(T)$, the *slide* of T into x . The sequence V_1, V_2, \dots is the *switch sequence* of (T, x) . Finally, define $\text{KEqrect}(T)$ by successively applying KEqjdt_x in *column rectification order*; that is, successively pick x to be the eastmost inner corner.

LEMMA 13.2. *For V_j in the switch sequence of (T, x) , we have the following.*

- (I) *The numerical box labels strictly increase along rows from left to right (ignoring \bullet s).*
- (II) *The numerical labels strictly increase down columns (ignoring \bullet s and reading labels of a given edge in increasing order).*
- (III) *Every numerical label southeast of a \bullet is at least j .*
- (IV) *Every numerical label northwest of a \bullet is strictly less than j .*

Proof. These are proved by simultaneous induction on j . In the inductive step, one considers any 2×2 local piece of V_j and studies the possible cases that can arise as one transitions from V_j to V_{j+1} ; we leave the straightforward details to the reader. \square

A set of labels is a *horizontal strip* if they are arranged in increasing order from southwest to northeast, with no two labels of the set in the same column.

LEMMA 13.3. *Let $T \in \text{EqInc}(v/\lambda, |\mu|)$, and let $x \in \lambda$ be an inner corner. Then, $\{i, i + 1, \dots, j\}$ forms a horizontal strip in V_k of the switch sequence of (T, x) if and only if it does so in V_{k+1} .*

Proof. This quickly reduces to consideration of the possibilities in a 2×2 local piece of V_k . Then, we proceed by straightforward case analysis using Lemma 13.2. \square

A label $s \in T$ is *special* if it is an edge label or lies in a box with a \star . At most one s appears in a column c . In column rectification order, each slide KEqjdt_x for $x \in c$ moves an s in c at most one step North (and it remains

in c). A special label s in c passes through x if it occupies x at any point during c 's rectification and initially $s \notin x$. Let x_1, \dots, x_s be the boxes s passes through, and let y_1, \dots, y_t be the numerically labeled boxes East of x_s in the same row. Set $\text{factor}_K(s) := 1 - \prod_{i=1}^s \hat{\beta}(x_i) \prod_{j=1}^t \hat{\beta}(y_j)$. If s does not move during the rectification of c , then $\text{factor}_K(s) := 0$. Now, set $\text{wt}_K(T) := \prod_s \text{factor}_K(s)$, where the product is over all special labels. Lastly, we define $\text{sgn}(T) := (-1)^{|\mu| - \#\star s \text{ in } T - \#\text{labels in } T}$.

Let $\mu[1] = (1, 2, 3, \dots, \mu_1)$, $\mu[2] = (\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2)$, and so on. Let T_μ be the *superstandard tableau* of shape μ ; that is, row i is filled by $\mu[i]$. The following is the conjecture of Thomas and Yong [ThYo13].

THEOREM 13.4. *We have $K_{\lambda, \mu}^v = \sum_T \text{sgn}(T) \cdot \text{wt}_K(T)$, where the sum is over*

$$\mathcal{A}_{\lambda, \mu}^v := \{T \in \text{EqInc}(v/\lambda, |\mu|) : \text{KEqrect}(T) = T_\mu\}.$$

We will prove Theorem 13.4 (after some preparation) by connecting to Theorem 1.3.

Let $\mathcal{B}_{\lambda, \mu}^v$ be the set of all $T \in \text{BallotGen}(v/\lambda)$ that have content μ . We need a *semistandardization map* $\Phi : \mathcal{A}_{\lambda, \mu}^v \rightarrow \mathcal{B}_{\lambda, \mu}^v$. Given $A \in \mathcal{A}_{\lambda, \mu}^v$, erase all $\star s$ and replace the labels $1, 2, \dots, \mu_1$ with $1_1, 1_2, \dots, 1_{\mu_1}$, respectively. Next, replace $\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2$ by $2_1, 2_2, \dots, 2_{\mu_2}$, respectively, and so on. The result is $\Phi(A)$. It should be noted that Φ is not bijective. Define a *standardization map* $\Psi : \mathcal{B}_{\lambda, \mu}^v \rightarrow \mathcal{A}_{\lambda, \mu}^v$ by reversing the above process in the obvious way; $\Psi(B)$ is \star -less.

LEMMA 13.5. *For $B \in \mathcal{B}_{\lambda, \mu}^v$, $\Psi(B) \in \text{EqInc}(v/\lambda, |\mu|)$.*

Proof. That $\Psi(B)$ has the desired shape and content is clear. Row strictness follows from (S.1), and column strictness from (S.2). □

LEMMA 13.6. *For $B \in \mathcal{B}_{\lambda, \mu}^v$ and for each i , $\mu[i]$ forms a horizontal strip in $\Psi(B)$, and also in each tableau of any switch sequence during the column rectification of $\Psi(B)$.*

Proof. By (S.2–4), the labels i_1, \dots, i_{μ_i} form a horizontal strip of B . The claim for $\Psi(B)$ is then immediate by definition of Ψ . The claim for the tableaux of the switch sequences then follows by Lemma 13.3. □

LEMMA 13.7. Let $B \in \mathcal{B}_{\lambda, \mu}^v$. Then,

- (I) after column rectifying the eastmost j columns of $\Psi(B)$, there are no edge labels in these eastmost j columns; and
- (II) while rectifying the next column, there is never an edge label north of a \bullet and in the same column, in any tableau of any switch sequence.

Proof. (I): Suppose that there were such an edge label $\ell \in \underline{x}$ after rectifying the eastmost j columns. Then, $\ell \in \underline{x}$ in $\Psi(B)$, since rectification never adds a label to any edge. Suppose that \underline{x} is in the i th row from the top of $\Psi(B)$. Then, since no label of B is too high, $\ell \in \mu[k]$, where $k \leq i$. Let the boxes North of \underline{x} and in the same column be $x_1, \dots, x_i = \underline{x}$ from north to south. By Lemma 13.2(II), we have for each e that $\text{label}(x_e) \in \mu[f(e)]$ for some $f(e) \leq k$. However, then, by Lemma 13.6, $f : \{1, 2, \dots, i\} \rightarrow \{1, 2, \dots, k - 1\}$ is injective, which is a contradiction.

(II): Let c be the column currently being rectified. For the columns East of c , the claim follows from part (I), noting that rectification never adds a label to any edge. For column c itself, the claim is vacuous if there is no \bullet in c . If there is $\bullet \in c$, the claim follows from noting that every label of column c North of this \bullet must have participated in some *switch*, and that *switch* never outputs any edge labels. □

An equivariant increasing tableau T is *ballot* if $\Phi(T)$ is ballot in the sense of Section 1.3. That is, for every \widetilde{T} obtained by selecting one copy of each label in T , every initial segment of \widetilde{T} 's column reading word has, for each $i \geq 1$, at least as many labels from $\mu[i]$ as from $\mu[i + 1]$. We extend this definition to tableaux with \bullet s by ignoring the \bullet s.

LEMMA 13.8. Let $B \in \mathcal{B}_{\lambda, \mu}^v$. Then, $\Psi(B)$ is ballot, as is each tableau of any switch sequence during the column rectification of $\Psi(B)$.

Proof. Let $A = \Psi(B)$. Since B is ballot and $\Phi(A) = B$, A is ballot, by definition. Suppose that some V_q is ballot, but V_{q+1} is not. Then there exist i and a \widetilde{V}_{q+1} with a ballotness violation between $\mu[i]$ and $\mu[i + 1]$.

If $q \notin \mu[i] \cup \mu[i + 1]$, then the labels of $\mu[i]$ and $\mu[i + 1]$ appear in the same locations in V_q and V_{q+1} , contradicting that V_q is ballot.

If $q \in \mu[i + 1]$, then no $\mu[i]$ -label moves. For each $\ell \in \mu[i + 1]$ appearing in \widetilde{V}_{q+1} , there is an ℓ east of that position in V_q . Hence, we construct a nonballot \widetilde{V}_q by choosing those corresponding ℓ s, the same labels from $\mu[i]$ as in \widetilde{V}_{q+1} , and all other labels arbitrarily. This contradicts that V_q is ballot.

Finally, if $q \in \mu[i]$, then there is some x in column c of V_q with $\bullet \in x$ and $q \in x^\rightarrow$ such that the q moving into x violates ballotness in the columns East of c . That is, locally, the switch is

$$V_q \supseteq \begin{array}{|c|c|} \hline a & b \\ \hline \bullet & q \\ \hline d & e \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline a & b \\ \hline q & \bullet \\ \hline d & e \\ \hline \end{array} \subseteq V_{q+1} \quad \text{or} \quad V_q \supseteq \begin{array}{|c|c|} \hline a & \bullet \\ \hline \bullet & q \\ \hline d & e \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline a & q \\ \hline q & \bullet \\ \hline d & e \\ \hline \end{array} \subseteq V_{q+1},$$

where the x is the left box of the second row. The $q \in x^\rightarrow$ is Westmost in V_q , since otherwise the nonballotness of V_{q+1} contradicts that V_q is ballot. In particular, $q \neq d$. Hence, by Lemma 13.2(III), $q < d$.

Since V_q is ballot but V_{q+1} is not, there is a $\bar{q} \in \mu[i + 1]$ in c^\rightarrow in V_q , and hence in V_{q+1} . By Lemma 13.2(II) applied to V_q , this \bar{q} is below q in c^\rightarrow . By Lemma 13.7(I), there are no edge labels East of column c . Therefore, in fact, e and hence d both exist. Indeed, by Lemma 13.2(II) and Lemma 13.6, $e = \bar{q}$. By Lemma 13.6, q is the only label of $\mu[i]$ that appears in c in V_{q+1} . Hence, $d \notin \mu[i]$. Thus, by Lemma 13.2(I) applied to V_q , we conclude that $d \in \mu[i + 1]$. However, this again contradicts that V_q is ballot. \square

For $A \in \text{EqInc}(v/\lambda, |\mu|)$, let $A^{(k)}$ be the ‘partial’ tableau that is the column rectification of the eastmost k columns of A .

LEMMA 13.9. *Let $B \in \mathcal{B}_{\lambda, \mu}^v$, and let $A = \Psi(B)$. For each i , the i th row of $A^{(k)}$ consists of a (possibly empty) final segment from $\mu[i]$.*

Proof. By Lemma 13.2(I, II), $A^{(k)}$ has strictly increasing rows and columns. By Lemma 13.6, the labels $\mu[i]$ form a horizontal strip in $A^{(k)}$ for each i ; moreover, the labels of $\mu[i]$ appearing in $A^{(k)}$ are a final segment of $\mu[i]$. By Lemma 13.7(I), there are no edge labels in $A^{(k)}$. By Lemma 13.8, $A^{(k)}$ is ballot. The lemma follows. \square

COROLLARY 13.10. *The tableau A rectifies to T_μ .*

Proof. The proof is immediate from Lemma 13.9. \square

PROPOSITION 13.11. *For $B \in \mathcal{B}_{\lambda, \mu}^v$, $\Psi(B) \in \mathcal{A}_{\lambda, \mu}^v$.*

Proof. By Lemma 13.5, $\Psi(B) \in \text{EqInc}(v/\lambda, |\mu|)$. By Corollary 13.10, $\Psi(B)$ rectifies to T_μ . \square

LEMMA 13.12. For $A \in \mathcal{A}_{\lambda, \mu}^v$, $\mu[i]$ forms a horizontal strip in A and each A' in the column rectification of T .

Proof. This is true for T_μ , and hence true for A and each A' , by Lemma 13.3. □

LEMMA 13.13. For $A \in \mathcal{A}_{\lambda, \mu}^v$, $\Phi(A)$ is semistandard.

Proof. Row strictness of A implies that $\Phi(A)$ satisfies (S.1). Since, by Lemma 13.12, $\mu[i]$ is a horizontal strip in A for each i , (S.2)–(S.4) hold in $\Phi(A)$. □

LEMMA 13.14. For $A \in \mathcal{A}_{\lambda, \mu}^v$, $\tilde{\Phi}(A)$ is ballot.

Proof. Suppose that $\Phi(A)$ is not ballot. Then, by definition, A is not ballot. We assert that every tableau in every switch sequence in the column rectification of A is also *not* ballot, implying that T_μ is not ballot, which is a contradiction.

Suppose that V_ℓ is not ballot, but $V_{\ell+1}$ is. We derive a contradiction. Since V_ℓ is not ballot, we pick a nonballot \tilde{V}_ℓ . Suppose that this nonballotness can be blamed on positions $\mathbf{a}_1, \dots, \mathbf{a}_s$ containing labels of $\mu[i]$ and positions $\mathbf{b}_1, \dots, \mathbf{b}_{s+1}$ containing labels of $\mu[i + 1]$ (for some i). Suppose that $\mathbf{a}_1, \dots, \mathbf{a}_s$ and $\mathbf{b}_1, \dots, \mathbf{b}_{s+1}$ are left to right in \tilde{V}_ℓ ; no two \mathbf{a}_j s (respectively, \mathbf{b}_j s) are in the same column, by Lemma 13.12. We may assume that \mathbf{b}_1 is southwestmost among all of these positions, say in column c , and that among all offending choices of i and positions, we picked one so that c is eastmost.

Since $V_{\ell+1}$ is supposed ballot, there is a label $\ell \in \mu[i + 1]$ in \mathbf{b}_1 of V_ℓ that moved to column c^\leftarrow . Locally, the switch is

x	y
•	ℓ

 \mapsto

x	y
ℓ	•

. By Lemma 13.12,

$\mu[i + 1]$ forms a horizontal strip in V_ℓ . Hence, $x, y \notin \mu[i + 1]$. Moreover, no label in column c is in $\mu[i]$ since otherwise we contradict that c is chosen eastmost. Now, there is some label $m \in \mu[i]$ above the • in column c^\leftarrow of V_ℓ since $V_{\ell+1}$ is ballot. Using Lemma 13.7(II), it follows that $m = x$. Now, we have argued that $y \notin \mu[i] \cup \mu[i + 1]$. However, by Lemma 13.2(I, II) applied to V_ℓ , there are no other possibilities for y , which is a contradiction. □

PROPOSITION 13.15. For $A \in \mathcal{A}_{\lambda, \mu}^v$, $\Phi(A) \in \mathcal{B}_{\lambda, \mu}^v$.

Proof. By construction, $\Phi(A)$ is an edge-labeled genomic tableau of shape ν/λ and content μ . By Lemma 13.13, $\Phi(A)$ is semistandard. By Lemma 13.14, $\Phi(A)$ is ballot. Since A rectifies to T_μ , no label of $\Phi(A)$ is too high. □

Given a label ℓ in $A \in \mathcal{A}_{\lambda, \mu}^v$, let $\Phi(\ell)$ be the corresponding label in $\Phi(A) \in \mathcal{B}_{\lambda, \mu}^v$. We recall the definitions of Section 1.4.

LEMMA 13.16. (I) *If ℓ is an edge label, then $\text{factor}_K(\ell) = \text{edgefactor}(\Phi(\ell))$.*

(II) *If ℓ is in a box with a \star , then $\text{factor}_K(\ell) = 1 - \text{boxfactor}(\Phi(\ell))$.*

Proof. These follow from the definitions of the factors combined with Lemma 13.9. □

LEMMA 13.17. *If $B \in \mathcal{B}_{\lambda, \mu}^v$, then*

$$\text{boxwt}(B) = \sum_{A \in \Phi^{-1}(B)} (-1)^{\#\star \text{ in } A} \prod_{\text{special box label } \ell \text{ of } A} \text{factor}_K(\ell).$$

Proof. A box x is productive in B if and only if it may have a \star in $\Psi(B)$. We are done by Lemma 13.16(II) and the ‘inclusion–exclusion’ identity $\sum_{S \subset [N]} (-1)^{|S|} \prod_{s \in S} (1 - z_s) = z_1 z_2 \cdots z_N$. □

Proof of Theorem 13.4. We recall that Theorem 13.4 asserts $K_{\lambda, \mu}^v = \sum_{A \in \mathcal{A}_{\lambda, \mu}^v} \text{sgn}(A) \text{wt}_K(A)$. To see this, we observe that, by Propositions 13.11 and 13.15,

$$\begin{aligned} \sum_{A \in \mathcal{A}_{\lambda, \mu}^v} \text{sgn}(A) \text{wt}_K(A) &= \sum_{B \in \mathcal{B}_{\lambda, \mu}^v} \sum_{A \in \Phi^{-1}(B)} \text{sgn}(A) \text{wt}_K(A) \\ &= \sum_{B \in \mathcal{B}_{\lambda, \mu}^v} \sum_{A \in \Phi^{-1}(B)} (-1)^{|\mu| - \#\star \text{ in } A - \#\text{labels in } A} \\ &\quad \times \prod_{\text{edge label } \ell \text{ of } A} \text{factor}_K(\ell) \prod_{\text{special box label } \ell \text{ of } A} \text{factor}_K(\ell) \\ &= \sum_{B \in \mathcal{B}_{\lambda, \mu}^v} \sum_{A \in \Phi^{-1}(B)} (-1)^{|\mu| - \#\text{labels in } A} \\ &\quad \times \left(\prod_{\text{edge label } \ell \text{ of } A} \text{factor}_K(\ell) \right) (-1)^{\#\star \text{ in } A} \prod_{\text{special box label } \ell \text{ of } A} \text{factor}_K(\ell). \end{aligned}$$

The number of labels of A equals the number of labels of B for any $A \in \Phi^{-1}(B)$. Combination of this with Lemma 13.16(I) shows that the previous expression

equals

$$\begin{aligned}
 &= \sum_{B \in \mathcal{B}_{\lambda, \mu}^{\vee}} (-1)^{|\mu| - \#\text{labels in } B} \left(\prod_{\text{edge label } \ell \text{ of } B} \text{edgefactor}(\ell) \right) \\
 &\quad \times \sum_{A \in \Phi^{-1}(B)} (-1)^{\#\ast \text{ in } A} \prod_{\text{special box label } \ell \text{ of } A} \text{factor}_K(\ell).
 \end{aligned}$$

By Lemma 13.17, this equals

$$= \sum_{B \in \mathcal{B}_{\lambda, \mu}^{\vee}} (-1)^{|\mu| - \#\text{labels in } B} \text{edgewt}(B) \text{boxwt}(B) := L_{\lambda, \mu}^{\vee}.$$

Since, by Theorem 1.3, $L_{\lambda, \mu}^{\vee} = K_{\lambda, \mu}^{\vee}$, we are done. \square

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Appendix A. Proof of Proposition 7.7

We check that (G.1)–(G.13) are preserved. Let $U \in \text{swap}_G(T)$. We prove the conditions in the order (G.1), (G.2), (G.4), (G.5), (G.6), (G.7), (G.3), (G.8), (G.9), (G.11), (G.10), (G.12), (G.13). In this way, each proof depends only on previously proved conditions. It is also necessary to show that the prescribed virtual labels in H5.2 and T4.2 satisfy the rules (V.1)–(V.3) for virtual labels. This is done in Lemma A.7 as part of the discussion of (G.13).

Condition (G.1): By T 's (G.1), no label of T is too high. Hence, if some label \mathcal{P} of U is too high, it must be placed in such a location by some miniswap. We therefore consider all miniswaps that might place a label \mathcal{P} on x or \underline{x} in U , when there is no \mathcal{P} or $\textcircled{\mathcal{P}}$ north of \underline{x} in T . By inspection, the miniswaps that can do so are H1, T1, T3, T4 and T5.

H1: The first output of H1 is not problematic since, if the $\mathcal{G} \in x$ in the first output were too high, the $\mathcal{G} \in \underline{x}$ in T would also have been too high, in violation

of T 's (G.1). If the second output of H1 creates a label that is too high, then, by definition of γ , that output is produced with coefficient 0. Thus, H1 does not create a tableau violating (G.1).

T1: Suppose that T has $\bullet_{\mathcal{G}} \in x$, and that, after applying T1, U has $\mathcal{G} \in x$, which is too high. By Lemma 6.9(IV,VII), T has $\mathcal{G} \in x^\downarrow$. By T 's (G.1), this $\mathcal{G} \in x^\downarrow$ is not too high. Since $\mathcal{G} \in x^\downarrow$ is not too high, but $\mathcal{G} \in x$ is too high, x is in row $i - 1$, where $i = \text{family}(\mathcal{G})$. In particular, $i > 1$.

By T 's (G.2), $\bullet_{\mathcal{G}} \notin x^\rightarrow$ in T , so if x^\rightarrow is a box of T , then it contains a genetic label. Consider $\text{label}_T(x^\rightarrow)$. If $\text{label}_T(x^\rightarrow) < \mathcal{G}$, then $\text{label}_T(x^\rightarrow)$ is marked. Hence, by Lemma 5.13, T has $\textcircled{\mathcal{G}} \in x^\rightarrow$. This contradicts that we are applying T1, as then x^\rightarrow is adjoined to the snake containing x by (R.3). If $\text{label}_T(x^\rightarrow) = \mathcal{G}$, this again contradicts that we are applying T1. If $\text{label}_T(x^\rightarrow) > \mathcal{G}$, then $\text{family}(\mathcal{G}) \leq \text{family}(\text{label}_T(x^\rightarrow))$. Thus, if $\mathcal{G} \in x$ were too high in U , then $\text{label}_T(x^\rightarrow)$ would already be too high in T , violating T 's (G.1). Thus, x^\rightarrow must not be a box of T .

Let c be the column of x . Consider the labels of T of family $i - 1$. Suppose that such a label appears in column c of T . It cannot be South of x , as then it would be marked and violate T 's (G.11). It cannot be in x , since $\bullet_{\mathcal{G}} \in x$. It cannot be North of x , as then it would be too high and violate T 's (G.1). Thus, there is no label of family $i - 1$ in column c .

Since x^\rightarrow is not a box of T , every column East of c contains at most $i - 2$ boxes. Therefore, any label of family $i - 1$ East of column c would be too high, contradicting T 's (G.1). Consider a genotype of T involving the $\mathcal{G} \in x^\downarrow$. No label of family $i - 1$ is read before the $\mathcal{G} \in x^\downarrow$. This contradicts T 's (G.8). Thus, this miniswap cannot create a label that is too high.

T3: If the $\mathcal{G} \in x$ in either output of T3 were too high, the $\mathcal{G}^+ \in x^\rightarrow$ in T with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$ would violate T 's (G.1).

T4 and T5: Suppose that either T4 or T5 created a label \mathcal{P} that was too high in U . Then, in the notation of those miniswaps, U has \mathcal{P} in the edge \bar{x} , and we must have $\mathcal{P} \in \{\mathcal{F}\} \cup Z$. By T 's (G.13), there is a label $\mathcal{Q} \in \underline{x^\rightarrow}$ (possibly virtual) in T with $\text{family } \mathcal{Q} = \text{family}(\mathcal{P}) + 1$. This $\mathcal{Q} \in \underline{x^\rightarrow}$ is then too high in T , contradicting T 's (G.1).

Condition (G.2): Consider a snake S in T . Since S is a short ribbon (Lemma 6.5), in the region of U defined by S , (G.2) can only be violated by having two $\bullet_{\mathcal{G}^+}$ s in the same row or column. By inspection of the miniswaps, no two $\bullet_{\mathcal{G}^+}$ s can appear in the same row. If two $\bullet_{\mathcal{G}^+}$ s appear in the same column, the top $\bullet_{\mathcal{G}^+}$ arose from a T4.1 or T4.2 miniswap. However, in those cases, the edge label $\mathcal{G} \in \text{tail}(S)$ implies $\text{tail}(S) = S$ by T 's (G.7).

Thus, we check U 's (G.2) for pairs of snakes S, S' . By Lemma 6.6, say S is southwest of S' . If S is entirely SouthWest of S' , (G.2) preservation is clear. It remains to consider the situations where S and S' share a row or a column.

Suppose that the snakes are in the configuration of Lemma 6.6(II). Here, $S = \{x\} = \boxed{\bullet_g} (\mathcal{G}, \mathcal{G}) \notin x$. Therefore, x takes part in a trivial H3 miniswap. By T 's (G.2), the southmost row r' of S' (assumed to be in x 's row) does not contain \bullet_g . Thus, r' takes part in an H9, B1 or B3 miniswap. The miniswaps H9 and B1 do not introduce a \bullet_{g+} , so (G.2) holds here. We claim that B3 is not possible. If r' participates in a B3 miniswap, then, by definition, $r' = \{y\} = \boxed{\mathcal{G}}$. It cannot be that $y^\leftarrow = x$, as then S and S' would be the same snake. Let $\mathcal{F} = \text{label}(y^\leftarrow)$. By (G.3), $\mathcal{F} < \mathcal{G}$. Hence, the $\bullet_g \in x$ means that the $\mathcal{F} \in y^\leftarrow$ is marked. However, then y^\leftarrow was adjoined to S' by (R.3); that is, $r' = \{y^\leftarrow, y\} = \boxed{\mathcal{F}^1 \mathcal{G}}$, which is a contradiction.

Finally, suppose that the snakes are in the configuration of Lemma 6.6(III). The two adjacent rows of S and S' are $\begin{matrix} & \bullet_g \\ \bullet_g & \mathcal{G}^+ \end{matrix}$ or $\begin{matrix} \bullet_g & \mathcal{G}^+ \\ \bullet_g & \mathcal{G}^+ \end{matrix}$. Hence, S' takes part in an H3 or H8 trivial miniswap. Let x be the east box of the northmost row of S . The box x takes part in miniswap H6, H7, H8 or T3. If it is miniswap H8, $\bullet_{g+} \notin x$ in U . In the other cases, by definition of α , the tableau produced with $\bullet_{g+} \in x$ appears with coefficient 0.

Condition (G.4): We show that U does not violate (G.4) in a given column c .

Case 1 ($\bullet_g \notin c$ in T): By inspection of the miniswaps, c either has labels removed or else a box label of c is pushed onto a lower edge of the same box (and a \bullet_{g+} comes into c).

Subcase 1.1 (c is strictly increasing in T): By the above, c is strictly increasing in U .

Subcase 1.2 (c is not strictly increasing in T): Here, we have that c contains

$\{x, x^\uparrow\} = \begin{matrix} \mathcal{F} \\ \mathcal{F}^1 \end{matrix}$ in T . By the above observation, it suffices to show that $\{x, x^\uparrow\}$ is not $\begin{matrix} \mathcal{F}^1 \\ \mathcal{F}^1 \end{matrix}$ or $\begin{matrix} \mathcal{F} \\ \mathcal{F} \end{matrix}$ in U . Since \mathcal{F}^1 appears in T , $\mathcal{F} < \mathcal{G}$. Since $\mathcal{F} \in x^\uparrow$ in T , there is no \bullet_g in T northwest of x^\uparrow . Thus, in U , no \bullet_{g+} can appear northwest of x^\uparrow . Hence, $\mathcal{F}^1 \notin x^\uparrow$ in U . This rules out the first scenario.

We now rule out the second scenario. By Lemma 5.4, in T , there is a \bullet_g in some box y West of x in the same row, and furthermore $\mathcal{E}^1 \in y^\rightarrow$ in T . By Lemma 5.5, T has $\mathcal{G}' \in y^\rightarrow$ (possibly marked), $(\mathcal{G}') \in y^\rightarrow$ or $(\mathcal{G}')^! \in y^\rightarrow$, where $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$. Let S be the snake containing y .

Subcase 1.2.1 ($y^{\rightarrow} \in S$): Since $\mathcal{E}^! \in y^{\rightarrow}$, we have $\{y, y^{\rightarrow}\} = \text{tail}(S)$. Thus, U has $\bullet_{\mathcal{G}^+} \in y$ or $\bullet_{\mathcal{G}^+} \in y^{\rightarrow}$. Hence, $\mathcal{F} \notin x$ in U .

Subcase 1.2.2 ($y^{\rightarrow} \notin S$ and neither \mathcal{G} nor \mathcal{G} appears in y 's column): Then, $S = \{y\}$ undergoes H3 and $\text{label}_U(y) = \bullet_{\mathcal{G}^+}$. Hence, $\mathcal{F}^! \in x$ in U .

Subcase 1.2.3 ($y^{\rightarrow} \notin S$ and either \mathcal{G} or \mathcal{G} appears in y 's column): By Lemma 5.13 applied to T , $\mathcal{G}' = \mathcal{G}$. Hence, $y^{\rightarrow} \in S$, violating the assumption of Subcase 1.2.3.

Case 2 ($\bullet_{\mathcal{G}} \in x$, where x is a box of c in T): Let S be the snake containing x .

Subcase 2.1 ($x \in \text{head}(S)$): Clearly, there is no (G.4) violation except possibly if we apply H5.2 or H5.3, where $\text{label}(x^{\rightarrow}) = \mathcal{G}$; thus, we assume that we are using one of these miniswaps. Let \mathcal{F} be the \leftarrow -greatest label appearing in x^{\uparrow} or x^{\downarrow} . Let \mathcal{H} be the \leftarrow -least label appearing in x^{\downarrow} or x^{\uparrow} . After the miniswap, \mathcal{G} appears in x . We show that $\mathcal{F} < \mathcal{G} < \mathcal{H}$. Since, in T , \mathcal{F} is northwest of $\bullet_{\mathcal{G}}$, $\mathcal{F} < \mathcal{G}$, by T 's (G.9). If $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$, then the $\mathcal{F} \in x^{\uparrow}$ or x^{\downarrow} and the $\mathcal{G} \in x^{\rightarrow}$ violate T 's (G.12). Hence, $\mathcal{F} < \mathcal{G}$. If $\mathcal{H} < \mathcal{G}$, then the $\mathcal{H} \in x^{\downarrow}$ or x^{\uparrow} is marked in T , violating (G.11). If $\mathcal{H} = \mathcal{G}$, then, since we are using H5.2 or H5.3, $\mathcal{H} = \mathcal{G} \in x^{\downarrow}$, so $x^{\downarrow} \in S$, contradicting $x \in \text{head}(S)$. Hence, $\mathcal{G} < \mathcal{H}$. Therefore, by T 's (G.6), $\mathcal{G} < \mathcal{H}$.

Subcase 2.2 ($x \in \text{tail}(S)$):

Subcase 2.2.1 ($\text{tail}(S)$ is T1, T2 or T3): By Lemma 6.9(IV,V,VII), S has at least two rows, and $\text{label}_T(x^{\downarrow}) = \mathcal{G}$. Suppose that there were a label \mathcal{Q} on x^{\downarrow} in T . By T 's (G.4), $\mathcal{Q} < \mathcal{G}$. However, then this $\mathcal{Q} \in x^{\downarrow}$ is marked, violating T 's (G.11). Hence, x^{\downarrow} is empty. Let \mathcal{F} be the \leftarrow -greatest label appearing in x^{\uparrow} or x^{\downarrow} . Let \mathcal{H} be the \leftarrow -least label appearing in x^{\downarrow} or x^{\uparrow} . Since there is $\mathcal{G} \in x^{\downarrow}$ in T , by T 's (G.4), we have $\mathcal{F} < \mathcal{G} < \mathcal{H}$.

Each of T1, T2 or T3 puts $\mathcal{G} \in x$. The swap does not affect \mathcal{F} or \mathcal{H} in column c . Now, if the swap puts $\bullet_{\mathcal{G}^+} \in x^{\downarrow}$ in U , we are done, since $\mathcal{F} < \mathcal{G} < \mathcal{H}$. Therefore, we assume otherwise. Then, x^{\downarrow} takes part in a miniswap H5.1, H5.2 (choosing the first output) or H9. In these cases, U has $\mathcal{G} \in x$ and $\mathcal{G} \in x^{\downarrow}$. Since $\mathcal{F} < \mathcal{G} < \mathcal{H}$, to show that (G.4) holds, we need $\mathcal{G} \in x^{\downarrow}$ in U to be marked. If the miniswap was H5.1 or H5.2, then $\bullet_{\mathcal{G}^+} \in x^{\downarrow \leftarrow}$ in U , so U has $\mathcal{G}^! \in x^{\downarrow}$. If the miniswap is H9, then there is some marked label $\mathcal{E}^! \in x^{\downarrow \leftarrow}$. By T 's (G.10), there is some $\bullet_{\mathcal{G}}$ West of $x^{\downarrow \leftarrow}$ and in its row of T . By Lemma 6.6, this $\bullet_{\mathcal{G}}$ is part of a single-box snake, which undergoes miniswap H3, the $\bullet_{\mathcal{G}}$ becoming $\bullet_{\mathcal{G}^+}$ in the same position. Hence, U has $\mathcal{G}^! \in x^{\downarrow}$.

Subcase 2.2.2 ($\text{tail}(S)$ is T4 or T5): The output of T4.1 and the first output of T4.2 leave c unaffected, so, since no other box of c is part of a snake, we are

done. The three remaining possibilities (second outputs of T4.2, T4.3 and T5) are similar, so we argue them together. In these cases, it should be noticed that x^\uparrow is not part of a snake. Let \mathcal{F} and Z be as in the description of these miniswaps. Each places $\mathcal{G} \in x$ and $\mathcal{F} \cup Z \in \bar{x}$. Let \mathcal{H} be the \prec -least label on x^\downarrow or \bar{x}^\downarrow . By T 's (G.11) and since $\text{label}_T(x) = \bullet_{\mathcal{G}}$, we have $\mathcal{G} \preceq \mathcal{H}$. If $\mathcal{G} < \mathcal{H}$, this \mathcal{H} is not moved by the swap, and column c of U satisfies (G.4) at least south of x . Otherwise, $\text{family}(\mathcal{G}) = \text{family}(\mathcal{H})$. Since each of these miniswaps says \mathcal{G} or $(\mathcal{G}) \in \underline{x^\rightarrow}$, if $\mathcal{G} < \mathcal{H}$, then by T 's (G.6), $\mathcal{G} = \mathcal{H}$. If $\mathcal{G} = \mathcal{H} \in \bar{x}^\downarrow$, we are in T6, which is a contradiction. Hence, $\mathcal{G} = \mathcal{H} \in x^\downarrow$ (and \bar{x}^\downarrow is empty). Consider what happens to x^\downarrow during the swap. The analysis to show that (G.4) is satisfied south of x is essentially the same as in Subcase 2.2.1, so we omit the details.

Finally, we show (G.4) for U north of x . Let \mathcal{E} be the \prec -greatest label appearing in T in x^\uparrow or \bar{x}^\uparrow . By T 's (G.9), $\mathcal{E} < \mathcal{G}$. Each of the miniswaps of interest asserts \mathcal{G} or $(\mathcal{G}) \in \underline{x^\rightarrow}$. Hence, by T 's (G.12), $\mathcal{E} < \mathcal{G}$. Indeed, by T 's (G.12), $\text{family}(\mathcal{E}) \neq \text{family}(\mathcal{F})$ and $\text{family}(\mathcal{E}) \neq \text{family}(\mathcal{Z})$ for any $\mathcal{Z} \in Z$. Hence, by Lemma 5.5 applied to x^\rightarrow , $\mathcal{E} < \mathcal{F}$. Therefore, (G.4) holds in U in c north of x .

Subcase 2.2.3 ($\text{tail}(S)$ is T6): No tableau is produced.

Subcase 2.3 ($x \in \text{body}(S)$): By Definition-Lemma 6.8 and Lemma 6.9(III), T has $\mathcal{G} \in x^\downarrow$ and $\mathcal{G} \in x^\rightarrow$. The swap places $\mathcal{G} \in x$, and either replaces the $\mathcal{G} \in x^\downarrow$ with a $\bullet_{\mathcal{G}^+}$ or else leaves the \mathcal{G} in place there. For the rest of the analysis, one proceeds exactly in the manner given in Subcase 2.2.1.

Condition (G.5): If U violates (G.5), the violation occurs on a horizontal edge e bounding a box of a snake S in T (here, e may possibly be a northern boundary edge of S , although only the edge labels of the southern boundary edges are defined as part of S). First, assume that we are not in the case of Lemma 6.6(III), so S does not share a column with any other snake.

We break our analysis based on where e appears in relation to S .

Case 1 (e bounds a box of $\text{body}(S)$ but not a box of $\text{head}(S)$ or $\text{tail}(S)$): There is no change of labels on e between T and U . Hence, there is no (G.5) violation on e in U .

Case 2 ($e = \underline{x}$ for some $x \in \text{head}(S)$): The only head miniswaps that could introduce new edge labels onto e are H6 and H7. In these cases, T has a $\mathcal{G}^+ \in x$ that moves to $e = \underline{x}$ in U . If $\mathcal{G}' \in e$ in T with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$, we violate T 's (G.4). Hence, the $\mathcal{G}^+ \in e$ in U is the only label of its family on e , as desired.

Case 3 ($e = \bar{x}$ for some $x \in \text{head}(S)$): If x is the only box of $\text{head}(S)$, we use H1 to move a \mathcal{G} from \underline{x} to \bar{x} . If there is a label $\mathcal{G}' \in \bar{x}$ in T with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$, we violate T 's (G.4). If $|\text{head}(S)| = 2$, no miniswap introduces edges onto a northern edge.

Case 4 ($e = \underline{x}$ for some $x \in \text{tail}(S)$): If $|\text{tail}(S)| = 1$, no new edge labels occur during any miniswap (namely T1), so we are done. Therefore, we assume that $|\text{tail}(S)| = 2$. New edge labels on e can only occur when using T3 (second output). Here, T has $\mathcal{G}^+ \in x$, while U has $\mathcal{G}^+ \in \underline{x}$. If U violates (G.5), there is $\mathcal{G}' \in \underline{x}$ in T with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$, but this contradicts T 's (G.4).

Case 5 ($e = \bar{x}$ for some $x \in \text{tail}(S)$): The miniswaps that could introduce edge labels onto e are T4.2, T4.3 and T5. In the notation of those miniswaps, T has $\bullet_{\mathcal{G}} \in x$, an $\mathcal{F}^1 \in x^{\rightarrow}$, and a set Z of labels ℓ on $\underline{x}^{\rightarrow}$ such that $\mathcal{F} < \ell < \mathcal{G}$. In U , all of these labels $\{\mathcal{F}\} \cup Z$ may have moved to \bar{x} . Here, U violates (G.5) only if it has a label $\mathcal{Q} \in \bar{x}$ with $\text{family}(\mathcal{Q}) = \text{family}(\mathcal{Z})$ for some $\mathcal{Z} \in \{\mathcal{F}\} \cup Z$. However, this \mathcal{Q} and \mathcal{Z} would violate T 's (G.12).

Finally, suppose that we are in the case of Lemma 6.6(III). In T , the adjacent rows of the snakes are $\begin{array}{|c|c|} \hline \bullet_{\mathcal{G}} & \mathcal{G} \\ \hline \bullet_{\mathcal{G}^+} & \mathcal{G}^+ \\ \hline \end{array}$ or $\begin{array}{|c|c|} \hline \bullet_{\mathcal{G}^+} & \mathcal{G}^+ \\ \hline \bullet_{\mathcal{G}} & \mathcal{G} \\ \hline \end{array}$. Let x be the east box of the south row in either case. If $i_j, k_\ell \in \bar{x}$ in U , then $i_j, k_\ell \in \bar{x}$ in T , since no miniswap affects \bar{x} . By T 's (G.5), $i \neq k$.

Condition (G.6): Consider $\mathcal{H}, \mathcal{H}'$ in T with $\mathcal{H} < \mathcal{H}'$ and $\text{family}(\mathcal{H}) = \text{family}(\mathcal{H}')$. Say that the eastmost \mathcal{H} in T appears in column c , while the westmost \mathcal{H}' appears in column d . By T 's (G.4) and (G.6), c is West of d . By the swaps' construction, in any $U \in \text{swap}_{\mathcal{G}}(T)$, the westmost \mathcal{H}' in U appears at most one column west of d , while the eastmost \mathcal{H} in U is no further east than column c . In any case, no \mathcal{H}' can be West of an \mathcal{H} in U .

Condition (G.7): Let e be an edge with $\mathcal{H} \in e$ in U . We must show that there is no \mathcal{H} West of e in U .

CLAIM A.1. *Let \mathcal{R} be the region consisting of the leftmost $c - 1$ columns of T (equivalently U). If U has an \mathcal{H} in \mathcal{R} , then T has an \mathcal{H} either in \mathcal{R} or in column c .*

Proof. By inspection of the miniswaps, if there is an \mathcal{H} in column d of U , then there was an \mathcal{H} or (\mathcal{H}) in either d or d^{\rightarrow} in T . By definition of virtual labels, the existence of (\mathcal{H}) implies the existence of \mathcal{H} further West. The claim follows. \square

Case 1 ($\mathcal{H} \notin e$ in T): We list the miniswaps that put $\mathcal{H} \in e$ in U : H1, H6, H7, T3, T4.2, T4.3, T5. In what follows, \underline{x} refers to the notation of the miniswap

discussed. For H1, locally, we have $T = \begin{bmatrix} \bullet & \\ \mathcal{H} & \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H} & \\ \bullet & \end{bmatrix} = U$ (in fact, $\mathcal{H} = \mathcal{G}$). By T 's (G.7), the $\mathcal{H} \in \underline{x}$ is westmost in T . If the $\mathcal{H} \in \bar{x}(= e)$ is not westmost in U , then, by Claim A.1, there is some \mathcal{H} in e 's column in T that takes part in a miniswap leading to an \mathcal{H} West of e in U . Clearly, this \mathcal{H} is not the $\mathcal{H} \in e$. Thus, there are two \mathcal{H} s in e 's column, violating T 's (G.4). For H6, H7 and T3, we have $\mathcal{H} = \mathcal{G}^+$ and $\mathcal{G} \in \underline{x}$ in U . Thus by U 's (G.4), (G.5) and (G.6), there is not $\mathcal{H} = \mathcal{G}^+$ West of $e = \underline{x}^>$ in U . For the remaining cases, $\mathcal{H} \in \{\mathcal{F}\} \cup Z$ (in the notation of the miniswaps). These labels in $\underline{x}^>$ and $\underline{x}^>$ of T are marked. Hence, by Lemma 5.7, they are all westmost in their respective genes in T . Therefore, the same labels of e in U are westmost by Claim A.1.

Case 2 ($\mathcal{H} \in e$ in T): No miniswap involving $\mathcal{H} \in e$ both keeps an $\mathcal{H} \in e$ and puts an \mathcal{H} West of e . Thus, if there is an \mathcal{H} West of e in U , then, by T 's (G.7) combined with Claim A.1, there is an \mathcal{H} in the column of e in T other than the $\mathcal{H} \in e$. This contradicts T 's (G.4).

Condition (G.3): Consider a row r of T .

Case 1 (The labels of r strictly \prec -increase from left to right, ignoring $\bullet_{\mathcal{G}}$): If there is no $\bullet_{\mathcal{G}}$ in r , then only H9, B1 and B3 miniswaps could involve labels of r . Therefore, either r is unchanged by $\text{swap}_{\mathcal{G}}$ (or only some labels became marked), or a \mathcal{G} in r is replaced by $\bullet_{\mathcal{G}^+}$. Thus, U 's (G.3) holds for r in this situation.

Otherwise, T has $\bullet_{\mathcal{G}}$ in r , say in box x . By assumption, the exceptional configuration of (G.3) does not occur here. We consider all miniswaps involving a $\bullet_{\mathcal{G}}$. It should be noted that T and U are identical in r both West of x and East of $\underline{x}^>$. Hence, it suffices to study the effect of a miniswap locally at $\{x^<, x, x^>\}$.

If H1 or H2 applies at x , then, locally, at x , T looks like $\begin{bmatrix} \mathcal{F} & \bullet_{\mathcal{G}} & \mathcal{H} \\ & \mathcal{G} & \end{bmatrix}$ or $\begin{bmatrix} \mathcal{F} & \bullet_{\mathcal{G}} & \mathcal{H} \\ & \mathcal{G}^+ & \end{bmatrix}$, where x is the center box. (If \mathcal{F} or \mathcal{H} does not exist, the argument is simplified.) It remains to show that $\mathcal{F} \prec \mathcal{G} \prec \mathcal{H}$. By T 's (G.9), $\mathcal{F} \prec \mathcal{G}$. We cannot have $\mathcal{G} = \mathcal{H}$, as then we would apply H4 or H5.3 instead of H1 or H2. Suppose that $\mathcal{H} \prec \mathcal{G}$. Then, the $\mathcal{H} \in \underline{x}^>$ is marked, and, by Lemma 5.13, $\mathcal{G} \in \underline{x}^>$. Hence, we would apply T5 or T6 instead of H1 or H2.

Applying miniswaps H3–H5, H8, B2, B3, T2 or T6 at x clearly preserves (G.3) for r .

Suppose that H6, H7 or T3 applies at x in T . Then, locally, at x , T looks like $\begin{bmatrix} \mathcal{F} & \bullet_{\mathcal{G}} & \mathcal{G}^+ \\ & \mathcal{G} & \end{bmatrix}$, $\begin{bmatrix} \mathcal{F} & \bullet_{\mathcal{G}} & \mathcal{G}^+ \\ & \mathcal{G} & \end{bmatrix}$ or $\begin{bmatrix} \mathcal{F} & \bullet_{\mathcal{G}} & \mathcal{G}^+ \\ & \mathcal{G}^+ & \end{bmatrix}$, respectively. (If there is no \mathcal{F} , there is nothing to show.) By T 's (G.9), $\mathcal{F} \prec \mathcal{G}$. Hence, (G.3) holds for r in U .

Suppose that T1 applies at x . Then, at x , T locally looks like $\begin{array}{|c|c|c|} \hline \mathcal{F} & \bullet_{\mathcal{G}} & \mathcal{H} \\ \hline & \mathcal{G} & \\ \hline \end{array}$ (where

the $\mathcal{G} \in x^\downarrow$ is guaranteed by Lemma 6.9(IV,VII); again, if \mathcal{F} or \mathcal{H} does not exist, the argument is only easier). We must show that $\mathcal{F} < \mathcal{G} < \mathcal{H}$. By T 's (G.9), $\mathcal{F} < \mathcal{G}$. Since we are applying T1, $\mathcal{H} \neq \mathcal{G}$. Now, we repeat the above argument for H1, H2 above *verbatim*.

Suppose that T4 or T5 applies at x . Locally, at x , T looks like $\begin{array}{|c|c|c|} \hline \bullet_{\mathcal{G}} & \mathcal{E}' & \mathcal{F} \\ \hline & \mathcal{G} & \\ \hline \end{array}$ or $\begin{array}{|c|c|c|} \hline \bullet_{\mathcal{G}} & \mathcal{E}' & \mathcal{F} \\ \hline & \mathcal{G} & \bullet_{\mathcal{G}'} & \mathcal{F} \\ \hline \end{array}$, while U looks like $\begin{array}{|c|c|c|} \hline \mathcal{G} & \bullet_{\mathcal{G}'} & \mathcal{F} \\ \hline & & \\ \hline \end{array}$. By assumption, $\text{label}_T(x^{\leftarrow}) < \mathcal{E}$. Thus, if $\mathcal{G} < \mathcal{F}$ (or there is no \mathcal{F}), U 's (G.3) is satisfied. Otherwise, $\mathcal{F} \preceq \mathcal{G}$. Then, $\mathcal{F} \in x^{\rightarrow}$ is marked in U . By Lemma 5.5, $N_{\mathcal{G}} = N_{\mathcal{E}}$, so, by T 's Lemma 5.6, $\mathcal{F} \neq \mathcal{G}$. Therefore, $\mathcal{F} < \mathcal{G}$. By T 's (G.6), we have $\mathcal{F} < \mathcal{G}$. This three-box configuration $\{x, x^{\rightarrow}, x^{\rightarrow}\}$ of U is the exceptional configuration of (G.3).

Case 2 (The labels of r do not strictly increase): Thus, r contains the local configuration $\begin{array}{|c|c|c|} \hline \mathcal{H} & \bullet_{\mathcal{G}} & \mathcal{F}' \\ \hline & & \\ \hline \end{array}$, where $\mathcal{H} > \mathcal{F}$. We call the middle box x . By Lemma 5.5, there is \mathcal{G}' or \mathcal{G} in x^{\rightarrow} with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$ and $N_{\mathcal{G}'} = N_{\mathcal{F}}$.

If $\mathcal{G}' \neq \mathcal{G}$, then x^{\rightarrow} is not part of the snake containing x . Further, by Lemma 5.13, there is no \mathcal{G} or \mathcal{G} in x^\downarrow or x in T . Hence, x takes part in an H3 miniswap, r is unchanged (except for the subscript on the \bullet) and (G.3) is preserved.

If $\mathcal{G}' = \mathcal{G}$, we apply T4, T5 or T6. It should be recalled that T6 produces no tableau. In the case of T4.1 and the first output of T4.2, we make no local changes in row r , so U 's (G.3) follows from T 's. The remaining considerations are the second output of T4.2 and the outputs of T4.3 and T5. By T 's (G.9), $\mathcal{H} < \mathcal{G}$. Let $\mathcal{E} := \text{label}_T(x^{\rightarrow})$ (if \mathcal{E} does not exist, the argument is trivialized). Since $N_{\mathcal{E}} = N_{\mathcal{G}'} = N_{\mathcal{F}}$, by T 's Lemma 5.6, $\mathcal{E} \neq \mathcal{G}$. If $\mathcal{G} < \mathcal{E}$, then U 's (G.3) holds. Otherwise, $\mathcal{E} < \mathcal{G}$, and the $\mathcal{E} \in x^{\rightarrow}$ in T is marked. Given T 's \mathcal{G} or $\mathcal{G} \in x^{\rightarrow}$, it follows, by T 's (G.6), that $\mathcal{E} < \mathcal{G}$. Therefore, U has the exceptional (G.3) configuration in r .

Condition (G.8): Consider any two genes \mathcal{E} and \mathcal{F} with $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$ and $N_{\mathcal{E}} = N_{\mathcal{F}}$. It suffices to show that in U , every \mathcal{E} is read before every \mathcal{F} ; that is, we have the following.

CLAIM A.2. *Given (nonvirtual) instances e, f of \mathcal{E} and \mathcal{F} , respectively, in U , either e is East of f or else e is north of f in the same column.*

Proof. Suppose that the claim fails for some fixed choice of e and f . Thus, in U , either f is North of e in its column or else f is East of e . The first scenario contradicts U 's (G.4), so we assume that the second occurs.

If U has a label \mathcal{Q} in column c , then T has \mathcal{Q} or \mathcal{Q} in c or c^\rightarrow . Thus, since U has f East of e , by T 's Lemma 5.6, T has e and f in the same column. By T 's (G.4), e is north of f in T .

Case 1 ($\text{family}(\mathcal{E}) < \text{family}(\mathcal{G})$): We may assume that a miniswap moves e West. Since $\text{family}(\mathcal{E}) < \text{family}(\mathcal{G})$, the only such miniswaps are T4 and T5. Hence, T has a box x with $\mathcal{G} \in \underline{x}$ and either $e \in \underline{x}$ or $e \in x$. By T 's (G.4), $f \in \underline{x}$. These miniswaps may move e to \bar{x}^\leftarrow , but then, by definition, they will also move f to x^\leftarrow or \bar{x}^\leftarrow .

Case 2 ($\text{family}(\mathcal{E}) = \text{family}(\mathcal{G})$): We are done unless $\text{swap}_{\mathcal{G}}$ moves e West. The possible miniswaps are H5, B2, B3, T2 and T4. By inspection of the miniswaps, $\mathcal{E} = \mathcal{G}$.

Case 2.1 ($f \in x$): By T 's (G.4), either $e \in \bar{x}$ or else $e \in x^\uparrow$ with \bar{x} empty. In all of the miniswaps of interest, $\bullet_{\mathcal{E}} \in x^{\uparrow\leftarrow}$. Therefore, by T 's (G.2), x^\leftarrow contains a

genetic label h of some gene \mathcal{H} . The local picture is either

$\bullet_{\mathcal{E}}$	e
h	f

 or

$\bullet_{\mathcal{E}}$	\star
h	f

 (where \star is some genetic label). By T 's (G.11), h is not marked; hence, $\mathcal{H} \succeq \mathcal{E}$. Thus, by T 's (G.6), either $\mathcal{H} > \mathcal{E}$ or $\mathcal{H} = \mathcal{E}$. If $\mathcal{H} = \mathcal{E}$, h and f violate Lemma 5.6 for T . Thus, $\mathcal{E} < \mathcal{H}$. By T 's (G.3), $\mathcal{H} < \mathcal{F}$. Since $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$, we have $\text{family}(\mathcal{H}) = \text{family}(\mathcal{F})$. Indeed, by T 's (G.6), $\mathcal{F} = \mathcal{H}^+$. Now, since $N_{\mathcal{E}} = N_{\mathcal{F}}$, we violate T 's (G.8) unless there is an \mathcal{E}' above h in its column with $\text{family}(\mathcal{E}') = \text{family}(\mathcal{E})$. If this $\mathcal{E}' \in \bar{x}^\leftarrow$, then, by (G.11) and (G.6), $\mathcal{E}' = \mathcal{E}$. However, then f and this $\mathcal{E} \in \bar{x}^\leftarrow$ violate Lemma 5.6 for T . Otherwise, this \mathcal{E}' is North of $x^{\uparrow\leftarrow}$. However, then this violates T 's (G.12) together with the $e \in x^\uparrow$ or \bar{x} .

Case 2.2 ($f \in \underline{x}$): By T 's (G.4), either $e \in \underline{x}$ or $e \in x$. In the relevant miniswaps, $\bullet_{\mathcal{G}} \in x^\leftarrow$. If B2, B3 or T2 applies, then T has $\mathcal{E} \in x^{\leftarrow\downarrow}$; together with f , this violates Lemma 5.6 for T . Hence, H5 or T4 applies. Since $f \in \underline{x}$, any H5 miniswap used is H5.1, while any T4 miniswap used is T4.1; both of these fix e , which is a contradiction.

Case 3 ($\text{family}(\mathcal{E}) > \text{family}(\mathcal{G})$): Neither e nor f is affected by $\text{swap}_{\mathcal{G}}$, so the Claim holds. □

Condition (G.9): Suppose that \mathcal{F} is northwest of $\bullet_{\mathcal{G}^+} \in y$ in U and $\mathcal{G}^+ \preceq \mathcal{F}$; we seek a contradiction. We may suppose that such \mathcal{F} is in x or \underline{x} in U .

Case 1 (\mathcal{F} appears in the same position in T): By inspection of the miniswaps, T has $\bullet_{\mathcal{G}} \in y, y^\leftarrow$ or y^\uparrow . By T 's (G.9), the $\mathcal{F} \in x$ or \underline{x} is not northwest of this box. Hence, one of the following subcases occurs.

Subcase 1.1 (T has $\bullet_{\mathcal{G}} \in y^{\leftarrow}$ and y is South of x in its column): Since $\mathcal{G} < \mathcal{F}$, T contains no \mathcal{F}^{\dagger} , so, by T 's (G.4), $\mathcal{F} < \text{label}_T(y)$, implying that $\mathcal{G}^+ < \text{label}_T(y)$. Hence, y is not part of any snake in T , contradicting $\bullet_{\mathcal{G}^+} \in y$ in U .

Subcase 1.2 (T has $\bullet_{\mathcal{G}} \in y^{\uparrow}$ and y is East of x in its row): By T 's (G.3) and (G.9), $\mathcal{F} < \text{label}_T(y)$. Then, $\mathcal{G}^+ < \text{label}_T(y)$, so y is not part of any snake in T , contradicting $\bullet_{\mathcal{G}^+} \in y$ in U .

Case 2 (\mathcal{F} does not appear in the same position in T): By inspection of the miniswaps, no label $\mathcal{H} > \mathcal{G}^+$ is affected by $\text{swap}_{\mathcal{G}}$. Furthermore, labels \mathcal{G}^+ can only be affected if $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$. Since $\mathcal{G}^+ \preceq \mathcal{F}$, this implies that $\mathcal{F} = \mathcal{G}^+$, with $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$. By inspection of the miniswaps that affect \mathcal{G}^+ , T has $\mathcal{F} = \mathcal{G}^+ \in x$, while U has $\bullet_{\mathcal{G}^+} \in x$ and $\mathcal{F} = \mathcal{G}^+ \in \underline{x}$. Since $\mathcal{F} \in \underline{x}$ is Southeast of the $\bullet_{\mathcal{G}^+} \in x$ in U , by U 's (G.2), it cannot also be northwest of a $\bullet_{\mathcal{G}^+}$, contradicting our assumption.

Condition (G.11): Consider a label $\mathcal{F}^{\dagger} \in x$ or \underline{x} in U . By T 's (G.2) and inspection of the miniswaps, T has either \mathcal{F} or \mathcal{F}^{\dagger} in the same position.

Case 1 (This \mathcal{F} is marked in T): By definition, there is a $\bullet_{\mathcal{G}^+}$ northwest of \mathcal{F}^{\dagger} in U . If U has a $\bullet_{\mathcal{G}^+}$ South of x and in its column, this contradicts U 's (G.2).

We now show that U has no $\bullet_{\mathcal{G}^+}$ North of x and in its column. By inspection of the miniswaps, if $\bullet_{\mathcal{G}^+} \in y$ in U , then T has a $\bullet_{\mathcal{G}}$ northwest of y . By T 's (G.11), T has no $\bullet_{\mathcal{G}}$ in x 's column. Since $\mathcal{F} < \mathcal{G}$, by Lemma 5.11, T has no $\bullet_{\mathcal{G}}$ NorthWest of x .

Case 2 (This \mathcal{F} is unmarked in T): By definition, U has a $\bullet_{\mathcal{G}^+}$ northwest of \mathcal{F}^{\dagger} . By inspection of the miniswaps, if $\bullet_{\mathcal{G}^+} \in y$ in U , then T has a $\bullet_{\mathcal{G}}$ northwest of y . Therefore, T has a $\bullet_{\mathcal{G}}$ northwest of said \mathcal{F} . Since the \mathcal{F} is unmarked, it must be that $\mathcal{G} \preceq \mathcal{F}$. However, since \mathcal{F}^{\dagger} appears in U , $\mathcal{F} < \mathcal{G}^+$. Hence, $\mathcal{F} = \mathcal{G}$. By Lemma 5.11, T has no $\bullet_{\mathcal{G}}$ NorthWest of x . Therefore, T has $\bullet_{\mathcal{G}}$ either West of x and in its row or else North of x and in its column. Since $\mathcal{F} = \mathcal{G}$, only the latter case is a concern. In that case, by T 's (G.4) and (G.11), in fact, $\mathcal{F} = \mathcal{G} \in x$ and $\bullet_{\mathcal{G}} \in x^{\uparrow}$. Hence, by inspection of the miniswaps, U has $\mathcal{F} = \mathcal{G} \in x^{\uparrow}$. (It should be noted that T4 does not apply at x^{\uparrow} by T 's (G.7).) Thus, U has no $\bullet_{\mathcal{G}^+}$ in x 's column.

Condition (G.10): Consider a label $\mathcal{F}^{\dagger} \in x$ or \underline{x} in U . By T 's (G.2) and inspection of the miniswaps, T has either \mathcal{F} or \mathcal{F}^{\dagger} in the same position.

Case 1 (This \mathcal{F} is marked in T): By inspection of the miniswaps, if $\bullet_{\mathcal{G}^+} \in y$ in U , then T has a $\bullet_{\mathcal{G}}$ northwest of y . By T 's (G.11), T has no $\bullet_{\mathcal{G}}$ in x 's column. Since $\mathcal{F} < \mathcal{G}$, by Lemma 5.11, T has no $\bullet_{\mathcal{G}}$ NorthWest of x . Hence, U has no

•_{G+} Northwest of x. However, by definition, U has a •_{G+} northwest of \mathcal{F}^\dagger , so it must be in x's row.

Case 2 (This \mathcal{F} is unmarked in T): Since \mathcal{F}^\dagger appears in U, $\mathcal{F} \preceq \mathcal{G}$. By definition, U has •_{G+} northwest of said \mathcal{F}^\dagger . By inspection of the miniswaps, if •_{G+} ∈ y in U, then •_G appeared northwest of y in T. Hence, T has •_G northwest of this \mathcal{F} . Since this \mathcal{F} is unmarked in T, $\mathcal{F} \succeq \mathcal{G}$. Thus, $\mathcal{F} = \mathcal{G}$.

By Lemma 5.11, T has no •_G NorthWest of x. Therefore, U has no •_{G+} NorthWest of x. However, U has some •_{G+} northwest of x, so it is either West of x in x's row or North of x in x's column. In the former case, we are done; in the latter case, we contradict U's (G.11).

Condition (G.12): Define the neighborhood of a box u to be $\text{Neigh}(u) := \{u, u^\leftarrow, u^\uparrow, \underline{u}, \bar{u}^\leftarrow\}$. For a lower edge \underline{u} , let $\text{Neigh}(\underline{u}) := \{\underline{u}, u, u^\leftarrow, \bar{u}, \bar{u}^\leftarrow\}$. Given a (possibly virtual) instance $q \in u$ or \underline{u} in T of the gene Q, let the children of q be all (nonvirtual) Qs in U in $\text{Neigh}(u)$ or $\text{Neigh}(\underline{u})$, respectively. Finally, define the children of a •_G ∈ u in T to be those •_{G+} in u, $u^\rightarrow, u^\downarrow$ in U. Clearly, we have the following.

LEMMA A.3. Every q in $U \in \text{swap}_G(T)$ is a child of at least one (possibly virtual) q in T. Moreover,, every •_{G+} in $U \in \text{swap}_G(T)$ is a child of at least one •_G in T. □

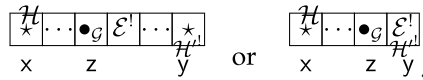
Suppose that H and H' are instances in T of genes of the same family. By Lemma A.3, it suffices to confirm U's (G.12) for ℓ a child of H and ℓ' a child of H'. To do this, we break into cases depending on the relative position of H and H'. By relabeling, we may assume that H is west of H'. Specifically, Cases 1–3 below concern the situation H NorthWest of H'. Cases 4–7 consider the case H southwest of H'.

For the first three cases, let x, y be boxes in the same row r of T with x West of y. By T's (G.12), we may assume that $\mathcal{H} \in \bar{x}$ or $\mathcal{H} \in x$, as well as $\mathcal{H}' \in y$ or $\mathcal{H}' \in \underline{y}$. By T's (G.12), there is a •_G in some box z of row r appearing East of x and west of y.

Case 1 (In T, we have H or $\mathcal{H} \in \bar{x}$ and H' or $\mathcal{H}' \in \underline{y}$): By T's (G.4), $\mathcal{H} < \text{label}_T(x)$. By T's (G.9), $\text{label}_T(x) < \mathcal{G}$. Hence, $\mathcal{H} < \mathcal{G}$. Since $\text{family}(\mathcal{H}) = \text{family}(\mathcal{H}')$, also $\mathcal{H}' < \mathcal{G}$. Therefore, the $\mathcal{H}' \in \underline{y}$ is marked (and is not virtual). By T's (G.11), this forces $z \neq y$. For convenience, we assume that $\mathcal{H} \in \bar{x}$. (The argument where this label is virtual is strictly easier.)

By Lemma 5.4, the box labels in y and in every box strictly between y and z are also marked. Let $\text{label}_T(z^\rightarrow) := \mathcal{E}^\dagger$ and note that $\mathcal{E} < \mathcal{H}' < \mathcal{G}$ (the first inequality by a combination of T's (G.3) and (G.4)). Summarizing, T locally

looks like one of the following at r :



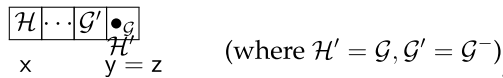
By T 's (G.2), $\bullet_{\mathcal{G}} \notin x^{\uparrow}, x^{\uparrow\leftarrow}$. Therefore, \bar{x} is not part of a snake, and so the only child of $\mathcal{H} \in \bar{x}$ in U is in \bar{x} .

Suppose that z is the only box in its snake section; that is, we apply H1, H2, H3 or T1. If the miniswap is H1, H2 or T1, then \mathcal{G} or \mathcal{G} appears in z^{\downarrow} or \underline{z} . Hence, by Lemma 5.13, $\mathcal{G} \in \underline{z}^{\rightarrow}$, so z^{\rightarrow} is adjoined by (R.3), contradicting z being the only box in its snake section. Thus, the miniswap is H3, and the unique child of $\bullet_{\mathcal{G}}$ is in z . Moreover, y is not part of a snake, or takes part in a trivial H9 miniswap. Hence, the unique child of $\mathcal{H}' \in \underline{y}$ is at \underline{y} in U . Thus, U 's (G.12) holds in this scenario.

Otherwise, the miniswap at z involves z and z^{\rightarrow} . Then, the miniswap is T4, T5 or T6. In these cases, the child of $\bullet_{\mathcal{G}}$ is in either z or z^{\rightarrow} in U . If the child of $\mathcal{H}' \in \underline{y}$ is in \underline{y} , we are done. If not, $y = z^{\rightarrow}$ and $\mathcal{H}' \in Z$ (in the notation of the miniswaps). Then, the child of \mathcal{H}' is at \bar{z} in U , so the child of \mathcal{H} is not North of the child of \mathcal{H}' and (G.12) holds vacuously.

Case 2 (In T , we have $\mathcal{H} \in x$ and \mathcal{H}' or $\mathcal{H}' \in \underline{y}$): By T 's (G.9), $\mathcal{H} < \mathcal{G}$.

Subcase 2.1 ($z = y$): By T 's (G.11), the \mathcal{H}' or $\mathcal{H}' \in \underline{y}$ is not marked; hence, $\mathcal{G} \leq \mathcal{H}'$. Thus, $\mathcal{H} < \mathcal{G} \leq \mathcal{H}'$, so $\text{family}(\mathcal{G}) = \text{family}(\mathcal{H})$. Combined with T 's (G.2), this implies that the unique child of $\mathcal{H} \in x$ is in x . By T 's (G.3) and (G.9), T has $\mathcal{G}' \in z^{\leftarrow}$ with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G}) = \text{family}(\mathcal{H})$ and $\mathcal{G}' < \mathcal{G}$. Hence, by T 's (G.6), $\mathcal{G}' = \mathcal{G}^-$ and $\mathcal{H}' = \mathcal{G}$; moreover, by T 's (V.2), this $\mathcal{H}' = \mathcal{G}$ is not virtual, since it is westmost. Hence, locally at r , T is

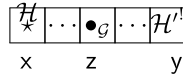


Thus, the miniswap involving z is one of H1, H4, H6 and T6. Now, H4 and T6 produce no output. If H1 or H6 applies, the child of $\mathcal{H}' \in \underline{y} = \underline{z}$ is northEast of the child of $\mathcal{H} \in x$, so (G.12) is confirmed vacuously.

Subcase 2.2 ($z \neq y$): By T 's (G.4), $\text{label}_T(y) < \mathcal{H}'$. Hence, by $\text{family}(\mathcal{H}) \leq \text{family}(\mathcal{G})$, $\text{label}_T(y)$ is marked. By Lemma 5.4, some $\mathcal{E}^! \in z^{\rightarrow}$. The remainder of this case is argued exactly as Case 1.

Case 3 (In T , we have \mathcal{H} or $\mathcal{H} \in \bar{x}$ and $\mathcal{H}' \in y$): Since $\mathcal{H}' \in y$, $\bullet_{\mathcal{G}} \notin y$, so $z \neq y$. By T 's (G.4) and (G.9), $\mathcal{H} < \text{label}_T(x) < \mathcal{G}$. Therefore, also $\mathcal{H}' < \mathcal{G}$,

and so $\mathcal{H}' \in \mathbf{y}$ is marked. Since \bar{x} does not participate in the swap, if $\mathcal{H} \in \bar{x}$, this \mathcal{H} has no children, so the (G.12) confirmation is vacuous here. Therefore, we assume that $\mathcal{H} \in \bar{x}$ in T ; since \bar{x} does not participate in the swap, its only child is in the same position in U . In summary, locally at r , T is



Subcase 3.1 ($z = \mathbf{y}^{\leftarrow}$): Consider the miniswap involving z . First, suppose that z is not the only box in its snake section. By Definition-Lemma 6.8(I,II), it involves \mathbf{y} and is T4, T5 or T6. The last miniswap produces no output. For the first two miniswaps, one possibility is that the child of $\mathcal{H}'^1 \in \mathbf{y}$ is at \bar{z} in U . Here, the (G.12) confirmation is vacuous. Otherwise, U 's unique child of $\mathcal{H}'^1 \in \mathbf{y}$ is in \mathbf{y} ; here, the unique child of $\bullet_{\mathcal{G}} \in \mathbf{z}$ is in \mathbf{z} , and so U 's (G.12) holds.

Otherwise, z is the only box in its snake section. Thus, H1, H2, H3 or T1 applies. If H1, H2 or T1 applies, then, by definition or Lemma 6.9(IV,VII), T has \mathcal{G} or \mathcal{G} in \mathbf{z}^{\downarrow} or \underline{z} . Hence, by Lemma 5.13, $\mathcal{G} \in \mathbf{y}$, so \mathbf{y} is adjoined by (R.3), contradicting z being the only box in its snake section. Thus, it is H3, and U 's unique children of $\mathcal{H}' \in \mathbf{y}$ and $\bullet_{\mathcal{G}} \in \mathbf{z}$ are in \mathbf{y} , \mathbf{z} , respectively; hence, (G.12) is confirmed here.

Subcase 3.2 ($z \neq \mathbf{y}^{\leftarrow}$): In this case, $\mathcal{H}' \in \mathbf{y}$ is not part of a snake in T or takes part in a trivial H9 miniswap; thus, its unique child is in \mathbf{y} in U . Hence, it suffices to check that U has a $\bullet_{\mathcal{G}^+}$ between x and y . By Lemma 5.4(II), since $\text{label}_T(\mathbf{y})$ is marked, $\text{label}_T(\mathbf{z}^{\rightarrow})$ is also marked. Consider the miniswap involving z . If T has no \mathcal{G} or \mathcal{G} in \mathbf{z}^{\downarrow} or \underline{z} , then the miniswap must be one of H3, T4 or T5 (it cannot be T1 by Lemma 6.9(IV,VII)). For each of these, a child of $\bullet_{\mathcal{G}} \in \mathbf{z}$ appears in \mathbf{z} or \mathbf{z}^{\rightarrow} in U . If T has \mathcal{G} or \mathcal{G} in \mathbf{z}^{\downarrow} or \underline{z} , then, by Lemma 5.13, $\mathcal{G} \in \underline{\mathbf{z}}^{\rightarrow}$. Hence, the miniswap is T5 or T6. In the former case, a child of $\bullet_{\mathcal{G}} \in \mathbf{z}$ appears in \mathbf{z}^{\rightarrow} in U . In the latter case, U does not exist.

Case 4 (In T , $\mathcal{H} \in \mathbf{a}$ is southwest of $\mathcal{H}' \in \mathbf{b}$): We will use the following.

CLAIM A.4. *In U , each child of $\mathcal{H} \in \mathbf{a}$ is west of each child of $\mathcal{H}' \in \mathbf{b}$.*

Proof. If \mathbf{a} is West of \mathbf{b} , then the claim holds by the definition of children. Therefore, assume that \mathbf{a} and \mathbf{b} are in the same column. By T 's (G.4), $\mathcal{H} = \mathcal{H}'$, and $\mathcal{H} \in \mathbf{a}$ is marked. Hence, $\mathcal{H} < \mathcal{G}$. By Lemma 5.4(II), T has a $\bullet_{\mathcal{G}}$ in some box \mathbf{z} West of \mathbf{a} and in its row. By Lemma 5.4(II), every box label strictly between \mathbf{z} and \mathbf{a} is also marked. Thus, by T 's (G.11), T has no $\bullet_{\mathcal{G}}$ in any column East

of \mathbf{z} and west of \mathbf{a} . Furthermore, by T 's (G.2), T has no $\bullet_{\mathcal{G}}$ Northwest of \mathbf{z} .

Summarizing, T is locally $\begin{matrix} & & \mathcal{H} \\ & \bullet_{\mathcal{G}} & \cdots & \mathcal{H}' \\ \mathbf{z} & & & \mathbf{a} \end{matrix}$. Hence, \mathbf{b} is not part of any snake, and so U 's unique child of $\mathcal{H}' \in \mathbf{b}$ is in \mathbf{b} . \square

No child of $\mathcal{H} \in \mathbf{a}$ is North of \mathbf{a}^\uparrow , and no child of $\mathcal{H}' \in \mathbf{b}$ is South of \mathbf{b} . Hence, if \mathbf{a} is at least two rows below \mathbf{b} , the (G.12) confirmation is vacuous.

Subcase 4.1 (\mathbf{a} is exactly one row south of \mathbf{b}): By inspection of the miniswaps, a child of $\mathcal{H} \in \mathbf{a}$ can only appear North of a child of $\mathcal{H}' \in \mathbf{b}$ if $\bullet_{\mathcal{G}} \in \mathbf{a}^\uparrow, \mathbf{b}^\leftarrow$. Then, by T 's (G.2), $\mathbf{a}^\uparrow = \mathbf{b}^\leftarrow$. Here, the $\mathcal{H}' \in \mathbf{b}$ has a child South of \mathbf{b} only if T is locally $\begin{matrix} \bullet_{\mathcal{G}} & \mathcal{H}' \\ \mathcal{H} & \end{matrix} = \begin{matrix} \bullet_{\mathcal{G}} & \mathcal{G}^+ \\ \mathcal{G} & \end{matrix}$ and $\mathcal{H}' = \mathcal{G}^+ \in \mathbf{b}$ is part of a T3 miniswap;

that is, $\begin{matrix} \bullet_{\mathcal{H}} & \mathcal{H}' \\ \mathcal{H} & \end{matrix} \mapsto \begin{matrix} \mathcal{H} & \bullet_{\mathcal{H}'} \\ \star & \mathcal{H}' \end{matrix}$ (here $\star \in \{\mathcal{H}', \bullet_{\mathcal{H}'}\}$; the uncertainty is irrelevant). By T 's (G.3), it follows that $\text{label}_U(\mathbf{b}^\rightarrow)$ is not marked and $\text{label}_U(\mathbf{b}^\rightarrow) \neq \mathcal{H}'$. Thus, we confirm (G.12).

Subcase 4.2 (\mathbf{a} is in the same row r as \mathbf{b}): Suppose $\mathcal{H}' \in \mathbf{b}$ has a child South of \mathbf{b} . Then, \mathbf{b} is part of an H6, H7 or T3 miniswap. Therefore, U 's unique child of $\mathcal{H}' \in \mathbf{b}$ is at \mathbf{b} and U has $\bullet_{\mathcal{G}} \in \mathbf{b}$. By T 's (G.2), T has no $\bullet_{\mathcal{G}} \in \mathbf{a}^\uparrow$, so no child of $\mathcal{H} \in \mathbf{a}$ is in \mathbf{a}^\uparrow . Thus, U 's (G.12) is confirmed.

Otherwise, no child of $\mathcal{H}' \in \mathbf{b}$ is South of \mathbf{b} . If a child of $\mathcal{H} \in \mathbf{a}$ is in $\overline{\mathbf{a}^\leftarrow}$, we used T4 or T5 at \mathbf{a} ; thus, U has $\bullet_{\mathcal{G}^+} \in \mathbf{a}$ and (G.12) is confirmed. Thus, it remains to consider the scenario that a child of $\mathcal{H} \in \mathbf{a}$ is at \mathbf{a}^\uparrow in U . This scenario is impossible. By inspection of the miniswaps, $\mathcal{G} = \mathcal{H}$ and T has $\bullet_{\mathcal{G}} \in \mathbf{a}^\uparrow$. Let

$\text{label}_T(\mathbf{b}^\uparrow) := \mathcal{E}$ (by T 's (G.2), $\bullet_{\mathcal{G}} \notin \mathbf{b}^\uparrow$). Locally, T is $\begin{matrix} \bullet_{\mathcal{H}} & \cdots & \mathcal{E} \\ \mathcal{H} & \cdots & \mathcal{H}' \end{matrix}$ (where $\mathcal{H} = \mathcal{G}$). By T 's (G.4), either $\mathcal{E} < \mathcal{H}'$ or $\mathcal{E} = \mathcal{H}'$. If $\mathcal{E} = \mathcal{H}'$, then, by T 's (G.4), the $\mathcal{H}' \in \mathbf{b}$ is marked; hence, $\mathcal{H}' < \mathcal{H}$, contradicting T 's (G.6). Thus, $\mathcal{E} < \mathcal{H}'$. Then, $\mathcal{E} < \mathcal{H}$, so $\mathcal{E} \in \mathbf{b}^\uparrow$ is marked. By Lemma 5.5, T has a label \mathcal{H}'' or $(\mathcal{H}'') \in \mathbf{b}^\uparrow$ with $\text{family}(\mathcal{H}'') = \text{family}(\mathcal{H})$. With T 's $\mathcal{H}' \in \mathbf{b}$, this violates T 's (G.4).

Case 5 (In T , $\mathcal{H} \in \mathbf{a}$ is southwest of $\mathcal{H}' \in \mathbf{b}$): By T 's (G.4), \mathbf{a} is West of \mathbf{b} .

Subcase 5.1 (\mathbf{a} South of \mathbf{b}): By the definition of children, every child of $\mathcal{H} \in \mathbf{a}$ is southwest of every child of $\mathcal{H}' \in \mathbf{b}$, so there is nothing to confirm here.

Subcase 5.2 (\mathbf{a} and \mathbf{b} are in the same row): By the definition of children, every child of $\mathcal{H} \in \mathbf{a}$ is west of every child of $\mathcal{H}' \in \mathbf{b}$. Every child of $\mathcal{H}' \in \mathbf{b}$ is north of \mathbf{b} . Thus, we are only concerned with the cases where a child of $\mathcal{H} \in \mathbf{a}$

is north of \mathbf{a} , so we assume this. Moreover, by inspection of the miniswaps, we may assume that T has $\bullet_G \in \mathbf{a}$ or $\bullet_G \in \mathbf{a}^\leftarrow$.

If $\bullet_G \in \mathbf{a}^\leftarrow$, then \mathbf{a} is part of a T4 or T5 miniswap. Hence, the unique child of $\mathcal{H} \in \underline{\mathbf{a}}$ is in \mathbf{a}^\leftarrow or $\overline{\mathbf{a}^\leftarrow}$; the unique child of $\mathcal{H}' \in \mathbf{b}$ is in \mathbf{b} ; and U has $\bullet_{G^+} \in \mathbf{a}$. Therefore, (G.12) is confirmed.

Otherwise, T has $\bullet_G \in \mathbf{a}$ and, moreover, $\mathcal{H} = \mathcal{G}$. Let $\mathcal{E} := \text{label}_T(\mathbf{a}^\rightarrow)$.

Locally, between \mathbf{a} and \mathbf{b} , T is $\begin{matrix} \bullet_{\mathcal{H}} & \mathcal{E} & \cdots & \mathcal{H}' \\ \mathcal{H} & & & \end{matrix} \mathbf{b}$. If $\mathcal{E}' \in \mathbf{a}^\rightarrow$, then, by Lemma 5.13, $(\mathcal{H}) \in \underline{\mathbf{a}^\rightarrow}$, and so \mathbf{a} swaps by T6 and there is no tableau U . Thus, $\mathcal{E} \in \mathbf{a}^\rightarrow$ is not marked, and by T 's (G.3), $\text{family}(\mathcal{E}) = \text{family}(\mathcal{H})$. By T 's (G.6), either $\mathcal{E} = \mathcal{H}$ or $\mathcal{E} = \mathcal{H}^+$. In the former case, H4 applies at \mathbf{a} , producing no U . In the latter case, H6 applies and one confirms (G.12) by inspection. (If $\mathbf{b} = \mathbf{a}^\rightarrow$, one checks, as was done in Case 4, that the configuration of the final sentence of (G.12) does not occur.)

Case 6 (In T , $\mathcal{H} \in \mathbf{a}$ is southwest of $\mathcal{H}' \in \mathbf{b}$): By T 's (G.4), \mathbf{a} is SouthWest of \mathbf{b} . We may assume that \mathbf{a} is one row South of \mathbf{b} and U has a child of $\mathcal{H} \in \mathbf{a}$ in \mathbf{a}^\uparrow (otherwise, the (G.12) check is vacuous). The unique child of $\mathcal{H}' \in \mathbf{b}$ is in \mathbf{b} . Hence, $\mathcal{H} = \mathcal{G}$ and T has $\bullet_G \in \mathbf{a}^\uparrow$. Consider the miniswap involving \mathbf{a}^\uparrow . It is B2, B3, T1, T2, T3, T4 or T5. If it is B2, B3, T2 or T3, then, locally, T is

$$\begin{matrix} \bullet_{\mathcal{H}} & \mathcal{H} & \cdots & * \\ \mathcal{H} & & & \mathcal{H}' \end{matrix} \mathbf{b} \quad \text{or} \quad \begin{matrix} \bullet_{\mathcal{H}} & \mathcal{H}^+ & \cdots & * \\ \mathcal{H} & & & \mathcal{H}' \end{matrix} \mathbf{b}$$

Since, by T 's (G.2), the $\bullet_{\mathcal{H}} \in \mathbf{a}^\uparrow$ is the only $\bullet_{\mathcal{H}}$ in its row, this contradicts T 's (G.12). Thus, it is T1, T4 or T5. Let $\mathcal{E} := \text{label}_T(\mathbf{a}^{\uparrow\rightarrow})$. By T 's (G.3) and (G.4), $\mathcal{E} < \mathcal{H}'$. Hence, $\mathcal{E} < \mathcal{H}$, and so $\mathcal{E}' \in \mathbf{a}^{\uparrow\rightarrow}$. By Lemma 5.13, T has $(\mathcal{H}) \in \underline{\mathbf{a}^{\uparrow\rightarrow}}$. Therefore, the miniswap is T5 and one confirms (G.12) directly.

Case 7 (T has $\mathcal{H} \in \underline{\mathbf{a}}$ southwest of $\mathcal{H}' \in \mathbf{b}$): By T 's (G.4) and (G.5), \mathbf{a} is West of \mathbf{b} . Hence, by the definition of children, U has every child of $\mathcal{H} \in \underline{\mathbf{a}}$ west of every child of $\mathcal{H}' \in \mathbf{b}$. We may assume that \mathbf{a} and \mathbf{b} are in the same row and that some child of $\mathcal{H} \in \underline{\mathbf{a}}$ is north of \mathbf{a} , as otherwise the (G.12) confirmation is vacuous. Then, T has $\bullet_G \in \mathbf{a}$ or $\bullet_G \in \mathbf{a}^\leftarrow$.

Subcase 7.1 ($\bullet_G \in \mathbf{a}$): Here, $\mathcal{G} = \mathcal{H}$. By T 's (G.4), $\text{label}_T(\mathbf{b}) < \mathcal{H}'$, whence $\text{label}_T(\mathbf{b}) < \mathcal{H}$. Therefore, $\text{label}_T(\mathbf{b})$ is marked. By Lemma 5.4(II), $\text{label}_T(\mathbf{a}^\rightarrow)$ is also marked, and so, by Lemma 5.13, $(\mathcal{H}) \in \underline{\mathbf{a}^\rightarrow}$. Thus, the miniswap involving \mathbf{a} is T6, and U does not exist.

Subcase 7.2 ($\bullet_G \in \mathbf{a}^\leftarrow$): The miniswap involving \mathbf{a}^\leftarrow is either T4 or T5. Here, U has $\bullet_{G^+} \in \mathbf{a}$ and the unique child of $\mathcal{H} \in \underline{\mathbf{a}}$ is in \mathbf{a}^\leftarrow or $\overline{\mathbf{a}^\leftarrow}$. The (G.12) confirmation is therefore clear.

Condition (G.13): By inspection of the miniswaps, if $\mathcal{E}^! \in \underline{x}$ or \underline{x} in U , then T has \mathcal{E} or $\mathcal{E}^!$ in the same location. Thus, there are two cases.

Case 1 (This \mathcal{E} is marked in T): By T 's (G.13), there is an \mathcal{F} or \mathcal{F} in \underline{x} with $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$. If $\mathcal{F} \in \underline{x}$ is nonvirtual, then, since it appears in the same place in U , U 's (G.13) holds. Thus, suppose that T has \mathcal{F} in \underline{x} . We check the conditions for \mathcal{F} to appear in \underline{x} in U . Let U^* be U with \mathcal{F} added in \underline{x} .

((V.1) holds; that is, $\mathcal{F} \in \underline{x}$ is not marked in U^*): By Lemma 5.9, T has $\bullet_{\mathcal{G}} \in \underline{x}^{\leftarrow}$. It should be noted that $\mathcal{F} \succeq \mathcal{G}$, since otherwise $\mathcal{F} \in \underline{x}$ would be marked in T , which is a contradiction. If $\mathcal{F} = \mathcal{G}$, then T5 or T6 would apply at $\{\underline{x}^{\leftarrow}, \underline{x}\}$, contradicting $\mathcal{E}^! \in \underline{x}$ or $\mathcal{E}^! \in \underline{x}$ in U . Thus, $\mathcal{F} \succ \mathcal{G}$, as desired.

((V.2) holds; that is, U has an \mathcal{F} West of \underline{x}): By T 's (V.2), T has an \mathcal{F} West of \underline{x} . This remains true for U since no swap removes a nonvirtual genetic label without putting one further west.

((G.1) holds for U^*): This is immediate from T 's $\mathcal{F} \in \underline{x}$ and U 's (G.1).

((G.4) holds for U^*): We have $\mathcal{E} < \mathcal{F} < \text{label}_T(\underline{x}^{\downarrow})$. Since $\mathcal{F} \succ \mathcal{G}$, $\text{label}_T(\underline{x}^{\downarrow}) = \text{label}_U(\underline{x}^{\downarrow})$. Hence, U^* does not violate (G.4) locally, so, by U 's (G.4), we are done.

((G.5) holds for U^*): Since $\mathcal{E}^! \in \underline{x}$ or \underline{x} in U , by inspection of the miniswaps, T and U have the same set of nonvirtual labels on \underline{x} .

((G.6) holds for U^*): If there is a $\tilde{\mathcal{F}}$ West of \underline{x} 's column in U , there is one west of \underline{x} 's column in T . Hence, we are done, by T 's (V.3).

((G.8) holds for U^*): It suffices to show that, in the column reading word of U^* (with \mathcal{F} placed in \underline{x}), no \mathcal{E} is read after the $\mathcal{F} \in \underline{x}$. By T 's (V.3), this is true in T . By inspection of the miniswaps, \mathcal{E} does not appear West of \underline{x} in U . Hence, by U^* 's (G.4), we are done.

((G.9) holds for U^*): Since U has $\mathcal{E}^! \in \underline{x}$ or \underline{x} , U has a $\bullet_{\mathcal{G}^+}$ northwest of \underline{x} . Therefore, by U 's (G.2), there is no $\bullet_{\mathcal{G}^+}$ southeast of \underline{x} .

((G.12) holds for U^*): This follows from the following two claims.

CLAIM A.5. *If $\text{family}(\mathcal{F}) = \text{family}(\tilde{\mathcal{F}})$, then U has no $\tilde{\mathcal{F}}$ SouthEast of \underline{x} .*

Proof. By T 's (G.11) and T 's $\mathcal{E}^!$ in \underline{x} 's column, T has a $\bullet_{\mathcal{G}}$ northWest of \underline{x} . Hence, by T 's (G.2), T has no $\bullet_{\mathcal{G}}$ SouthEast of \underline{x} . Thus, by the (G.12) condition of T 's (V.3), T has no $\tilde{\mathcal{F}}$ SouthEast of $\mathcal{F} \in \underline{x}$. A child of $\tilde{\mathcal{F}}$ will only be South of its parent if T has $\tilde{\mathcal{F}} \in \underline{y}$ and U has $\tilde{\mathcal{F}} \in \underline{y}$. Thus, U has no $\tilde{\mathcal{F}}$ SouthEast of \underline{x} that is a child of a nonvirtual $\tilde{\mathcal{F}}$ in T .

The remaining concern is $\mathcal{F} \in \underline{y}$ in T with a child SouthEast of \underline{x} in U .

By inspection of the miniswaps, this $(\tilde{\mathcal{F}})$ cannot have a child South or East of \underline{y} . Therefore, \underline{y} must be SouthEast of \underline{x} in T . Again, by inspection of the miniswaps, T has a $\bullet_{\mathcal{G}} \in \underline{y}^{\leftarrow}$ or a $\bullet_{\mathcal{G}} \in \underline{y}$. This is impossible by T 's (G.2), recalling T 's $\bullet_{\mathcal{G}}$ northWest of \underline{x} . \square

CLAIM A.6. *If U has an $\tilde{\mathcal{F}}$ NorthWest of \underline{x} with $\text{family}(\mathcal{F}) = \text{family}(\tilde{\mathcal{F}})$, then this $\tilde{\mathcal{F}}$ and the $\mathcal{F} \in \underline{x}$ satisfy U^* 's (G.12).*

Proof. By assumption, \mathcal{E}^{\downarrow} remains in \underline{x} or \underline{x} in U . By Lemma 5.9, T has $\bullet_{\mathcal{G}} \in \underline{x}^{\leftarrow}$ and either $\text{family}(\mathcal{G}) = \text{family}(\mathcal{F})$ or $\text{family}(\mathcal{G}) = \text{family}(\mathcal{E})$. By T 's (V.1), $\mathcal{G} \preceq \mathcal{F}$. If $\mathcal{G} = \mathcal{F}$, then $\{\underline{x}^{\leftarrow}, \underline{x}\}$ is a tail of type T5 or T6; however, these miniswaps do not leave \mathcal{E}^{\downarrow} in place, which is a contradiction. Hence, $\mathcal{G} \prec \mathcal{F}$. Therefore, by T 's (G.4) and (G.5) and Lemma 5.13, we observe that T has no \mathcal{G} or (\mathcal{G}) in $\underline{x}^{\leftarrow}$'s column. Hence, H3 applies at $\underline{x}^{\leftarrow}$, and U has $\bullet_{\mathcal{G}^+} \in \underline{x}^{\leftarrow}$.

By T 's (G.12), there is no $\tilde{\mathcal{F}}$ NorthWest of \bar{x} . Since no miniswap moves a label more than one box north, it suffices to consider an $\tilde{\mathcal{F}}$ in T that is in either \underline{y} , \underline{y} , \bar{y} or $\underline{y}^{\downarrow}$, where \underline{y} is in \underline{x} 's row. By T 's (G.2) and $\bullet_{\mathcal{G}} \in \underline{x}^{\leftarrow}$, T has no $\bullet_{\mathcal{G}}$ strictly northwest of $\underline{x}^{\leftarrow}$. Hence, no such $\tilde{\mathcal{F}}$ in these four positions can move North. Therefore, any $\tilde{\mathcal{F}}$ NorthWest of $\mathcal{F} \in \underline{x}$ in U satisfies (G.12), in view of the $\bullet_{\mathcal{G}^+} \in \underline{x}^{\leftarrow}$. It should be noted that the forbidden configuration from the final sentence of (G.12) cannot occur since $\mathcal{F} \neq \mathcal{G}$, whereas the forbidden configuration forces $\mathcal{F} = \mathcal{G}$. \square

Case 2 (This \mathcal{E} is unmarked in T): By Lemma A.3, since there is a $\bullet_{\mathcal{G}^+}$ northwest of this \mathcal{E} in U , there is a $\bullet_{\mathcal{G}}$ northwest of \underline{x} in T . Moreover, by U 's (G.11), $\underline{x}^{\leftarrow}$ is a box of U , and hence of T . Since this \mathcal{E} is unmarked in T , $\mathcal{E} \succeq \mathcal{G}$. Since this \mathcal{E} is marked in U , $\mathcal{E} \prec \mathcal{G}^+$. Thus, $\mathcal{E} = \mathcal{G}$. Let \mathcal{H} be the gene (if it exists) with $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$. If T has $\mathcal{H} \in \underline{x}$, then, since it appears in the same place in U , U 's (G.13) holds. Thus, we assume that $\mathcal{H} \notin \underline{x}$ in T .

Subcase 2.1 (In T , $\bullet_{\mathcal{G}} \in \underline{x}^{\leftarrow}$): The miniswap applied at $\underline{x}^{\leftarrow}$ is H5.2 or T4.2. We are done by the following lemma.

LEMMA A.7. *If T is a \mathcal{G} -good tableau where H5.2 or T4.2 applies and $U \in \text{swap}_{\mathcal{G}}(T)$, then all prescribed (\mathcal{H}) s from the outputs of H5.2 and T4.2 are valid virtual labels in the sense of (V.1)–(V.3).*

Proof. Consider such a miniswap. We may assume that U contains an output with a prescribed (\mathcal{H}) , say in \underline{x} . Let U^* be U with \mathcal{H} added in \underline{x} .

((V.1) holds; that is, $\mathcal{H} \in \underline{x}$ is not marked in U^*): By assumption, $\mathcal{G} < \mathcal{H}$ and so $\mathcal{G}^+ \preceq \mathcal{H}$.

((V.2) holds; that is, some \mathcal{H} appears in U West of x): Since $(\mathcal{H}) \in \underline{x}$ in T , by T 's (V.2), some \mathcal{H} appears in T West of x . By inspection of the miniswaps, this \mathcal{H} has a child, which is West of x in U .

((G.1) holds in U^*): This is immediate from the (G.1) condition of T 's (V.3).

((G.4) holds in U^*): By U 's (G.4), the only concern is an $\tilde{\mathcal{H}}$ in x 's column of U with $\text{family}(\tilde{\mathcal{H}}) = \text{family}(\mathcal{H})$. By Lemma A.3, this $\tilde{\mathcal{H}}$ is a child of an \mathcal{H} in T . Since $\mathcal{G} < \tilde{\mathcal{H}}$, this $\tilde{\mathcal{H}}$ in T is in the same location as its unique child in U , contradicting the (G.4) condition of T 's (V.3).

((G.5) holds in U^*): Neither miniswap in question affects the nonvirtual labels on \underline{x} , so we are done by the (G.5) condition of T 's (V.3).

((G.6) holds in U^*): All labels of this family appear in the same places in T and U , so we are done by the (G.6) condition of T 's (V.3).

((G.8) holds in U^*): First, suppose that there is a nonballot genotype G_{U^*} of U^* , with the $\mathcal{H} \in \underline{x}$ taken. Since no labels of $\text{family}(\mathcal{H}) + k$ (for $k \geq 0$) are moved by $\text{swap}_{\mathcal{G}}$, by T 's (G.8), there is no violation among those families. By U 's (G.8), it suffices to consider the possibility that in $\text{word}(G_{U^*})$ the selected \mathcal{G} appears after the \mathcal{H} . However, if such a G_{U^*} exists, then, by T 's (G.4) and inspection of the miniswaps, it follows that there is a \mathcal{G} West of x in T ; this contradicts $(\mathcal{H}) \in \underline{x}$ in T .

((G.9) holds in U^*): By U 's (G.2), U has no $\bullet_{\mathcal{G}^+}$ southeast of x^{\leftarrow} .

((G.12) holds in U^*): By T 's (G.2) and (G.12), T has no $\tilde{\mathcal{H}}$ SouthEast of \underline{x} with $\text{family}(\tilde{\mathcal{H}}) = \text{family}(\mathcal{H})$. The same is true in U , as $\text{swap}_{\mathcal{G}}$ does not affect $\tilde{\mathcal{H}}$. It remains to consider such $\tilde{\mathcal{H}}$ in U that are NorthWest of \underline{x} . Since $\text{swap}_{\mathcal{G}}$ does not affect $\tilde{\mathcal{H}}$, by T 's (G.12), no $\tilde{\mathcal{H}}$ is North of \bar{x} in either T or U . Thus, we assume that $\tilde{\mathcal{H}}$ is West of x and either in its row or on a top edge of its row. By assumption, T 's $\bullet_{\mathcal{G}}$ in x^{\leftarrow} becomes the desired $\bullet_{\mathcal{G}^+} \in x^{\leftarrow}$ in U . \square

Subcase 2.2 (In T , $\mathcal{D}^{\downarrow} \in x^{\leftarrow}$): If $\mathcal{G} \in \underline{x}$, this \mathcal{G} is not in a snake of T . Otherwise, $\mathcal{G} \in x$ and H9 applies at x . By Lemma 5.5, T has some $\tilde{\mathcal{G}} \in \underline{x}^{\leftarrow}$ or $(\tilde{\mathcal{G}}) \in \underline{x}^{\leftarrow}$ with $\text{family}(\tilde{\mathcal{G}}) = \text{family}(\mathcal{G})$ and $N_{\tilde{\mathcal{G}}} = N_{\mathcal{D}}$. By Lemma 5.6, $\tilde{\mathcal{G}} \neq \mathcal{G}$. Therefore, by T 's (G.6), $\tilde{\mathcal{G}} = \mathcal{G}^{\leftarrow}$. By T 's (G.11), T has a $\bullet_{\mathcal{G}}$ northWest of x^{\leftarrow} . Hence, this $\tilde{\mathcal{G}} \in \underline{x}^{\leftarrow}$ is nonvirtual and marked. Thus, by T 's (G.13), there is some $\tilde{\mathcal{H}} \in \underline{x}^{\leftarrow}$ or $(\tilde{\mathcal{H}}) \in \underline{x}^{\leftarrow}$ with $\text{family}(\tilde{\mathcal{H}}) = \text{family}(\tilde{\mathcal{G}}) + 1$ and $N_{\tilde{\mathcal{H}}} = N_{\tilde{\mathcal{G}}}$. We claim that there is an \mathcal{H} South of x and in its same column, with $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1 = \text{family}(\tilde{\mathcal{H}})$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$. Certainly, by $N_{\tilde{\mathcal{H}}} = N_{\tilde{\mathcal{G}}} = N_{\mathcal{D}}$, there is such an \mathcal{H} somewhere in T . By Lemma 5.6, it is located as described. Now, if $\mathcal{H} \in \underline{x}$, we are done. Otherwise, by T 's (G.4), $\mathcal{H} \in x^{\downarrow}$. However, now

$\text{label}_T(x^{\leftarrow}) < \mathcal{H}$, by T 's (G.3), and $\text{label}_T(x^{\leftarrow}) > \tilde{\mathcal{H}}$, by T 's (G.4), which is a contradiction. \square

Appendix B. Proof of Proposition 9.5

We check that (G.1)–(G.13) are preserved. Let $T \in \text{revswap}_{\mathcal{G}^+}(U)$, where U is \mathcal{G}^+ -good. Below, the proof of property (G. j) only possibly depends on earlier properties (G. i). We also show that the virtual labels prescribed by reverse miniswaps L2.3, L4.3 and L4.5 are valid virtual labels in the sense of (V.1)–(V.3). This appears as Lemma B.4 in the section ‘Consistency of the prescribed virtual labels’, located between the arguments for (G.12) and (G.13).

Condition (G.1): Suppose that T has $Q \in x$ or $Q \in \underline{x}$ that is too high. By U 's (G.1), the label Q does not appear in the same place in U . Hence, Q is placed in x or \underline{x} in T by some reverse miniswap. We consider which reverse miniswap this might be. By U 's (G.1), Q does not appear anywhere in U north of \underline{x} . Hence, by visual inspection, the only miniswap to consider is L1.1. However, to apply L1.1, we have, by assumption, $Q \in x^\uparrow$ in U . Since this is impossible by U 's (G.1), T cannot have any label too high.

Condition (G.2): By Lemma 8.4, ladders lie in distinct rows and columns; hence, T has no $\bullet_{\mathcal{G}}$ NorthWest of another. Since revswap is defined by its action on rows, T has at most one $\bullet_{\mathcal{G}}$ in any row.

Suppose that T has $\bullet_{\mathcal{G}} \in x$ North of $\bullet_{\mathcal{G}} \in y$ and in the same column. These boxes are in ladders of U . By Lemma 8.4, x and y are in the same ladder of U .

By Lemma 8.3, $x = y^\uparrow$, and the two boxes are $\begin{matrix} \mathcal{G} \\ \bullet_{\mathcal{G}^+} \end{matrix}$ in U . Hence, in order to have

$\bullet_{\mathcal{G}} \in y$ in T , we must apply L1.2 or L3 to y . By definition, U 's $\mathcal{G} \in x$ means that L1.2 does not apply. Reverse miniswap L3 requires $\mathcal{G} \in \bar{y}$, contradicting U 's (G.4).

Condition (G.3): It is enough to confirm (G.3) for an arbitrary fixed row \mathcal{R} of T . If \mathcal{R} does not intersect any ladder of U , then T 's (G.3) is confirmed in \mathcal{R} by U 's (G.3).

Otherwise, by Lemma 8.4, \mathcal{R} intersects a single ladder L . Let $r := \mathcal{R} \cap L$. Let x be the westmost box of r .

If r is L1, confirmation is trivial unless $\mathcal{G} \in x^\uparrow$. In that case, locally at x ,

$\text{revswap}_{\mathcal{G}^+}$ results in $\begin{matrix} \mathcal{G} \\ \mathcal{F} \bullet_{\mathcal{G}^+} \mathcal{H} \end{matrix} \mapsto \begin{matrix} \star \\ \mathcal{F} \mathcal{G} \mathcal{H} \end{matrix}$ (here, $\star = \bullet_{\mathcal{G}}$, but this is not

important to us). (If either \mathcal{F} or \mathcal{H} does not exist, the argument is simpler.) We need $\mathcal{F} < \mathcal{G} < \mathcal{H}$. By U 's (G.9), $\mathcal{F} \leq \mathcal{G}$. Since r is L1, $\mathcal{F} \neq \mathcal{G}$, so $\mathcal{F} < \mathcal{G}$.

If $Q \in x^{\uparrow\rightarrow}$ in U , by U 's (G.3) and (G.4), then $\mathcal{G} < Q \leq \mathcal{H}$. Otherwise, $\bullet_{\mathcal{G}^+} \in x^{\uparrow\rightarrow}$ in U . By U 's (G.11), $\mathcal{H} \in x^{\rightarrow}$ is not marked in U . Thus, $\mathcal{G}^+ \leq \mathcal{H}$ and $\mathcal{G} < \mathcal{H}$, as desired.

Suppose that r is L2. Then, $\text{label}_U(x) = \mathcal{G}$, while $\text{label}_T(x) \in \{\mathcal{G}, \bullet_{\mathcal{G}}\}$. Hence, T 's (G.3) is confirmed from U 's (G.3).

If r is L3, r does not change and we are done.

Suppose that r is L4.1, L4.2 or L4.3. Locally, at r , the reverse miniswap is

$$\boxed{\mathcal{F} \mid \mathcal{G} \mid \bullet_{\mathcal{G}^+} \mid \mathcal{H}} \mapsto \boxed{\mathcal{F} \mid \bullet \mid \mathcal{G}^+ \mid \mathcal{H}} \quad (\text{the labels of } \underline{x} \text{ in } T \text{ are not displayed}).$$

We need to show that $\mathcal{F} < \mathcal{G}^+ < \mathcal{H}$. (If \mathcal{F} or \mathcal{H} does not exist, the argument is simpler.) By U 's (G.3), $\mathcal{F} < \mathcal{G}$, so $\mathcal{F} < \mathcal{G}^+$. By U 's (G.12) (final sentence), $\mathcal{H} \neq \mathcal{G}^+$ and $\mathcal{H} \in x^{\rightarrow}$ is not marked in U . Thus, $\mathcal{G}^+ < \mathcal{H}$.

Lastly, suppose that r is L4.4 or L4.5. Locally, at r , $\boxed{\mathcal{E} \mid \mathcal{G} \mid \bullet \mid \mathcal{H}} \mapsto \boxed{\mathcal{E} \mid \bullet \mid \mathcal{F} \mid \mathcal{H}}$, where \mathcal{F} is as in the definitions of L4.4 and L4.5.

First, we check that T has no (G.3) violation between \mathcal{E} and \mathcal{F} . If $\mathcal{E} < \mathcal{F}$, this is obvious. If $\mathcal{E} > \mathcal{F}$, \mathcal{E} and \mathcal{F} form the exceptional configuration of (G.3) in T . Suppose that $\text{family}(\mathcal{E}) = \text{family}(\mathcal{F})$. By U 's (G.3) and (G.7), $\mathcal{E} \neq \mathcal{F}$. Hence, by U 's (G.6), $\mathcal{E} < \mathcal{F}$, as desired.

Now, we check $\mathcal{F} < \mathcal{H}$.

Case 1 ($\mathcal{G} < \mathcal{H}$): Since $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F} < \mathcal{H}$ follows.

Case 2 ($\mathcal{G} \geq \mathcal{H}$): By U 's (G.3), $\mathcal{G} > \mathcal{H}$ and $\mathcal{H}^! \in x^{\rightarrow\rightarrow}$ in U .

Subcase 2.1 ($\text{family}(\mathcal{F}) < \text{family}(\mathcal{H})$): Then $\mathcal{F} < \mathcal{H}$, and we are done.

Subcase 2.2 ($\text{family}(\mathcal{F}) = \text{family}(\mathcal{H})$): By Lemma 5.10 applied to U , $\mathcal{F} \neq \mathcal{H}$. Hence, by U 's (G.6), $\mathcal{F} < \mathcal{H}$.

Subcase 2.3 ($\text{family}(\mathcal{F}) > \text{family}(\mathcal{H})$): We derive a contradiction. By Lemma 5.5, U has $\mathcal{G}' \in x^{\rightarrow\rightarrow}$ or $\mathcal{G}' \in \mathcal{G} \in x^{\rightarrow\rightarrow}$ with $N_{\mathcal{H}} = N_{\mathcal{G}'}$ and $\text{family}(\mathcal{G}) = \text{family}(\mathcal{G}')$. If $\mathcal{G} = \mathcal{G}'$, then, by U 's (G.7), $\mathcal{G}' \in x^{\rightarrow\rightarrow}$ in U . However, $x^{\rightarrow\rightarrow}$ is southeast of a $\bullet_{\mathcal{G}^+}$ in x^{\rightarrow} in U , contradicting U 's (V.1). Hence, $\mathcal{G} \neq \mathcal{G}'$, so, by U 's (G.6), $\mathcal{G}' \geq \mathcal{G}^+$. If $\mathcal{G}' > \mathcal{G}^+$, then, by U 's (G.6), there must be a \mathcal{G}^+ in the column of x^{\rightarrow} . This \mathcal{G}^+ is not South of x^{\rightarrow} by U 's (G.12). It is also not north of x^{\rightarrow} by U 's (G.9). Thus, $\mathcal{G}' = \mathcal{G}^+$.

Since $N_{\mathcal{G}^+} = N_{\mathcal{H}}$, $N_{\mathcal{G}} = N_{\mathcal{H}^-}$. By U 's (G.8), since U has $\mathcal{G} \in x$, every \mathcal{H}^- appears before x in column reading order. By U 's (G.4) and (G.6), \mathcal{H}^- does not appear east of $x^{\rightarrow\rightarrow}$'s column. By U 's (G.12), \mathcal{H}^- does not appear North of x^{\rightarrow} and in its column. Since $\bullet_{\mathcal{G}^+} \in x^{\rightarrow}$, \mathcal{H}^- is not in x^{\rightarrow} . By U 's (G.11), \mathcal{H}^- is not

South of x^\rightarrow and in its column. Thus, \mathcal{H}^- appears only in x 's column and is north of x . By U 's (G.12), it then follows that $\mathcal{H}^- \in \bar{x}$ in U . However, then since $N_{\mathcal{H}^-} = N_{\mathcal{G}}$ and $\mathcal{H}^- < \mathcal{F}$, this contradicts the definition of \mathcal{F} .

Condition (G.4): Consider an arbitrary column c of U ; we show that (G.4) holds for c in T .

Case 1 (U has no $\bullet_{\mathcal{G}^+}$ in c):

Subcase 1.1 (c is $<$ -increasing in U): By inspection of the reverse miniswaps, it is clear that c is $<$ -increasing in T .

Subcase 1.2 (c contains $\begin{matrix} \boxed{\mathcal{F}} \\ \boxed{\mathcal{F}^\downarrow} \end{matrix}$): Suppose that the depicted \mathcal{F} is in x in U . Since $\mathcal{F}^\downarrow \in x^\downarrow$ in U , $\mathcal{F} \preceq \mathcal{G}$. Therefore, there are two possibilities.

Subcase 1.2.1 ($\mathcal{F} < \mathcal{G}$): Since $\mathcal{F} < \mathcal{G}$, x and x^\downarrow are not in ladders of U . Therefore, $\mathcal{F} \in x$ and $\mathcal{F} \in x^\downarrow$ in T ; however, we do not know *a priori* which of these \mathcal{F} 's are marked. We must show that T has unmarked $\mathcal{F} \in x$ and $\mathcal{F}^\downarrow \in x^\downarrow$. We will need the following lemma.

LEMMA B.1. *Let U be \mathcal{G}^+ -good, and let $T \in \text{revswap}_{\mathcal{G}^+}(U)$. Suppose that $\bullet_{\mathcal{G}} \in y$ in T . Then, in U , either $\mathcal{G} \in y$ or $\bullet_{\mathcal{G}^+} \in y$.*

Proof. Since $\bullet_{\mathcal{G}} \in y$ in T , y is in a ladder of U . Thus, the lemma is immediate from the definition of ladders. □

Since unmarked $\mathcal{F} \in x$ in U , U has no $\bullet_{\mathcal{G}^+}$ northwest of x . By U 's (G.3) or (G.4), U has no \mathcal{G} northwest of x . Hence, by Lemma B.1, T has no $\bullet_{\mathcal{G}}$ northwest of x ; hence, unmarked $\mathcal{F} \in x$ in T .

By definition of marked labels, U has $\bullet_{\mathcal{G}^+}$ in some box z northwest of x^\downarrow in U . By inspection of the miniswaps, there is a $\bullet_{\mathcal{G}}$ northwest of z in T . Hence, $\mathcal{F}^\downarrow \in x^\downarrow$ in T .

Subcase 1.2.2 ($\mathcal{F} = \mathcal{G}$): By inspection of the miniswaps, $\bullet_{\mathcal{G}} \in x$ in T , and no other box or edge of the column was changed. Hence, we are done, by U 's (G.4).

Case 2 (U has a $\bullet_{\mathcal{G}^+}$ in c): By U 's (G.11), U has no marked labels in c . Hence, by U 's (G.4), the labels of U strictly $<$ -increase down c , ignoring the $\bullet_{\mathcal{G}^+}$. Suppose that the $\bullet_{\mathcal{G}^+}$ in c is in x . By inspection, a (G.4) violation in c can only occur from L1.1, L4.4 or L4.5 applied at x . If we apply L1.1 at x , then, locally, at c , $\begin{matrix} \boxed{\mathcal{G}} \\ \bullet \end{matrix} \mapsto \begin{matrix} \bullet \\ \boxed{\mathcal{G}} \end{matrix}$, while the rest of the column is unchanged, so c satisfies (G.4) in T .

Suppose that we apply L4.4 or L4.5 at x . By definition, U has $\mathcal{G} \in x^\leftarrow$ and there is no $\mathcal{G}^+ \notin \underline{x}$ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$. Let \mathcal{E} be the \leftarrow -greatest gene in c north of \bar{x} , and let \mathcal{H} be the \leftarrow -least gene in c south of \underline{x} . (If either of these fails to exist, the argument is simplified.) Let \mathcal{F} be the \leftarrow -least element of $Z^\#$ in the notation of L4.4 and L4.5. It remains to show the following.

CLAIM B.2. *We claim that*

(I) $\mathcal{G} < \mathcal{H}$; and

(II) either $\mathcal{E} < \mathcal{F}$, or $\mathcal{E} = \mathcal{F}$, with \mathcal{E} an unmarked label in x^\uparrow and $\mathcal{F} < \mathcal{G}$.

Proof. (I): By U 's (G.11), the \mathcal{H} in c is not marked in U , so $\mathcal{G} < \mathcal{H}$. If $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G})$, then, by U 's (G.12), $\mathcal{H} \in \underline{x}$ rather than x^\uparrow . Then, by U 's (G.7), $\mathcal{H} \neq \mathcal{G}$. However, then, by U 's (G.6), $\mathcal{H} = \mathcal{G}^+ \in \underline{x}$, contradicting our assumption. Thus, $\mathcal{G} < \mathcal{H}$.

(II): By U 's (G.9), $\mathcal{E} \preceq \mathcal{G}$. Hence, by U 's (G.6), either $\mathcal{E} = \mathcal{G}$ or else $\mathcal{E} < \mathcal{G}$.

Case A ($\mathcal{E} = \mathcal{G}$): By Lemma 5.6, $Z = \emptyset$ (in the notation of L4.4 and L4.5).

Further, by U 's (G.7), $\mathcal{E} \in x^\uparrow$ (rather than \bar{x}). Hence, locally, c changes as $\begin{matrix} \boxed{\mathcal{G}} \\ \bullet \end{matrix} \mapsto$

$\begin{matrix} \bullet \\ \boxed{\mathcal{G}} \end{matrix}$, while the rest of c is unchanged. Then, c satisfies (G.4) in T .

Case B ($\mathcal{E} < \mathcal{G}$): If $\mathcal{F} = \mathcal{G}$, then $\mathcal{E} < \mathcal{F}$, as desired. Therefore, by U 's (G.4), we may assume that $\mathcal{F} < \mathcal{G}$ and furthermore that $\text{family}(\mathcal{E}) \geq \text{family}(\mathcal{F})$. Then, by Lemma 5.8, there is an $\tilde{\mathcal{E}} \in Z$ with $\text{family}(\tilde{\mathcal{E}}) = \text{family}(\mathcal{E})$.

If $\tilde{\mathcal{E}} \neq \mathcal{E}$, then, by U 's (G.6), $\mathcal{E} = \tilde{\mathcal{E}}^+$. Since $N_{\tilde{\mathcal{E}}} = N_{\mathcal{G}} \geq 1$, $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$ and $N_{\mathcal{E}} = N_{\mathcal{G}^+}$. Hence, by U 's (G.6) and (G.8), U has \mathcal{G}^+ in c south of \bar{x} . By U 's (G.9), it is south of \underline{x} . By U 's (G.12), it is not South of \underline{x} . Hence, $\mathcal{G}^+ \in \underline{x}$ in U , contradicting the assumptions of L4.4 and L4.5.

Thus, $\tilde{\mathcal{E}} = \mathcal{E}$. By Lemma 5.6, $\tilde{\mathcal{E}} = \min Z$, so $\tilde{\mathcal{E}} = \mathcal{F}$. By U 's (G.7), the \mathcal{E} in c in U is in x^\uparrow . Since, in T , $\bullet_{\mathcal{G}} \in x^\leftarrow$ and $\mathcal{E} < \mathcal{G}$, $\mathcal{E}^! \in x$ in T .

We must show that unmarked $\mathcal{E} \in x^\uparrow$ in T . By U 's (G.2), U has no $\bullet_{\mathcal{G}^+}$ northwest of x . By U 's (G.3) and (G.4), U has no \mathcal{G} northwest of x^\uparrow . Hence, by Lemma B.1, T has no $\bullet_{\mathcal{G}}$ northwest of x^\uparrow , so T has unmarked $\mathcal{E} \in x^\uparrow$. \square

Condition (G.5): We are only concerned with reverse miniswaps that produce or relocate edge labels. That is L2.2, L3, L4.2, L4.4 and L4.5. In L2.2, we create an edge label \mathcal{G} on \underline{x} , while U has $\mathcal{G} \in x$. Hence, by U 's (G.4), there is no other label of \mathcal{G} 's family on \underline{x} . This verifies T 's (G.5) in this situation. A similar argument applies for L3 and L4.2.

The arguments for L4.4 and L4.5 are similar. Consider any $Q \in A''$ or \widetilde{A}'' . If (G.5) fails in T , we may assume that U has $Q' \in \underline{x}^{\rightarrow}$ with $\text{family}(Q) = \text{family}(Q')$. If $Q \neq \mathcal{G}$, then, by U 's (G.4), $Q < \mathcal{G}$, so $Q' \in \underline{x}^{\rightarrow}$ is marked in U , violating U 's (G.11). Thus, $Q = \mathcal{G}$. By U 's (G.7), $Q' \neq \mathcal{G}$. Hence, by U 's (G.6), $Q' = \mathcal{G}^+$, contradicting that L4.4 or L4.5 applies.

Condition (G.6): We will use the following.

LEMMA B.3. *Let U be \mathcal{G}^+ -good. If \mathcal{H} appears in column c of $T \in \text{revswap}_{\mathcal{G}^+}(U)$, then U has \mathcal{H} in column c or column c^{\leftarrow} .*

Proof. This is proved by inspection of the reverse miniswaps. □

Suppose that i_b appears West of i_a in T . Then, by Lemma B.3, i_b appears west of i_a in U . Therefore, by U 's (G.4), either $b = a$ or i_b is West of i_a in U . Thus, by U 's (G.6), $b \leq a$.

Condition (G.7): Let $Q \in \underline{x}$ in column c of T . First, suppose that U has $Q \in \underline{x}$. By U 's (G.7), U has no Q West of column c . Hence, by Lemma B.3, T has no Q West of column c , as desired.

Therefore, suppose that U has $Q \notin \underline{x}$. Thus, T 's $Q \in \underline{x}$ was created by one of the reverse miniswaps L2.2, L3, L4.2, L4.4 or L4.5. In each case, we are done by Lemma B.3, provided that we know that U has no Q West of c . This is by assumption in L2.2 and L4.2. This holds for L3 by U 's (G.7). For L4.4, $Q \in A''$ (in the terminology of L4.4). If $Q = \mathcal{G}$, we are done by assumption; otherwise, we are done by U 's (G.7), since $Q \in A$ (in the terminology of L4.4). For L4.5, $Q \in \widetilde{A}''$ (in the terminology of L4.5), and we are done by U 's (G.7), since $Q \in A$ (in the terminology of L4.5).

Condition (G.8): Suppose that $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$. By T 's (G.4), to show T 's (G.8), it suffices to show that no \mathcal{F} is East of any \mathcal{E} in T .

Let e be the westmost instance of \mathcal{E} , and let f be the eastmost instance of \mathcal{F} in U . By U 's (G.8), e is east of f in U . Swapping does not move e West, and moves f at most one column East. We may therefore assume that e and f are in the same column c of U . We may also assume that the swap moves f East; that is, the swap involving f is L4.4 or L4.5 (say at $\{x, x^{\rightarrow}\}$). If $e \in \bar{x}$, then T 's westmost \mathcal{E} is in x^{\rightarrow} or $\underline{x}^{\rightarrow}$, and there is no (G.8) violation. Hence, by U 's (G.4), we may assume that $e \in x^{\uparrow}$. It should be noted that $N_{\mathcal{F}} = N_{\mathcal{G}}$, U has $\mathcal{G} \in x$, and either $\mathcal{F} = \mathcal{G}$ or $\mathcal{F} \in \bar{x}$.

By U 's (G.3), $\mathcal{E} < Q := \text{label}_U(x^{\uparrow})$. By U 's (G.9), $Q \leq \mathcal{G}$. By Lemma 5.6, $Q \neq \mathcal{G}$. Hence, $Q < \mathcal{G}$, so, by U 's (G.6), $Q < \mathcal{G}$. By U 's (G.4) and Lemma 5.8,

for every $\text{family}(\mathcal{F}) < i < \text{family}(\mathcal{G})$, there is a label $\mathcal{H}^i \in \bar{x}$ with $\text{family}(\mathcal{H}^i) = i$ and $N_{\mathcal{H}^i} = N_{\mathcal{E}} = N_{\mathcal{F}} = N_{\mathcal{G}}$. By Lemma 5.6, $\mathcal{Q} \neq \mathcal{F}$ and $\mathcal{Q} \neq \mathcal{H}^i$. Hence, by U 's (G.6), \mathcal{Q} is one of \mathcal{E}^+ , \mathcal{F}^+ , $(\mathcal{H}^i)^+$. Hence, U has a \mathcal{G}^+ in c^{\rightarrow} with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$. By U 's (G.9) and (G.12), it follows that $\mathcal{G}^+ \in \underline{x}^{\rightarrow}$; this contradicts the assumptions of L4.4 or L4.5.

Condition (G.9): Let $\mathcal{F} \succeq \mathcal{G}$. Suppose that $\mathcal{F} \in x$ (or $\mathcal{F} \in \underline{x}$) is northwest of $\bullet_{\mathcal{G}} \in y$ in T . By inspection of the reverse miniswaps, U has an \mathcal{F} northwest of \underline{x} .

Suppose that $\mathcal{F} \succ \mathcal{G}$. By U 's (G.9), U has no $\bullet_{\mathcal{G}^+}$ southeast of x . Hence, by Lemma B.1, $\mathcal{G} \in y$ in U . Thus, by U 's (G.3) and (G.4), $\mathcal{F} \preceq \mathcal{G}$, contradicting $\mathcal{F} \succ \mathcal{G}$.

Thus, suppose that $\mathcal{F} = \mathcal{G}$. By Lemma B.1, in U , either $\bullet_{\mathcal{G}^+} \in y$ or $\mathcal{G} \in y$. Suppose that $\mathcal{G} \in y$. By Lemma 5.10, y is not southEast of x ; that is, x and y are in the same column. Hence, by U 's (G.4), U has $\mathcal{G}^! \in y$. Hence, y is not in a ladder of U , contradicting T 's $\bullet_{\mathcal{G}} \in y$. Thus, U has $\bullet_{\mathcal{G}^+} \in y$. By Lemma 5.12, it follows that x and y are in the same row or column. By U 's (G.9), in fact, $y = x^{\downarrow}$ or $y = x^{\rightarrow}$. Now, by inspection of the reverse miniswaps, we conclude that $\mathcal{F} \notin x$ (respectively, $\mathcal{F} \notin \underline{x}$) or $\bullet_{\mathcal{G}} \notin y$ in T , contradicting our initial assumptions.

Condition (G.10): Consider $\mathcal{F}^! \in x$ or $\mathcal{F}^! \in \underline{x}$ in T .

Case 1 (This \mathcal{F} is not in the same location in U): By assumption, $\mathcal{F} \prec \mathcal{G}$. By inspection of the reverse miniswaps, the $\mathcal{F}^!$ in question appears in T as a result of L4.4 or L4.5. By definition, $\bullet_{\mathcal{G}} \in x^{\leftarrow}$ in T and we are done.

Case 2 (This \mathcal{F} is in the same location in U): By U 's (G.3) and (G.4), U has no \mathcal{G} northwest of x . However, T has a $\bullet_{\mathcal{G}} \in z$ northwest of x . Hence, by Lemma B.1, U has a $\bullet_{\mathcal{G}^+}$ northwest of x . Since $\mathcal{F} \prec \mathcal{G} \prec \mathcal{G}^+$, this means that U 's $\mathcal{F} \in x$ or \underline{x} is marked.

By Lemma 5.4, U has a $\bullet_{\mathcal{G}^+} \in w$ West of x and in the same row. If T has $\bullet_{\mathcal{G}} \in w$ or w^{\leftarrow} , we are done. Otherwise, L1.1 applies at w . Then, U has $\mathcal{G} \in w^{\uparrow}$, and, by Lemma 5.5, U has a $\tilde{\mathcal{G}} \in \underline{x}$ (possibly virtual) in U such that $\text{family}(\mathcal{G}) = \text{family}(\tilde{\mathcal{G}})$; this contradicts U 's (G.12).

Condition (G.11): Consider $\mathcal{F}^! \in x$ or $\mathcal{F}^! \in \underline{x}$ in T .

Case 1 (\mathcal{F} is not in the same location in U): By assumption, $\mathcal{F} \prec \mathcal{G}$. By inspection of the miniswaps, the $\mathcal{F}^!$ in question appears in T as a result of L4.4 or L4.5. Therefore, we have $\mathcal{F} \prec \mathcal{G}$ and $N_{\mathcal{F}} = N_{\mathcal{G}}$.

Suppose that $\bullet_{\mathcal{G}} \in y$ in T , where y is in x 's column. By U 's (G.2), $\bullet_{\mathcal{G}^+} \notin y$ in U . Hence, by Lemma B.1, U has $\mathcal{G} \in y$. This contradicts Lemma 5.6 (applied to U), since U has $\mathcal{G} \in y$ and $\mathcal{F} \in \bar{x}^{\leftarrow}$, where $\mathcal{F} \prec \mathcal{G}$ and $N_{\mathcal{F}} = N_{\mathcal{G}}$.

Case 2 (\mathcal{F} is in the same location in U): By the first paragraph of the argument of (G.10) Case 2, this \mathcal{F} is marked in U . Suppose that T has $\bullet_{\mathcal{G}} \in y$, where y is in x 's column. By U 's (G.11), U has no $\bullet_{\mathcal{G}^+}$ in x 's column. Hence, by Lemma B.1, U has $\mathcal{G} \in y$. By U 's (G.4), since $\mathcal{F} \prec \mathcal{G}$, y is South of x and in its column. By assumption, T has some $\bullet_{\mathcal{G}}$ northwest of x . With $\bullet_{\mathcal{G}} \in y$, this contradicts T 's (G.2).

Condition (G.12): Let x, z be boxes in row r with x West of z . Suppose that $\text{family}(\mathcal{F}) = \text{family}(\mathcal{F}')$. Cases 1–3 consider the case where U has \mathcal{F} NorthWest of \mathcal{F}' . In these cases, by U 's (G.12), we may assume that U has $\mathcal{F} \in \bar{x}$ or $\mathcal{F} \in x$ as well as $\mathcal{F}' \in z$ or $\mathcal{F}' \in \underline{z}$. Moreover, by U 's (G.12), U has a $\bullet_{\mathcal{G}^+}$ in some box y of row r that is East of x and west of z . Cases 4–7 consider the case where U has \mathcal{F} southwest of \mathcal{F}' .

Case 1 (U has \mathcal{F} or $\mathcal{F} \in \bar{x}$ and \mathcal{F}' or $\mathcal{F}' \in \underline{z}$): By U 's (G.4), $\mathcal{F} \prec \text{label}_U(x)$. By U 's (G.9), $\text{label}_U(x) \prec \mathcal{G}^+$. Hence, $\mathcal{F} \prec \mathcal{G}^+$. Since $\text{family}(\mathcal{F}) = \text{family}(\mathcal{F}')$, also $\mathcal{F}' \prec \mathcal{G}^+$. Therefore, U 's $\mathcal{F}' \in \underline{z}$ is marked (and is not virtual). By U 's (G.11), it follows that $y \neq z$. Moreover, by U 's Lemma 5.4, $\text{label}_U(y^{\rightarrow})$ is marked. By U 's (G.4), $\mathcal{G} \notin z$. Hence, z is not in a ladder, and so T has $\mathcal{F}' \in \underline{z}$. For convenience, we assume that $\mathcal{F} \in \bar{x}$. (The argument where this label is virtual is strictly easier.)

Case 1.1 (x and y are not in the same ladder of U): By Lemma 8.4, x is not in any ladder. Hence, T has $\mathcal{F} \in \bar{x}$. By inspection, unless the reverse miniswap applied at y is L1.1, T has $\bullet_{\mathcal{G}} \in y$ or $\bullet_{\mathcal{G}} \in y^{\leftarrow} \neq x$. Hence, although T has \mathcal{F} NorthWest of \mathcal{F}' , there is no (G.12) violation. If the reverse miniswap is L1.1, then U has $\mathcal{G} \in y^{\uparrow}$. By Lemma 5.13, U has $\mathcal{G} \in \underline{y^{\rightarrow}}$; with the $\mathcal{G} \in y^{\uparrow}$, this violates the (G.12) condition of U 's (V.3).

Case 1.2 (x and y are in the same ladder of U): Then, $y = x^{\rightarrow}$ and the ladder row is type L4. If it is L4.1, L4.2 or L4.3, then U has $\mathcal{G} \in y^{\leftarrow}$, $\bullet_{\mathcal{G}^+} \in y$ and $\mathcal{G}^+ \in \underline{y}$ with $\text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$. Since $\text{label}_U(y^{\rightarrow})$ is marked, this contradicts U 's (G.12).

Hence, it is L4.4 or L4.5. By Lemma 5.5, $\underline{y^{\rightarrow}}$ contains a label \mathcal{F}'' with $\text{family}(\mathcal{F}'') = \text{family}(\mathcal{F})$ and a (possibly virtual) label \mathcal{G}'' with $\text{family}(\mathcal{G}'') = \text{family}(\mathcal{G})$. By Lemma 5.5, $N_{\mathcal{F}''} = N_{\mathcal{G}''}$. By U 's (G.12), there is no label of \mathcal{F} 's family North of y and in y 's column. By U 's (G.11), there is no label of that family South of y and in y 's column. Hence, there is no label of \mathcal{F} 's family in y 's column. Hence, by U 's (G.6) and (G.7), $\mathcal{F}'' = \mathcal{F}^+$. By U 's (G.12), there is no label in y 's column of the same family as \mathcal{G} . Hence, by U 's (G.6) and Lemma 5.10, $\mathcal{G}'' = \mathcal{G}^+$. Therefore, $N_{\mathcal{F}} = N_{\mathcal{G}}$ and $\mathcal{F} \in Z$ (in the notation of L4.4/L4.5). Thus, T has $\mathcal{F} \in \underline{y}$ or $\mathcal{F} \in y$. Therefore, by T 's (G.3) and (G.4), T has $\mathcal{F} \in \underline{y}$, and so has \mathcal{F} southWest of \mathcal{F}' , in agreement with (G.12).

Case 2 (U has $\mathcal{F} \in x$ and \mathcal{F}' or $(\mathcal{F}') \in z$): By U 's (G.9), $\mathcal{F} < \mathcal{G}^+$.

Subcase 2.1 ($y = z$): By U 's (G.11), $(\mathcal{F}')^! \notin z$; hence, $\mathcal{G}^+ \preceq \mathcal{F}'$. Thus, $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G}^+) = \text{family}(\mathcal{G})$. Hence, by U 's (G.9), $\text{family}(\text{label}_U(y^{\leftarrow})) = \text{family}(\mathcal{F})$. Therefore, by U 's (G.6) and (G.9), $\text{label}_U(y^{\leftarrow}) = (\mathcal{F}')^-$ and $\mathcal{F}' = \mathcal{G}^+$. Hence, by U 's (G.6) and (G.7), U has (nonvirtual) $\mathcal{G}^+ \in y$. The applicable reverse miniswap is then L4.1, L4.2 or L4.3. Hence, in T , \mathcal{F} is not NorthWest of \mathcal{F}' , and (G.12) holds.

Subcase 2.2 ($y \neq z$): By U 's (G.4), $\text{label}_U(z) < \mathcal{F}'$. Therefore, $\text{label}_U(z)$ is marked; by Lemma 5.4, $\text{label}_U(y^{\rightarrow})$ is marked. The reverse swap does not affect z . Hence, we may assume that T has $\mathcal{F}' \in z$.

If the reverse swap moves the $\mathcal{F} \in x$ South, then T has \mathcal{F} southWest of \mathcal{F}' , in accordance with (G.12). No reverse swap can move the $\mathcal{F} \in x$ North. Hence, we may assume that T has $\mathcal{F} \in x$ or $\mathcal{F} \in x^{\rightarrow}$. If $\text{label}_T(x^{\rightarrow}) = \mathcal{F}$, then $y = x^{\rightarrow}$ and $\mathcal{F} = \mathcal{G}$, so T has $y \ni \mathcal{G} \succeq \text{label}_T(y^{\rightarrow})$, contradicting T 's (G.3). Thus, $\text{label}_T(x) = \mathcal{F}$. By Lemma 5.5, U has a label of the same family as \mathcal{G} on y^{\rightarrow} . Hence, the reverse miniswap involving y is not L1.1, since $\mathcal{G} \in y^{\uparrow}$ would violate U 's (G.12). Hence, by inspection of the reverse miniswaps, T has $\bullet_{\mathcal{G}} \in y$ or $\bullet_{\mathcal{G}} \in y^{\leftarrow}$, in accordance with T 's (G.12).

Case 3 (U has \mathcal{F} or $(\mathcal{F}) \in \bar{x}$ and $\mathcal{F}' \in z$): Since $\mathcal{F}' \in z$, $\bullet_{\mathcal{G}^+} \notin z$, so $y \neq z$. By U 's (G.4), $\mathcal{F} < \text{label}_U(x)$, whereas, by U 's (G.9), $\text{label}_U(x) < \mathcal{G}^+$; hence, $\mathcal{F} < \mathcal{G}^+$. Therefore, also $\mathcal{F}' < \mathcal{G}$, and so $\mathcal{F}' \in z$ is marked in U . The box z is not part of a ladder, so T has $\mathcal{F}' \in z$. No reverse swap can move the $\mathcal{F} \in \bar{x}$ North. If it moves South, it will be southWest of z in T , so no (G.12) violation ensues. Hence, we may assume that T has $\mathcal{F} \in \bar{x}$. By Lemma 5.4, y^{\rightarrow} contains a marked label $\mathcal{E}^!$, and so, by Lemma 5.5, U has a (possibly virtual) $\mathcal{G}' \in y^{\rightarrow}$ with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$.

The reverse miniswap involving y is not L1.1, as $\mathcal{G} \in y^{\uparrow}$ violates U 's (G.12), together with $\mathcal{G}' \in y^{\rightarrow}$. Thus, T has $\bullet_{\mathcal{G}} \in y$ or $\bullet_{\mathcal{G}} \in y^{\leftarrow}$. Unless T has $\bullet_{\mathcal{G}} \in y^{\leftarrow}$ and $x = y^{\leftarrow}$, this does not violate T 's (G.12). Suppose that $x = y^{\leftarrow}$ and T has $\bullet_{\mathcal{G}} \in x$. The reverse miniswap involving y is L4. Hence, U has $\mathcal{G} \in x$. By Lemma 5.5, either $\text{family}(\mathcal{E}) = \text{family}(\mathcal{F})$ or U has some $\mathcal{F}'' \in y^{\rightarrow}$ with $\text{family}(\mathcal{F}'') = \text{family}(\mathcal{F})$. By Lemma 5.5, $N_{\mathcal{G}'} = N_{\mathcal{E}} = N_{\mathcal{F}''}$. By U 's (G.11) and (G.12), y 's column does not contain a label of \mathcal{F} 's family. Hence, by U 's (G.6) and (G.7), $\mathcal{F}'' = \mathcal{F}^+$. By U 's (G.9) and (G.12), if y 's column contains a label of the same family as \mathcal{G} , it is on y . By U 's (G.6) and (G.7), it can only be \mathcal{G}^+ ; this contradicts U 's (G.12) (last sentence). Thus, U has $\mathcal{G}^+ \notin y$, and so $\mathcal{G}' = \mathcal{G}^+$. The reverse miniswap is L4.4 or L4.5. We have $N_{\mathcal{F}} = N_{\mathcal{G}}$. Thus, $\mathcal{F} \in z$ (in the notation of L4.4 and L4.5), contradicting that T has $\mathcal{F} \in \bar{x}$.

Case 4 (U has $\mathcal{F} \in \mathbf{a}$ southwest of $\mathcal{F}' \in \mathbf{b}$): Say that \mathbf{b} is in row r . No reverse miniswap can move \mathcal{F} North or move \mathcal{F}' further South than \mathbf{b}^\downarrow . Hence, unless \mathbf{a} is in row r , T has \mathcal{F} southwest of \mathcal{F}' and no (G.12) violation ensues. Therefore, assume that \mathbf{a} is in row r .

If the reverse swap moves \mathcal{F} , then it cannot also move \mathcal{F}' ; hence, T has \mathcal{F} southwest of \mathcal{F}' and no (G.12) violation ensues. Thus, assume that T has $\mathcal{F} \in \mathbf{a}$. To violate (G.12), the reverse swap must move \mathcal{F}' South. Therefore, in T , $\mathcal{F}' \in \underline{\mathbf{b}}$, $\mathcal{F}' \in \underline{\mathbf{b}}^\rightarrow$ or $\mathcal{F}' \in \mathbf{b}^\downarrow$.

Subcase 4.1 (T has $\mathcal{F}' \in \underline{\mathbf{b}}$): Here, $\mathcal{F}' = \mathcal{G}$, and the reverse miniswap involving \mathbf{b} is L2.2 or L4.2. Although T has \mathcal{F} NorthWest of \mathcal{F}' , T has $\bullet_{\mathcal{G}} \in \mathbf{b}$ to avoid violating (G.12). (We avoid violating the last sentence of T 's (G.12) by U 's (G.3) in the L2.2 case and by T 's $\bullet_{\mathcal{G}} \in \mathbf{b}$ in the L4.2 case.)

Subcase 4.2 (T has $\mathcal{F}' \in \underline{\mathbf{b}}^\rightarrow$): Here, U has $\bullet_{\mathcal{G}^+} \in \mathbf{b}^\rightarrow$, and the reverse miniswap involving \mathbf{b}^\rightarrow is L4.4 or L4.5. Then, T has $\bullet_{\mathcal{G}} \in \mathbf{b}$, so, although T has \mathcal{F} NorthWest of \mathcal{F}' , they do not violate (G.12).

Subcase 4.3 (T has $\mathcal{F}' \in \mathbf{b}^\downarrow$): Here, U has $\bullet_{\mathcal{G}^+} \in \mathbf{b}^\downarrow$ and $\mathcal{F}' = \mathcal{G}$. Hence, by U 's (G.2), $\text{label}_U(\mathbf{a}^\downarrow)$ is a genetic label. Since \mathbf{a}^\downarrow is northWest of a $\bullet_{\mathcal{G}^+}$, $\text{label}_U(\mathbf{a}^\downarrow)$ is not marked. Hence, by U 's (G.4), $\mathcal{F} < \text{label}_U(\mathbf{a}^\downarrow)$. However, by U 's (G.9), $\text{label}_U(\mathbf{a}^\downarrow) < \mathcal{G}^+$, which is a contradiction.

Case 5 (U has $\mathcal{F} \in \underline{\mathbf{a}}$ southwest of $\mathcal{F}' \in \mathbf{b}$): Say that \mathbf{b} is in row r . No reverse miniswap can move \mathcal{F} further North than \mathbf{a} or \mathcal{F}' further South than \mathbf{b}^\downarrow . Therefore, unless \mathbf{a} is in r , T has \mathcal{F} southwest of \mathcal{F}' and no (G.12) violation ensues. Hence, we assume that \mathbf{a} is in r .

Subcase 5.1 (The reverse swap moves \mathcal{F} North): Here, U has $\bullet_{\mathcal{G}^+} \in \mathbf{a}$; the reverse miniswap involving \mathbf{a} is L4.1, L4.2 or L4.3; and T has $\mathcal{F} \in \mathbf{a}$. By U 's (G.2), \mathcal{F}' takes part in no reverse miniswap, so T has $\mathcal{F}' \in \mathbf{b}$. Hence, T has \mathcal{F} southwest of \mathcal{F}' and no (G.12) violation ensues.

Subcase 5.2 (T has $\mathcal{F} \in \underline{\mathbf{a}}$): To have \mathcal{F} NorthWest of \mathcal{F}' in T , the reverse swap must move \mathcal{F}' to \mathbf{b}^\downarrow . Hence, U has $\bullet_{\mathcal{G}^+} \in \mathbf{b}^\downarrow$ and $\mathcal{F}' = \mathcal{G}$. Since U has $\mathcal{F} \in \underline{\mathbf{a}}$ and $\bullet_{\mathcal{G}^+} \in \mathbf{b}^\downarrow$, it has a label in \mathbf{a}^\downarrow . By U 's (G.2), it is a genetic label \mathcal{H} . By U 's (G.4), $\mathcal{F} < \mathcal{H}$. Hence, since $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$, $\mathcal{G} < \mathcal{H}$. However, by U 's (G.9), $\mathcal{H} < \mathcal{G}^+$, which is a contradiction.

Case 6 (U has $\mathcal{F} \in \mathbf{a}$ southwest of $\mathcal{F}' \in \underline{\mathbf{b}}$): Say that \mathbf{a} is in row r . No swap can move \mathcal{F} North. No swap can move \mathcal{F}' further South than $\underline{\mathbf{b}}^\downarrow$. Hence, unless $\mathbf{b} \in r^\uparrow$, T has \mathcal{F} southwest of \mathcal{F}' and no (G.12) violation ensues. Thus, we assume that $\mathbf{b} \in r^\uparrow$. To obtain a (G.12) violation, \mathcal{F}' must move South to $\underline{\mathbf{b}}^\downarrow$ or $\underline{\mathbf{b}}^\downarrow \rightarrow$.

Subcase 6.1 (T has $\mathcal{F}' \in \underline{\mathfrak{b}^\downarrow}$): Here, U has $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^\downarrow$, T has $\mathcal{F} \in \mathfrak{a}$ and $\mathcal{F}' = \mathcal{G}$. Therefore, T has $\bullet_{\mathcal{G}} \in \mathfrak{b}^\downarrow$. It remains to show that we do not violate the last sentence of (G.12). By U 's (G.12), $\text{family}(\text{label}_U(\mathfrak{b}^{\downarrow\rightarrow})) \neq \text{family}(\mathcal{G})$; hence, the same is true in T . Hence, if $\text{label}_T(\mathfrak{b}^{\downarrow\rightarrow})$ is marked, then $\text{label}_U(\mathfrak{b}^{\downarrow\rightarrow})$ is marked. Then, by Lemma 5.5, U has a label on $\underline{\mathfrak{b}^{\downarrow\rightarrow}}$ of the same family as \mathcal{G} , contradicting U 's (G.12). Thus, (G.12) is confirmed in T .

Subcase 6.2 (T has $\mathcal{F}' \in \underline{\mathfrak{b}^{\downarrow\rightarrow}}$): Here, U has $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^{\downarrow\rightarrow}$ and $\mathcal{G} \in \mathfrak{b}^\downarrow$, while T has $\mathcal{F} \in \mathfrak{a}$ and $\bullet_{\mathcal{G}} \in \mathfrak{b}^\downarrow$. Thus, although T has \mathcal{F} NorthWest of \mathcal{F}' , they do not violate (G.12).

Case 7 (U has $\mathcal{F} \in \underline{\mathfrak{a}}$ southwest of $\mathcal{F}' \in \underline{\mathfrak{b}}$): Say that \mathfrak{b} is in row r . No swap can move \mathcal{F} further North than \mathfrak{a} . No swap can move \mathcal{F}' further South than $\underline{\mathfrak{b}^\downarrow}$. Hence, if \mathfrak{a} is South of row r^\downarrow , then T has \mathcal{F} southwest of \mathcal{F}' and no (G.12) violation ensues.

Case 7.1 (\mathfrak{a} is in row r^\downarrow): Here, T has \mathcal{F} southwest of \mathcal{F}' , unless T has both $\mathcal{F} \in \mathfrak{a}$ and \mathcal{F}' in $\underline{\mathfrak{b}^\downarrow}$ or $\mathcal{F}' \in \underline{\mathfrak{b}^{\downarrow\rightarrow}}$. Suppose that both of these occur. Then, U has $\bullet_{\mathcal{G}^+} \in \mathfrak{a}$ and either $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^\downarrow$ or $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^{\downarrow\rightarrow}$. Since, by U 's (G.4), \mathfrak{a} is West of \mathfrak{b} , this contradicts U 's (G.2).

Case 7.2 (\mathfrak{a} is in row r): Suppose that T has both $\mathcal{F} \in \mathfrak{a}$ and \mathcal{F}' in $\underline{\mathfrak{b}^\downarrow}$, $\mathcal{F}' \in \mathfrak{b}^{\downarrow\rightarrow}$ or $\mathcal{F}' \in \underline{\mathfrak{b}^{\downarrow\rightarrow}}$. Then, U has $\bullet_{\mathcal{G}^+} \in \mathfrak{a}$ and either $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^\downarrow$ or $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^{\downarrow\rightarrow}$, contradicting U 's (G.2). Hence, the reverse swap cannot both move \mathcal{F} North and move \mathcal{F}' South.

Suppose that T has $\mathcal{F} \in \mathfrak{a}$. Then, T has $\mathcal{F}' \in \underline{\mathfrak{b}}$ and U has $\bullet_{\mathcal{G}^+} \in \mathfrak{a}$. Then, $\mathcal{F} = \mathcal{G}^+$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G}) = \text{family}(\mathcal{G}^+)$. The reverse miniswap involving \mathfrak{a} is L4.1, L4.2 or L4.3. Hence, U has $\mathcal{G} \in \mathfrak{a}^\leftarrow$. By U 's (G.4), $\text{label}_U(\mathfrak{b}) < \mathcal{F}'$, so it is marked. By Lemma 5.4, $\text{label}_U(\mathfrak{a}^\rightarrow)$ is marked. This contradicts the last sentence of U 's (G.12).

Suppose that T has \mathcal{F}' in $\underline{\mathfrak{b}^\downarrow}$, $\mathfrak{b}^{\downarrow\rightarrow}$ or $\underline{\mathfrak{b}^{\downarrow\rightarrow}}$. Then, T has $\mathcal{F} \in \underline{\mathfrak{a}}$ and $\bullet_{\mathcal{G}} \in \mathfrak{b}^\downarrow$, while U has $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^\downarrow$ or $\mathfrak{b}^{\downarrow\rightarrow}$. Hence, although T has \mathcal{F} NorthWest of \mathcal{F}' , they do not violate (G.12).

Consistency of the prescribed virtual labels:

LEMMA B.4. *Let U be a \mathcal{G}^+ -good tableau in which we apply L2.3, L4.3 or L4.5, and let $T \in \text{revswap}_{\mathcal{G}^+}(U)$. Then, all prescribed \mathcal{G} 's from the outputs of L2.3, L4.3 and L4.5 are valid virtual labels in the sense of (V.1)–(V.3).*

Proof. Consider such a reverse miniswap. We may assume that T contains an output with a prescribed \mathcal{G} , say in \underline{x} . Let T^* be T with \mathcal{G} added in \underline{x} .

((V.1) holds; that is, $\mathcal{G} \in \underline{x}$ is not marked in T^*): Every \bullet in T^* is $\bullet_{\mathcal{G}}$, so, by definition, no \mathcal{G} in T^* is marked.

((V.2) holds; that is, T has a \mathcal{G} West of \underline{x}): By assumption of the reverse miniswaps, U has some \mathcal{G} West of \underline{x} . Hence, by Lemma B.3, T also has a \mathcal{G} West of \underline{x} .

((G.1) holds in T^*): This is immediate from U 's (G.1).

((G.4) holds in T^*): Let $\mathcal{H} = \text{label}_T(x^\downarrow)$. We must show that $\mathcal{G} < \mathcal{H}$. By Lemma 8.4, x^\downarrow is not in a ladder of U , so $\mathcal{H} = \text{label}_U(x^\downarrow)$. Hence, by U 's (G.4), we get $\mathcal{G} < \mathcal{H}$ in the case of L2.3 and L4.3. In the case of L4.5, by U 's (G.11), $\text{family}(\mathcal{H}) \geq \text{family}(\mathcal{G})$. However, by U 's (G.12), $\text{family}(\mathcal{H}) \neq \text{family}(\mathcal{G})$, so $\mathcal{G} < \mathcal{H}$.

Let $\mathcal{F} = \text{label}_T(x^\uparrow)$. We must show that $\mathcal{F} < \mathcal{G}$ in the L2.3 and L4.3 cases. (In the L4.5 case, there is nothing to confirm here.) Since x^\uparrow is not part of a ladder in U , $\mathcal{F} = \text{label}_U(x^\uparrow)$ as well. Hence, $\mathcal{F} < \mathcal{G}$ follows from U 's (G.4).

((G.5) holds in T^*): Suppose the contrary. Then, there is some $\mathcal{G}' \in \underline{x}$ in T with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$. This \mathcal{G}' is not the result of any reverse miniswap, so $\mathcal{G}' \in \underline{x}$ in U . In the case of L2.3 or L4.3, this contradicts U 's (G.4). In the case of L4.5, it should be noted that, by U 's (G.6) and (G.7), $\mathcal{G}' = \mathcal{G}^+$; this contradicts the assumption of L4.5.

((G.6) holds in T^*): This follows from Lemma B.3, as in the proof of T 's (G.6) above.

((G.8) holds in T^*): In the case of L4.5, this is immediate from the assumption that $N_{\mathcal{G}} = N_{\mathcal{E}}$ for every $\mathcal{E} \in Z$. In the case of L2.3, consider the tableau \tilde{T} differing from T only in the box \underline{x} , where we choose the other output of L2.3. By \tilde{T} 's (G.8), \tilde{T} is ballot. However, \tilde{T} and T^* have the same column reading words. Hence, T^* is ballot.

Finally, we consider the case of L4.3. By T 's (G.8), T is ballot. Hence, if there is a genotype G of T^* that is not ballot, G uses $\mathcal{G} \in \underline{x}$. Let \mathcal{F} be the gene with $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G}) - 1$ and $N_{\mathcal{F}} = N_{\mathcal{G}}$. Since G is not ballot, \mathcal{F} appears after \mathcal{G} in $\text{word}(G)$. Say that \mathcal{F} appears in column c in G . By inspection of the reverse miniswaps, \mathcal{F} appears in U either in column c or in column c^{\leftarrow} . Thus, considering this \mathcal{F} and U 's $\mathcal{G} \in \underline{x}$, we contradict Lemma 5.6 for U .

((G.9) holds in T^*): Here, \underline{x} is southeast of a $\bullet_{\mathcal{G}}$ in T . Hence, by T 's (G.2), \underline{x} is not northwest of a $\bullet_{\mathcal{G}}$ in T , and this (G.9) verification is vacuous.

((G.12) holds in T^*): For L2.3, consider the tableau \tilde{T} differing from T only in the box \underline{x} , where we choose the other output of L2.3. By \tilde{T} 's (G.12), any label \mathcal{G}' of \tilde{T} with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$ that is West of \underline{x} must be no further North than the upper edge of \underline{x} 's row. Since T^* and \tilde{T} are identical outside of

x , this is also true of T^* . Since T^* has $\bullet_{\mathcal{G}} \in x$ and $\mathcal{G} \notin x^{\leftarrow}$, T^* 's (G.12) then follows.

For L4.3, we observe that in light of the $\mathcal{G}^+ \in x^{\rightarrow}$ in T , it follows from T 's (G.12) that any label \mathcal{G}' of T with $\text{family}(\mathcal{G}') = \text{family}(\mathcal{G})$ that is West of x must be no further North than the upper edge of x 's row. Since T^* has $\bullet_{\mathcal{G}} \in x$ and $\mathcal{G} \notin x^{\leftarrow}$, T^* 's (G.12) then follows as a consequence.

The L4.5 case is proved by repeating the arguments above for T 's (G.12) (Cases 3–5). □

Condition (G.13): For every marked label $\mathcal{E}^!$ in T , U has \mathcal{E} or $\mathcal{E}^!$ in the same position. Thus, our analysis splits into two cases.

Case 1 (U has $\mathcal{E}^! \in \underline{\mathbf{b}}$ or $\underline{\mathbf{b}}$): Let ℓ be this instance of $\mathcal{E}^!$, and let $\underline{\mathbf{b}}$ be in column c ; U also has some \mathcal{F} or $\mathcal{F} \in \underline{\mathbf{b}}$ with $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$. Since ℓ is marked, $\mathcal{E} \leq \mathcal{G}$.

Subcase 1.1 ($\text{family}(\mathcal{E}) \leq \text{family}(\mathcal{G}) - 2$): Such a marked label is not moved by any reverse miniswap. Hence, ℓ is in the same position in U and T . By Lemma 5.5, U has $\mathcal{F} \in \underline{\mathbf{b}}$ (rather than $\mathcal{F} \in \underline{\mathbf{b}}$). This \mathcal{F} is also not moved by any reverse miniswap, so T has $\mathcal{F} \in \underline{\mathbf{b}}$ as well, and (G.13) is satisfied.

Subcase 1.2 ($\text{family}(\mathcal{E}) = \text{family}(\mathcal{G}) - 1$): Here, $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$. As in Subcase 1.1, ℓ is in the same position in U and T . Suppose that U has $\mathcal{F} \in \underline{\mathbf{b}}$ (rather than $\mathcal{F} \in \underline{\mathbf{b}}$). By U 's (G.11), $\bullet_{\mathcal{G}^+} \notin c$ in U . Hence, there is no reverse miniswap that affects $\mathcal{F} \in \underline{\mathbf{b}}$. Therefore, T has $\mathcal{F} \in \underline{\mathbf{b}}$ and (G.13) holds.

Hence, assume that $\mathcal{F} \in \underline{\mathbf{b}}$ in U . By Lemma 5.9, $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G}^+)$ and U has $\bullet_{\mathcal{G}^+} \in \mathbf{b}^{\leftarrow}$. Let T^* be T with $\mathcal{F} \in \underline{\mathbf{b}}$. By U 's (V.1), $\mathcal{G} \prec \mathcal{F}$. Hence, T^* satisfies (V.1). By the proof of Lemma 9.4, it satisfies (V.2). It remains to show that T^* satisfies (G.1), (G.4)–(G.6), (G.8), (G.9) and (G.12).

(G.1): Since U has $\mathcal{F} \in \underline{\mathbf{b}}$, this follows from T 's (G.1).

(G.4): By T 's (G.4), since $\mathcal{E} < \mathcal{F}$, T has $\mathcal{Q} < \mathcal{F}$ for any \mathcal{Q} north of $\underline{\mathbf{b}}$ in c . By U 's (G.4), since no reverse miniswap affects labels of family greater than $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G})$, we also have $\mathcal{Q} > \mathcal{F}$ for any \mathcal{Q} appearing south of $\underline{\mathbf{b}}$ in c in T . Hence, (G.4) holds in T^* .

(G.5): By U 's (G.5), U has no label of family $\text{family}(\mathcal{F})$ on $\underline{\mathbf{b}}$. Indeed, by U 's (G.4), U has no label of that family in c . By U 's (G.11), U has $\bullet_{\mathcal{G}^+} \notin c$. Hence, no ladder of U intersects c , and T has no label of family $\text{family}(\mathcal{F})$ on $\underline{\mathbf{b}}$. Hence, (G.5) holds in T^* .

(G.6): As no ladder of U intersects c , the genes West of c in U are exactly the genes West of c in T , and the genes East of c in T are a subset of those East of c in U . Hence, (G.6) holds in T^* .

(G.8): No ladder of U intersects c , c is the same in U and T . Since $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$, it suffices to check that no \mathcal{E} is read in T^* after $\mathcal{F} \in \underline{\mathbf{b}}$. By U 's (G.4), there is no \mathcal{E} in U South of $\underline{\mathbf{b}}$ in c . Thus, this is also true in T . By Lemma 5.7, ℓ is the westmost \mathcal{E} in U . However, the genes that appear West of c in U are exactly the genes that appear West of c in T . Hence, no \mathcal{E} appears West of $\underline{\mathbf{b}}$ in T , no \mathcal{E} is read in T^* after $\underline{\mathbf{b}}$ and T^* 's (G.8) holds.

(G.9): Since U has $\bullet_{\mathcal{G}^+} \in \mathbf{b}^{\leftarrow}$, by U 's (G.2), $\underline{\mathbf{b}}$ is not northwest of a $\bullet_{\mathcal{G}^+}$. Hence, by inspection of the reverse miniswaps, $\underline{\mathbf{b}}$ is not northwest of a $\bullet_{\mathcal{G}}$ in T , so (G.9) holds in T^* .

(G.12): Suppose that i is a label of \mathcal{F} 's family appearing in \mathbf{a} or $\underline{\mathbf{a}}$ NorthWest of $\underline{\mathbf{b}}$ in T . First, suppose that i does not appear in the same position in U . Then, i was involved in a reverse miniswap. If it was L1.1, then $\mathcal{G} \in \mathbf{a}^{\uparrow}$ in U , contradicting U 's (G.12). It obviously was not L1.2. If it was an L2 miniswap, $i \in \mathbf{a}$ in U , contradicting U 's (G.12). The same holds for L3. For L4.1–3 to apply, $\mathbf{a} = \mathbf{b}^{\leftarrow}$, by U 's (G.2), but this contradicts U 's (G.12) (last sentence). In L4.4 and L4.5, $Z \neq \emptyset$, and the only labels of concern are the \mathcal{G} s. Since they satisfy (G.12) in U , they do in T^* .

Now, suppose that i is in the same position in U and T . Hence, i is NorthWest of $\underline{\mathbf{b}}$ in U . Let $\underline{\mathbf{b}}$ be in row r . By U 's (G.12), either U has $i \in \mathbf{a}$ and $\mathbf{a} \in r$, or else U has $i \in \underline{\mathbf{a}}$ and $\underline{\mathbf{a}} \in r^{\uparrow}$. Moreover, \mathbf{a} is West of \mathbf{b}^{\leftarrow} . Since the labels in question do not move, it remains to check that $\bullet_{\mathcal{G}}$ appears in row r in T East of \mathbf{a} and west of $\underline{\mathbf{b}}$. We are clearly only concerned when the reverse miniswap involving \mathbf{b}^{\leftarrow} is L1.1, or is L4 with $\mathbf{b}^{\leftarrow} = \mathbf{a}^{\rightarrow}$. If it is L1.1, U has $\mathcal{G} \in \mathbf{b}^{\leftarrow\uparrow}$, contradicting U 's (G.12). If it is L4 with $\mathbf{b}^{\leftarrow} = \mathbf{a}^{\rightarrow}$, then i would not appear in the same position in T , contradicting our assumption.

Now, that suppose i is a label of \mathcal{F} 's family appearing in \mathbf{a} or $\underline{\mathbf{a}}$ SouthEast of $\underline{\mathbf{b}}$ in T . First, suppose that i does not appear in the same position in U . Then, i was involved in a reverse miniswap. By U 's (G.2), this can only be an L2 reverse miniswap. It is obviously not L2.1. If it is L2.2 or L2.3, then, by U 's (G.12), the \mathcal{G} or \mathcal{G} in $\underline{\mathbf{a}}$ is not a (G.12) violation in T because of T 's $\bullet_{\mathcal{G}} \in \mathbf{a}$.

Otherwise, i is in the same position in U and T . Hence, i is SouthEast of $\underline{\mathbf{b}}$ in U . By U 's (G.12), U has a $\bullet_{\mathcal{G}^+}$ SouthEast of $\underline{\mathbf{b}}$. Given U 's $\bullet_{\mathcal{G}^+} \in \mathbf{b}^{\leftarrow}$, this contradicts U 's (G.2).

We conclude that the desired \mathcal{F} appears on $\underline{\mathbf{b}}$ in T .

Subcase 1.3 ($\text{family}(\mathcal{E}) = \text{family}(\mathcal{G})$): Suppose that ℓ is moved by the reverse swap. We recall that $\mathcal{E} \prec \mathcal{G}^+$. By inspection, no reverse swap will move

such an $\mathcal{E} \prec \mathcal{G}^+$ with $\text{family}(\mathcal{E}) = \text{family}(\mathcal{G})$ unless $\mathcal{E} = \mathcal{G}$. Since the \bullet s in T are $\bullet_{\mathcal{G}}$ s, no \mathcal{E} will be marked in T , so T 's (G.13) check is vacuous.

Hence, we assume that ℓ is unaffected by $\text{revswap}_{\mathcal{G}^+}$ and indeed that $\mathcal{E} \prec \mathcal{G}$. Since $\text{family}(\mathcal{F}) = \text{family}(\mathcal{G}) + 1$, no reverse miniswap affects any instance of \mathcal{F} . In particular, if U has $\mathcal{F} \in \underline{\mathfrak{b}}$ (instead of \mathfrak{F}), then T will also have $\mathcal{F} \in \underline{\mathfrak{b}}$ and satisfy (G.13).

Hence, we further assume that U has $\mathfrak{F} \in \underline{\mathfrak{b}}$. We need $\mathfrak{F} \in \underline{\mathfrak{b}}$ in T . By Lemma 5.9, U has $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^{\leftarrow}$. Let c be \mathfrak{b} 's column. By U 's (G.11), U has $\bullet_{\mathcal{G}^+} \notin c$. By U 's (G.4), since $\mathcal{E} \prec \mathcal{G}$, U has $\mathcal{G} \notin c$. Hence, no ladder of U intersects c , and so column c in T is identical to column c in U .

Let T^*, U^* be T, U , respectively, with $\mathcal{F} \in \underline{\mathfrak{b}}$ added. We show that T^* satisfies (V.1)–(V.3).

(V.1): Since $\mathcal{F} > \mathcal{G}$, this is obvious.

(V.2): By U 's (V.2), U has an \mathcal{F} West of c . Hence, there is an \mathcal{F} West of c in T , as needed.

(V.3): We show that T^* satisfies (G.1), (G.4), (G.5), (G.6), (G.8), (G.9) and (G.12). We know that T satisfies these.

(G.1): This is immediate from T 's (G.1), given U 's $\mathfrak{F} \in \underline{\mathfrak{b}}$.

(G.4) and (G.5): These hold in T^* since they hold in U^* , and column c of T is identical to column c of U .

(G.6): The genes West of c in U are exactly the genes West of c in T , and the genes East of c in T are a subset of those East of c in U . Now, T^* 's (G.6) follows.

(G.8): This is immediate from U 's $\mathfrak{F} \in \underline{\mathfrak{b}}$ and the facts that

- the genes West of c in T are exactly the genes West of c in U ;
- the genes East of c in T are a subset of those East of c in U ; and
- column c in T is identical to column c in U .

(G.9): Since ℓ is marked in T , T has a $\bullet_{\mathcal{G}}$ northwest of $\underline{\mathfrak{b}}$. Hence, by T 's (G.2), $\underline{\mathfrak{b}}$ is not northwest of a $\bullet_{\mathcal{G}}$ in T^* , so this condition is vacuous.

(G.12): Take \mathcal{F}' with $\text{family}(\mathcal{F}') = \text{family}(\mathcal{F})$. Suppose that \mathcal{F}' is NorthWest of $\underline{\mathfrak{b}}$ in T^* . Since $\text{family}(\mathcal{F}') > \text{family}(\mathcal{G})$, \mathcal{F}' appears in the same positions in both T^* and U^* . Hence, \mathcal{F}' is NorthWest of $\underline{\mathfrak{b}}$ in U' . However, then, this \mathcal{F}' is northwest of U^* 's $\bullet_{\mathcal{G}^+} \in \mathfrak{b}^{\leftarrow}$, contradicting U^* 's (G.9). Thus, T^* has no such \mathcal{F}' NorthWest of $\underline{\mathfrak{b}}$. Similarly, T^* has no \mathcal{F}' SouthEast of $\underline{\mathfrak{b}}$.

Case 2 (ℓ is a marked label in T that is not marked in U): Suppose that ℓ is an instance of $\mathcal{E}^!$ on \mathfrak{b} or $\underline{\mathfrak{b}}$ in T . Since ℓ is marked and every bullet in T is $\bullet_{\mathcal{G}}$, $\mathcal{E} \prec \mathcal{G}$. Hence, $\mathcal{E} \prec \mathcal{G}^+$, and any instance of \mathcal{E} southeast of a $\bullet_{\mathcal{G}^+}$ is marked in U .

Subcase 2.1 ($\text{family}(\mathcal{E}) = \text{family}(\mathcal{G})$): No reverse miniswap affects any instance of \mathcal{E} . Hence, ℓ is in the same position in U as in T . Since ℓ is unmarked in U , U has no $\bullet_{\mathcal{G}^+}$ northwest of $\underline{\mathbf{b}}$. Hence, since U has $\mathcal{E} \in \mathbf{b}$ or $\underline{\mathbf{b}}$, by U 's (G.3) and (G.4), U has no \mathcal{G} northwest of $\underline{\mathbf{b}}$. However, there is $\bullet_{\mathcal{G}}$ northwest of $\underline{\mathbf{b}}$ in T ; this contradicts Lemma B.1.

Subcase 2.2 ($\text{family}(\mathcal{E}) < \text{family}(\mathcal{G})$): If ℓ is not moved by $\text{revswap}_{\mathcal{G}^+}$, we obtain a contradiction exactly as in Subcase 2.1. Otherwise, it is moved by an L4.4 or L4.5 reverse miniswap. Then, by definition, $N_{\mathcal{E}} = N_{\mathcal{G}}$. Let \mathcal{F} be the gene (which must exist) with $N_{\mathcal{F}} = N_{\mathcal{E}} = N_{\mathcal{G}}$ and $\text{family}(\mathcal{F}) = \text{family}(\mathcal{E}) + 1$. By Lemma 5.8, \mathcal{F} appears in \mathbf{b}^{\leftarrow} 's column in U . Hence, by U 's (G.4), U has $\mathcal{F} \in \mathbf{b}^{\leftarrow}$ or $\underline{\mathbf{b}^{\leftarrow}}$. It is then in the set A'' or $A''' \cup \{\mathcal{G}\}$ (in the notation of L4.4/L4.5) and appears on $\underline{\mathbf{b}}$ in T (possibly virtual), as desired. \square

Appendix C. Block decomposition; completion of proof of Proposition 12.3(II)

Below, we define, for each $S \in \text{Snakes}_{\mathcal{G}}$, a set \mathcal{B}_S containing S . Clearly, $\cup_{S \in \text{Snakes}_{\mathcal{G}}} \mathcal{B}_S = \text{Snakes}_{\mathcal{G}}$. Along the way, we will argue that if $S, S' \in \text{Snakes}_{\mathcal{G}}$ and $S' \in \mathcal{B}_S$, then $\mathcal{B}_S = \mathcal{B}_{S'}$. This proves (D.1).

We recall that $\Gamma_i := \{T \in P_{\mathcal{G}} : T \text{ contains a snake from } \mathcal{B}_i\}$. From the construction, the following two additional conditions will also be essentially clear.

(D.3) Suppose that $A, B \in \Gamma_i$. For each snake in \mathcal{B}_i and A , there is a snake of \mathcal{B}_i in B and in the exact same location.

(D.4) The tableaux $A, B \in \Gamma_i$ are identical outside of the snakes in \mathcal{B}_i .

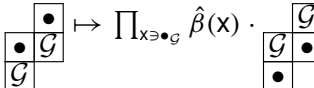

The bulk of the work is to establish (D.2). This will be done simultaneously with the description of each \mathcal{B}_S . To establish (D.2), we must verify (12.3) by considering the `boxfactors`, `edg factors` and `virtualfactors` from every box and edge of the common shape ν/λ . Except where otherwise noted, by inspection, these factors do not change for boxes/edges not in \mathcal{B}_S . Thus, the majority of our discussion concerns the region defined by \mathcal{B}_S . For simplicity, we assume that $\mathcal{B}_J = \emptyset$. The modifications for the general case are straightforward, using (D.3) and (D.4).

Assume that $\mathcal{G} := i_k$. Let $S \in \text{Snakes}_{\mathcal{G}}$ be in the tableau U . We break into cases according to the type of `head(S)`. We write T_1, T_2, \dots for the fine tableau in $\text{swap}_S(U)$, in the order illustrated in each case. We write U_j for T_j together with its coefficient in $\text{swap}_S(U)$ and S_i for the image of the snake S in T_i .

Case 0 ($\text{head}(S) = \emptyset$): By Definition-Lemma 6.8, either the southmost row of S contains a single box, or else it consists of two boxes x, x^{\rightarrow} with $\bullet \in x$ and a marked label in x^{\rightarrow} . Thus, $\text{body}(S)$ is either empty, or it falls under case B1 or B3.

Subcase 0.1 ($\text{body}(S)$ is B1): Let $\mathcal{B}_S = \{S\}$. Since S contains no $\bullet_{\mathcal{G}}$, $\text{swapset}_{\mathcal{B}_S}(U) = U$. Thus, $\text{wt}(\text{swapset}_{\mathcal{B}_S}(U)) = \text{wt}(U)$, which implies (12.3).

Subcase 0.2 ($\text{body}(S)$ is B3 or $\text{body}(S) = \emptyset$): By Definition-Lemma 6.8(III), either S has at least two rows, or else $S = \text{tail}(S)$. In either case, $\text{tail}(S) \neq \emptyset$.

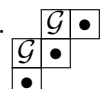
Subcase 0.2.1 ($\text{tail}(S)$ is T1): Let $\mathcal{B}_S = \{S\}$. Locally, at the snake S , this swap looks like  $\mapsto \prod_{x \ni \bullet_{\mathcal{G}}} \hat{\beta}(x) \cdot$ . It should be noted that $\text{body}(S)$

is nonempty in this case. This swap does not affect the locations or weights of edge labels or virtual labels in U . Hence, $\text{edgewt}(U) = \text{edgewt}(T_1)$ and $\text{virtualwt}(U) = \text{virtualwt}(T_1)$. One checks that a box outside S is productive in U if and only if it is productive in T_1 . (The critical checks are for the box immediately east of the northmost box of S and the box immediately west of the southmost box of S .) Moreover, each such productive box has the same boxfactor in U and T_1 . The productive boxes of S_1 are the boxes $\{x\}$ containing \mathcal{G} , while in S they are the boxes $\{x^{\downarrow}\}$ containing \mathcal{G} . For each productive box x of S_1 with $\text{boxfactor}(x) := w_x$, there is a corresponding productive box x^{\downarrow} in S with $\text{boxfactor}(x^{\downarrow}) = \hat{\beta}(x)w_x$.

Thus, $\text{wt } U = \text{wt } U_1$ follows from

$$(-1)^{d(U)} \prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} \hat{\beta}(x)w_x = \left(\prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} \hat{\beta}(x) \right) \cdot (-1)^{d(U)} \prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} w_x.$$

Subcase 0.2.2 ($\text{tail}(S)$ is T2): Let $\mathcal{B}_S = \{S\}$. This case is similar to Subcase 0.2.1; we have something like

$$\img alt="Diagram showing a swap of a box with G and a box with a dot in a 2x2 grid." data-bbox="330 770 410 825"/> $\mapsto - \prod_{x \ni \bullet} \hat{\beta}(x) \cdot$ .$$

This swap preserves locations and weights of edge labels. Hence, $\text{edgewt}(U) = \text{edgewt}(T_1)$. A box outside S is productive in U if and only if it is productive in T_1 . Moreover, each such productive box has the same boxfactor in U and T_1 . Let y be the box containing the Northmost \mathcal{G} in S . In S_1 , the

productive boxes are the boxes $\{x\}$ containing \mathcal{G} . The productive boxes of S are the boxes $\{x^\downarrow\}$ containing \mathcal{G} in all but the northmost row of S , and y if $\mathcal{G}^+ \notin y^\rightarrow$ with $\text{family}(\mathcal{G}) = \text{family}(\mathcal{G}^+)$. For each productive box x in S_1 with $\text{boxfactor}(x) := w_x$, there is a corresponding productive x^\downarrow of S with $\text{boxfactor}(x^\downarrow) = \hat{\beta}(x)w_x$.

The box y is productive in S with $\text{boxfactor}(y) := y$ if and only if $\textcircled{\mathcal{G}} \in \bar{y}$ in S_1 with $\text{virtualfactor}_{\mathcal{G}}(\bar{y}) = y$. The swap does not otherwise affect the location or weight contribution of virtual labels. Finally, it should be noted that $(-1)^{d(U)} = (-1)^{d(T_1)-1}$. If y is productive in S , then $\text{wt } U = \text{wt } U_1$ follows from

$$\begin{aligned} & (-1)^{d(U)} \cdot y \prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} \hat{\beta}(x)w_x \\ &= \left(- \prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} \hat{\beta}(x) \right) \cdot (-1)^{d(U)-1} \cdot y \cdot \prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} w_x. \end{aligned}$$

If y is not productive in S , we use the same identity without y .

Subcase 0.2.3 ($\text{tail}(S)$ is T3): This is again similar to Subcase 0.2.1. Locally, at S , we have something like

$$\begin{array}{|c|c|} \hline & \bullet \mathcal{G}^+ \\ \hline \bullet & \mathcal{G} \\ \hline \mathcal{G} & \\ \hline \end{array} \mapsto \prod_{x \ni \bullet} \hat{\beta}(x) \cdot \begin{array}{|c|c|} \hline \mathcal{G} & \mathcal{G}^+ \\ \hline \mathcal{G} & \bullet \\ \hline \bullet & \\ \hline \end{array} - \alpha \cdot \begin{array}{|c|c|} \hline \mathcal{G} & \bullet \\ \hline \mathcal{G} & \bullet \\ \hline \bullet & \\ \hline \end{array} \mathcal{G}^+$$

By Lemma 6.9(V), S has at least two rows. Let y be the box containing \mathcal{G}^+ in S . Here, $\alpha := \prod_{x \ni \bullet} \hat{\beta}(x)$ if $\bullet \notin y^\uparrow$ and $\alpha := 0$ otherwise.

The locations and weights of virtual labels are unaffected by the swap. Therefore, $\text{virtualwt}(U) = \text{virtualwt}(T_1) = \text{virtualwt}(T_2)$. Furthermore, the edgefactors and boxfactors from labels outside S are the same in each of U, T_1, T_2 , so we restrict attention to the boxfactors and edgefactors from labels inside S .

Subcase 0.2.3.1 ($\bullet_{\mathcal{G}} \notin y^\uparrow$): Let $\mathcal{B}_S = \{S\}$. The productive boxes of S_1 and S_2 are the boxes $\{x\}$ containing \mathcal{G} and not in the northmost row, and possibly also y (depending on what label, if any, appears in y^\rightarrow). The productive boxes of S are those boxes $\{x^\downarrow\}$ containing \mathcal{G} not in the second row from the top, the box z containing \mathcal{G} in the second row from the top, and possibly also y . One sees that y is productive in any of S, S_1, S_2 if and only if it is productive in all of them. Further, if it is productive, then it has the same boxfactor, say a , in each one.

For each productive box x in each of S_1, S_2 , with $\text{boxfactor}(x) := w_x$, there is a corresponding productive box x^\downarrow in S , with $\text{boxfactor}(x^\downarrow) = \hat{\beta}(x)w_x$. Let $\text{edgefactor}_{z \rightarrow \epsilon T_2}(\mathcal{G}^+) := 1 - b$. Then, $\text{boxfactor}_U(z) = \hat{\beta}(z^\uparrow)b$.

Now, (12.3) is the statement $\text{wt } U = \text{wt}(U_1 + U_2)$. If y is productive in U , this follows from the identity

$$a\hat{\beta}(z^\uparrow)b \prod_x \hat{\beta}(x)w_x = \left(\hat{\beta}(z^\uparrow) \prod_x \hat{\beta}(x) \right) \cdot \left[a \prod_x w_x - (1 - b) \cdot a \prod_x w_x \right].$$

Otherwise, we use the same identity without a .

Subcase 0.2.3.2 ($\bullet_{\mathcal{G}} \in y^\uparrow$): Here, $\alpha = 0$, so we ignore T_2 . Let S' be the snake containing y^\uparrow , and let $\mathcal{B}_S = \{S, S'\}$. By Lemma 6.6, $S' = \{y^\uparrow\}$ participates in a trivial H3 or H8 miniswap.

The productive boxes $\{x\}$ of S_1 are those containing \mathcal{G} (even the box y^\leftarrow) and possibly also y (depending on what label, if any, appears in y^\rightarrow). The productive boxes of S are the boxes $\{x^\downarrow\}$ containing \mathcal{G} and possibly also y . One checks that y is productive in S if and only if it is productive in S_1 . If it is productive, $\text{boxfactor}_U(y) = \text{boxfactor}_{T_1}(y) := y$. In S' , $y^{\uparrow\rightarrow}$ is productive if and only if it is in S'_1 . If it is productive, it contributes the same $\text{boxfactor}(y^{\uparrow\rightarrow}) := q$ to both.

For each productive x in S_1 with $\text{boxfactor}(x) := w_x$, there is a corresponding productive x^\downarrow in S with $\text{boxfactor}(x^\downarrow) = \hat{\beta}(x)w_x$. If y and $y^{\uparrow\rightarrow}$ are productive in U , (12.3) follows from

$$qy \prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} \hat{\beta}(x)w_x = q \left(\prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} \hat{\beta}(x) \right) \cdot y \cdot \prod_{x:\text{label}_U(x)=\bullet_{\mathcal{G}}} w_x.$$

Otherwise, we use the same identity without q, y or both.

Subcase 0.2.4 (tail(S) is T4): Let $\text{tail}(S) = \{x, y := x^\rightarrow\}$. By (G.7), the $\mathcal{G} \in \underline{y}$ is westmost in its gene, so $\text{body}(S) = \emptyset$.

Subcase 0.2.4.1 (tail(S) is T4.1): Set $\mathcal{B}_S = \{S\}$. Locally, at S , $\boxed{\bullet_{\mathcal{G}} \mathcal{F}^!}_{\mathcal{G}, \mathcal{H}} \mapsto \boxed{\bullet_{\mathcal{G}} \mathcal{F}^!}_{\mathcal{G}, \mathcal{H}}$, where we have $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$. Virtual labels appear in the same places in U and T_1 . In particular, neither U nor T_1 can have $(\mathcal{G}) \in \underline{x}$, since it would be West of every \mathcal{G} . Further, no labels move. As no weights change, trivially, $\text{wt } U = \text{wt } U_1$, which implies (12.3).

Subcase 0.2.4.2 (tail(S) is T4.2):

Subcase 0.2.4.2.1 ($\mathcal{G}^- \in x^-$ with family(\mathcal{G}^-) = family(\mathcal{G}): Set $\mathcal{B}_S = \{S\}$. Locally, at S , the swap is $\boxed{\mathcal{G}^- \bullet \mathcal{F}^1}_{\mathfrak{y} \oplus \mathfrak{x}} \mapsto \boxed{\mathcal{G}^- \bullet \mathcal{F}^1}_{\mathfrak{y} \oplus \mathfrak{x}} + \hat{\beta}(x) \cdot \boxed{\mathcal{G}^- \mathcal{G} \bullet}_{\mathfrak{y} \oplus \mathfrak{x}}$. Here, (12.3) is equivalent to $\text{wt } U = \text{wt } T_1 + \hat{\beta}(x) \text{wt } T_2$.

The $\mathcal{F}^1 \in \mathfrak{y}$ is productive in U and T_1 if and only if $(\mathcal{F}) \in \bar{\mathfrak{y}}$ in T_2 . If these boxes are productive,

$$\text{boxfactor}_U(\mathfrak{y}) = \text{boxfactor}_{T_1}(\mathfrak{y}) = \text{virtualfactor}_{\bar{\mathfrak{y}} \in T_2}(\mathcal{F}) := a.$$

The box x is not productive in U or T_1 , but it is in T_2 . Let $\text{boxfactor}_{T_2}(x) := u$. Then, $\text{virtualfactor}_{\mathfrak{y} \in U}(\mathcal{H}) = u$ and $\text{virtualfactor}_{\mathfrak{y} \in T_1}(\mathcal{H}) = u - 1$. Let $w := \text{edgefactor}_{\mathfrak{y} \in T}(\mathcal{G})$. Then, we have $\text{edgefactor}_{\mathfrak{y} \in T_1}(\mathcal{G}^1) = w$ and $\text{edgefactor}_{\bar{x} \in T_2}(\mathcal{F}) = w$. The box x^- is productive in U and T_1 , but not in T_2 . We have $\text{boxfactor}_U(x^-) = \text{boxfactor}_{T_1}(x^-) = \hat{\beta}(x)u$.

If \mathfrak{y} is productive in U , $\text{wt } U = \text{wt } T_1 + \hat{\beta}(x) \text{wt } T_2$ follows from the identity on \mathcal{B}_S -contributions: $\hat{\beta}(x)au^2w = \hat{\beta}(x)au(u - 1)w + \hat{\beta}(x)auw$. If \mathfrak{y} is not productive in U , it follows from the same identity after canceling as .

Subcase 0.2.4.2.2 ($\mathcal{G}^- \notin x^-$ or $\mathcal{G}^- < \mathcal{G}$): Let $\bar{T} := \phi_4^{-1}(T) \in P_G$. Let \bar{S} be the snake of \bar{T} containing x . Set $\mathcal{B}_S = \{S, \bar{S}\}$; thus, $\Gamma = \{T, \bar{T}\}$.

Locally, at S , the swap is $\boxed{\bullet \mathcal{F}^1}_{\mathfrak{y} \oplus \mathfrak{x}} \mapsto \boxed{\bullet \mathcal{F}^1}_{\mathfrak{y} \oplus \mathfrak{x}} + \hat{\beta}(x) \cdot \boxed{\mathcal{G} \bullet}_{\mathfrak{y} \oplus \mathfrak{x}}$. Locally, at \bar{S} , $\boxed{\bullet \mathcal{F}^1}_{\mathfrak{y} \oplus \mathfrak{x}} \mapsto 0$. By Proposition 12.21, $[T]P_G = [\bar{T}]P_G$. Hence, (12.3) is equivalent to $\text{wt } U + \text{wt } \bar{U} = \text{wt } U_1 + \text{wt } U_2$.

The $\mathcal{F}^1 \in \mathfrak{y}$ is productive in U , T_1 and \bar{U} if and only if $(\mathcal{F}) \in \bar{\mathfrak{y}}$ in T_2 . If these boxes are productive,

$$\begin{aligned} \text{boxfactor}_U(\mathfrak{y}) &= \text{boxfactor}_{T_1}(\mathfrak{y}) \\ &= \text{boxfactor}_{\bar{U}}(\mathfrak{y}) = \text{virtualfactor}_{\bar{\mathfrak{y}} \in T_2}(\mathcal{F}) := a. \end{aligned}$$

The box x is not productive in U , T_1 or \bar{U} . Let $w = \text{edgefactor}_{\mathfrak{y} \in T}(\mathcal{G})$. Then, we have $\text{virtualfactor}_{\mathfrak{y} \in \bar{T}}(\mathcal{G}) = -w$, $\text{edgefactor}_{\mathfrak{y} \in T_1}(\mathcal{G}^1) = w$ and $\text{edgefactor}_{\bar{x} \in T_2}(\mathcal{F}) = w$. Let $u = \text{virtualfactor}_{\mathfrak{y} \in U}(\mathcal{H})$. Then, $\text{virtualfactor}_{\mathfrak{y} \in T_1}(\mathcal{H}) = u - 1$. Let $1 - v = \text{edgefactor}_{\bar{x} \in \bar{U}}(\mathcal{G})$. Then, $\text{boxfactor}_{T_2}(x) = v/\hat{\beta}(x)$.

If \mathfrak{y} is productive in U , $\text{wt } U + \text{wt } \bar{U} = \text{wt } U_1 + \text{wt } U_2$ follows from the identity on \mathcal{B}_S -contributions: $auw - a(1 - v)w = a(u - 1)w + a\hat{\beta}(x)(v/\hat{\beta}(x))w$. The same is true if \mathfrak{y} is not productive in U , except that a does not appear.

Subcase 0.2.4.3 (tail(S) is T4.3):

Subcase 0.2.4.3.1 ($\mathcal{G}^- \in x^{\leftarrow}$ with $\text{family}(\mathcal{G}^-) = \text{family}(\mathcal{G})$): Set $\mathcal{B}_S = \{S\}$. Locally, at S , the swap is $\boxed{\mathcal{G}^- \bullet \mathcal{F}^1}_{Z \cup \mathcal{G}} \mapsto \hat{\beta}(x) \cdot \boxed{\mathcal{G}^- \mathcal{G} \bullet}_{Z \cup \mathcal{G}}$. Here, (12.3) is equivalent to $\text{wt } U = \hat{\beta}(x) \text{wt } T_1$.

For every label $\ell \in Z \cup \mathcal{G}$ in U , there is a unique label $\ell_1 \in \mathcal{F} \cup Z$ with $\text{family}(\ell_1) = \text{family}(\ell) - 1$. Moreover, $\text{edgefactor}_U(\ell) = \text{edgefactor}_{T_1}(\ell_1) := a_\ell$.

In U , y is productive if and only if y^\rightarrow does not contain a label of the same family as \mathcal{F} . Hence, y is productive in U if and only if $(\mathcal{F}) \in \bar{y}$ in T_1 . If y is productive in U , then $\text{boxfactor}_U(y) = \text{boxfactor}_{\bar{T}_1}(y) = \text{virtualfactor}_{\bar{y} \in T_1}((\mathcal{F})) := b$.

The box x is productive in T_1 , but not in U . Let $w := \text{boxfactor}_{T_1}(x)$. The box x^{\leftarrow} is productive in U , but not in T_1 . We have $\text{boxfactor}_U(x^{\leftarrow}) = \hat{\beta}(x)w$.

Hence, if y is productive in U , $\text{wt } U = \hat{\beta}(x) \text{wt } T_1$ follows from

$$\hat{\beta}(x)wb \prod_{\ell \in Z \cup \mathcal{G}} a_\ell = \hat{\beta}(x)bw \prod_{\ell_1 \in \mathcal{F} \cup Z} a_{\ell_1}.$$

Otherwise, we use the same identity after canceling b .

Subcase 0.2.4.3.2 ($\mathcal{G}^- \notin x^{\leftarrow}$ or $\mathcal{G}^- < \mathcal{G}$): Let $\bar{T} := \phi_4^{-1}(T) \in P_G$. Let \bar{S} be the snake of \bar{T} containing x , and set $\mathcal{B}_S = \{S, \bar{S}\}$.

Locally, at S , the swap is $\boxed{\bullet \mathcal{F}^1}_{Z \cup \mathcal{G}} \mapsto \hat{\beta}(x) \cdot \boxed{\mathcal{G} \bullet}_{Z \cup \mathcal{G}}$. Locally, at \bar{S} , $\boxed{\bullet \mathcal{F}^1}_{\mathcal{G} \cup \mathcal{Z}} \mapsto 0$. By Proposition 12.21, $[T]P_G = [\bar{T}]P_G$. Hence, (12.3) is equivalent to $\text{wt } U + \text{wt } \bar{U} = \text{wt } U_1$.

For every label $\ell \in Z \cup \mathcal{G}$ in U , there is a unique label $\ell_1 \in \mathcal{F} \cup Z$ with $\text{family}(\ell_1) = \text{family}(\ell) - 1$. Moreover, $\text{edgefactor}_U(\ell) = \text{edgefactor}_{T_1}(\ell_1) := a_\ell$. If $\ell \in Z$ in U , then there is unique $\bar{\ell} \in Z$ in \bar{U} with $\text{family}(\bar{\ell}) = \text{family}(\ell)$, and $\text{edgefactor}_{\bar{T}}(\bar{\ell}) = a_\ell$. Further, $\text{virtualfactor}_{y \in \bar{U}}((\mathcal{G})) = -a_G$.

In U , y is productive if and only if y^\rightarrow does not contain a label of the same family as \mathcal{F} . Hence, y is productive in U if and only if it is productive in \bar{U} and further if and only if $(\mathcal{F}) \in \bar{y}$ in T_1 . If y is productive in U , then $\text{boxfactor}_U(y) = \text{boxfactor}_{\bar{T}}(y) = \text{virtualfactor}_{\bar{y} \in T_1}((\mathcal{F})) := b$.

The box x is productive in T_1 . Let $w := \text{boxfactor}_{T_1}(x)$. Observe that $\text{edgefactor}_{x \in \bar{U}}(\mathcal{G}) = 1 - \hat{\beta}(x)w$. Hence, $\text{wt } U + \text{wt } \bar{U} = \text{wt } U_1$ follows from

$$b \prod_{\ell \in Z \cup \mathcal{G}} a_\ell + b(1 - \hat{\beta}(x)w)(-a_G) \prod_{\ell \in Z} a_\ell = \hat{\beta}(x)bw \prod_{\ell \in Z \cup \mathcal{G}} a_\ell$$

if y is productive in U . If it is not productive, we use the same identity without b .

Subcase 0.2.5 ($\text{tail}(S)$ is T5): Either $\text{body}(S) = \emptyset$ and $\text{tail}(S) = S$, or else $\text{body}(S) \neq \emptyset$. Set $B_S = \{S\}$. Let $\text{tail}(S) = \{x, y := x^\rightarrow\}$.

Subcase 0.2.5.1 ($\text{body}(S) = \emptyset$): Locally, at S , $\begin{bmatrix} \bullet & \mathcal{F}^\dagger \\ \oplus_{\mathbb{Z}} & \end{bmatrix} \mapsto \hat{\beta}(x) \cdot \begin{bmatrix} \mathcal{G} & \bullet \\ \mathbb{Z} & \end{bmatrix}$. In U , x is not productive, while y is productive if and only if y^\rightarrow does not contain a label of the same family as \mathcal{F} . In T_1 , x is productive, but y is not. In U , y is productive if and only if $(\mathcal{F}) \in \bar{y}$ in T_1 . If y is productive in U , then $\text{boxfactor}_U(y) = \text{virtualfactor}_{\bar{y} \in T_1}((\mathcal{F})) := a$.

For every edge label $\ell \in Z$ in U , there is a unique $\ell_1 \in \mathcal{F} \cup Z$ in T_1 with $\text{family}(\ell_1) = \text{family}(\ell) - 1$. Furthermore, $\text{edgefactor}_U(\ell) = \text{edgefactor}_{T_1}(\ell_1) := b_\ell$. Let ℓ_1^M be greatest label of Z in T_1 , and let $\text{edgefactor}_{T_1}(\ell_1^M) := w$. Then, $\text{virtualfactor}_{y \in U}(\mathcal{G}) = -w$.

Let $\text{boxfactor}_{T_1}(x) := c$. Then, $\text{virtualfactor}_{x \in U}(\mathcal{G}) = \hat{\beta}(x)c$. Since T_1 has one more \mathcal{G} than U , $d(U) = d(T_1) - 1$.

In this case, (12.3) is equivalent to $\text{wt } U = \hat{\beta}(x) \text{wt } T_1$. This follows from

$$(-1)^{d(U)} \cdot \left(\prod_{\ell \in Z} b_\ell \right) \cdot (-w) \hat{\beta}(x)c \cdot a = \hat{\beta}(x) \cdot (-1)^{d(U)+1} \cdot w \left(\prod_{\ell \in Z} b_\ell \right) \cdot a \cdot c,$$

if y is productive in U . If it is not productive, we use the same identity without a .

Subcase 0.2.5.2 ($\text{body}(S) \neq \emptyset$): Locally, at S , the swap looks like

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \mathcal{F}^\dagger \\ \hline \bullet & \mathcal{G} \\ \hline \mathcal{G} & \\ \hline \end{array} \mapsto - \prod_{z \in \bullet} \hat{\beta}(z) \cdot \begin{array}{|c|c|} \hline & \bullet \\ \hline \mathcal{G} & \bullet \\ \hline \mathcal{G} & \\ \hline \bullet & \\ \hline \end{array} .$$

Let A be the set of boxes of S_1 containing \mathcal{G} . The productive boxes of S are $\{z^\downarrow : z \in A\}$, as well as perhaps y , which is productive if and only if y^\rightarrow contains a label of the same family as \mathcal{F} in U . The productive boxes of S_1 are A . For each $z \in A$, let $a_z := \text{boxfactor}_{T_1}(z)$. Then, $\text{boxfactor}_U(z^\downarrow) = \hat{\beta}(z)a_z$. If y is productive in U , let $b := \text{boxfactor}_U(y)$. Observe that y is productive in U if and only if $(\mathcal{F}) \in \bar{y}$ in T_1 . Furthermore, if y is productive in U , then $\text{virtualfactor}_{\bar{y} \in T_1}((\mathcal{F})) = b$.

For every edge label $\ell \in Z$ in U , there is a unique $\ell_1 \in \mathcal{F} \cup Z$ in T_1 with $\text{family}(\ell_1) = \text{family}(\ell) - 1$. Furthermore, $\text{edgefactor}_U(\ell) = \text{edgefactor}_{T_1}(\ell_1) := c_\ell$. Let ℓ_1^M be greatest label of Z in T_1 , and let $\text{edgefactor}_{T_1}(\ell_1^M) := w$. Then, $\text{virtualfactor}_{y \in U}(\mathcal{G}) = -w$.

In this case, (12.3) is equivalent to $\text{wt } U = -\prod_{z \in A} \hat{\beta}(z) \text{wt } T_1$. This follows from

$$\left(\prod_{\ell \in Z} c_\ell\right) \cdot (-w) \cdot b \prod_{z \in A} (\hat{\beta}(z) a_z) = \left(-\prod_{z \in A} \hat{\beta}(z)\right) \cdot w \left(\prod_{\ell \in Z} c_\ell\right) \cdot b \cdot \prod_{z \in A} a_z.$$

Subcase 0.2.6 (tail(S) is T6): This case is covered by Subcases 0.2.4.2 and 0.2.4.3.

Case 1 (head(S) is H1): Set $B_S = \{S\}$. Let x be the unique box of S . Locally, at S , the swap is $\begin{smallmatrix} \bullet \\ \mathcal{G} \end{smallmatrix} \mapsto \beta(x) \cdot \begin{smallmatrix} \mathcal{G} \\ \bullet \end{smallmatrix} + \gamma \cdot \begin{smallmatrix} \mathcal{G} \\ \mathcal{G} \end{smallmatrix}$, where $\gamma := 0$ if x is in row i and $\gamma := 1$ otherwise.

Let $\text{boxfactor}_{T_1}(x) := a$. The box x is productive in U if and only if $\text{family}(\text{label}(x^\leftarrow)) = \text{family}(\mathcal{G})$, in which case $\text{boxfactor}_U(x) = a$. Observe that $\text{edgefactor}_{x \in U}(\mathcal{G}) = 1 - \hat{\beta}(x)a$, $\text{edgefactor}_{x \in T_2}(\mathcal{G}) = 1 - a$ and x is not productive in T_2 . Notice further that $\gamma = 0$ if and only if $a = 1$.

If x^\leftarrow is empty, then x^\leftarrow is not productive in any of U, T_1, T_2 . If x^\leftarrow is nonempty and $\text{label}(x^\leftarrow) < \mathcal{G}$, then x^\leftarrow is productive in all three tableaux, and $\text{boxfactor}_U(x^\leftarrow) = \text{boxfactor}_{T_1}(x^\leftarrow) = \text{boxfactor}_{T_2}(x^\leftarrow) := b$. If x^\leftarrow is nonempty and $\text{family}(\text{label}(x^\leftarrow)) = \text{family}(\mathcal{G})$, then x^\leftarrow is productive in T_2 , but not in U or T_1 . Moreover, by (G.6), $\mathcal{G}^- \in x^\leftarrow$, so $\text{boxfactor}_{T_2}(x^\leftarrow) = \hat{\beta}(x)a$.

In this case, (12.3) is equivalent to $\text{wt } U = \beta(x) \text{wt } T_1 + \gamma \text{wt } T_2$. Since $\gamma = 1$ whenever $\text{wt } T_2 \neq 0$, it suffices to show that $\text{wt } U = \beta(x) \text{wt } T_1 + \text{wt } T_2$. If x^\leftarrow is nonempty and $\text{label}(x^\leftarrow) < \mathcal{G}$, this follows from

$$(1 - \hat{\beta}(x)a) \cdot b = \beta(x) \cdot ab + (1 - a) \cdot b.$$

If x^\leftarrow is empty, we use the same identity without b . If $\text{family}(\text{label}(x^\leftarrow)) = \text{family}(\mathcal{G})$, we use the identity

$$(1 - \hat{\beta}(x)a) \cdot a = \beta(x) \cdot a + (1 - a) \cdot \hat{\beta}(x)a.$$

Case 2 (head(S) is H2): Set $B_S = \{S\}$. Let x be the unique box of S . Locally, at S , $\begin{smallmatrix} \bullet \\ \mathcal{G} \end{smallmatrix} \mapsto \begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix} + \beta(x) \cdot \begin{smallmatrix} \mathcal{G} \\ \bullet \end{smallmatrix}$.

Let $\text{boxfactor}_{T_2}(\mathbf{x}) := a$. Then, $\text{virtualfactor}_{\mathbf{x} \in U}(\mathcal{G}) := a\hat{\beta}(\mathbf{x})$. In T_1 , $(\mathcal{G}) \in \bar{\mathbf{x}}$ with $\text{virtualfactor}_{\mathbf{x} \in T_1}(\mathcal{G}) = a$. Due to T_2 's extra $\mathcal{G} \in \mathbf{x}$, $d(T_2) = d(U) + 1 = d(T_1) + 1$.

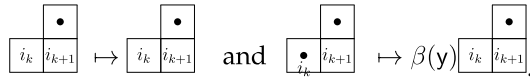
In this case, (12.3) is equivalent to $\text{wt } U = \text{wt } T_1 + \beta(\mathbf{x}) \text{wt } T_2$. This follows from

$$(-1)^{d(U)} \cdot a\hat{\beta}(\mathbf{x}) = (-1)^{d(U)} \cdot a + \beta(\mathbf{x}) \cdot (-1)^{d(U)+1} \cdot a.$$

Case 3 ($\text{head}(S)$ is H3): Here, $S = \{\mathbf{x}\}$. Let $\mathbf{y} = \mathbf{x}^{\downarrow\leftarrow}$. Locally, at S , $\begin{matrix} \square \\ \bullet_{i_k} \end{matrix} \mapsto \begin{matrix} \square \\ \bullet_{i_{k+1}} \end{matrix}$.

Subcase 3.1 ($i_{k+1} \in \mathbf{x}^{\downarrow} = \mathbf{y}^{\rightarrow}$, $i_k \in \mathbf{y}$, no \bullet West of \mathbf{y} in the same row): In U , the \mathbf{y} is not productive, whereas, in T_1 , \mathbf{y} is productive. Let $a := \text{boxfactor}_{T_1}(\mathbf{y})$.

Subcase 3.1.1 (\mathbf{y} contains the only i_k in T): Let $\bar{T} := \phi_2^{-1}(T)$. Let S' be the snake in T containing \mathbf{y} , \bar{S} be the snake in \bar{T} containing \mathbf{x} , and \bar{S}' be the snake in \bar{T} containing \mathbf{y} . Set $B_S = \{S, S', \bar{S}, \bar{S}'\}$. Locally, at $S \cup S'$ and $\bar{S} \cup \bar{S}'$, the swaps are, respectively,



where $\bullet = \bullet_{i_k}$ before the swap and $\bullet = \bullet_{i_{k+1}}$ after the swap.

By Proposition 12.12, $[T]P_{i_k} = -[\bar{T}]P_{i_k}$. Hence, (12.3) is equivalent to the following.

CLAIM C.1. We have $\text{wt}(U - \bar{U}) = \text{wt}(T_1 - \beta(\mathbf{y})T_1)$.

Proof. First, \mathbf{x} is not productive in U , \bar{U} or T_1 . Second, \mathbf{x}^{\downarrow} is productive in U , \bar{U} and T_1 , and, moreover,

$$\text{boxfactor}_U(\mathbf{x}^{\downarrow}) = \text{boxfactor}_{\bar{U}}(\mathbf{x}^{\downarrow}) = \text{boxfactor}_{T_1}(\mathbf{x}^{\downarrow}) := b.$$

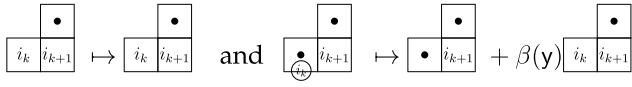
Third, \mathbf{y} is not productive in \bar{U} .

Next, $\text{edgefactor}_{\mathbf{y} \in \bar{U}}(i_k) = 1 - a\hat{\beta}(\mathbf{y})$. Hence, the claim follows from

$$b - (1 - a\hat{\beta}(\mathbf{y})) \cdot b = \hat{\beta}(\mathbf{y}) \cdot ab = (1 - \beta(\mathbf{y}))\text{wt}(T_1). \quad \square$$

Subcase 3.1.2 (Subcase 3.1.1 does not apply): Let $\bar{T} := \phi_{1,k}^{-1}(T)$. Let S' be the snake in T containing \mathbf{y} , \bar{S} be the snake in \bar{T} containing \mathbf{x} and \bar{S}' be the snake

in \overline{T} containing y . Set $B_S = \{S, S', \overline{S}, \overline{S'}\}$. Locally, at $S \cup S'$ and $\overline{S} \cup \overline{S'}$, the swaps are, respectively,



where $\bullet = \bullet_{i_k}$ before the swap and $\bullet = \bullet_{i_{k+1}}$ after the swap. It should be noted that $T_1 = \overline{T}_2$.

By Proposition 12.8, $[T]P_{i_k} = -[\overline{T}]P_{i_k}$. Thus, (12.3) is equivalent to the following.

CLAIM C.2. We have $\text{wt } U - \text{wt } \overline{U} = \text{wt } T_1 - \text{wt } \overline{T}_1 - \beta(y) \text{wt } T_1$.

Proof. First, x is not productive in U, \overline{U}, T_1 or \overline{T}_1 . Second, x^\downarrow is productive in U, \overline{U}, T_1 and \overline{T}_1 . Moreover,

$$\begin{aligned} \text{boxfactor}_U(x^\downarrow) &= \text{boxfactor}_{\overline{U}}(x^\downarrow) \\ &= \text{boxfactor}_{T_1}(x^\downarrow) = \text{boxfactor}_{\overline{T}_1}(x^\downarrow) := b. \end{aligned}$$

It should be noted that y is not productive in U, \overline{U} or \overline{T}_1 . Observe that $\text{virtualfactor}_{y \in \overline{U}}(i_k) = a\hat{\beta}(y)$. Finally, $d(U) = d(T_1) = d(\overline{U}) + 1 = d(\overline{T}_1) + 1$. The claim then follows from

$$\begin{aligned} &(-1)^{d(U)} \cdot b - (-1)^{d(U)-1} \cdot a\hat{\beta}(y) \cdot b \\ &= (-1)^{d(U)} \cdot ab - (-1)^{d(U)-1} \cdot b - \beta(y) \cdot (-1)^{d(U)} \cdot ab. \quad \square \end{aligned}$$

Subcase 3.2 (Subcase 3.1 does not apply):

Subcase 3.2.1 (x^\downarrow is part of a T3 tail): Let S' be the snake containing x^\downarrow , and let $B_S = B_{S'}$. The remaining discussion of this case is found with the discussion of S' ; see, for example, Subcase 0.2.3.

Subcase 3.2.2 ($i_{k+1} \in x^\downarrow, \bullet_{i_k} \in y, i_k \in y$): Let $\overline{T} := \phi_2(T)$. Let S' be the snake in T containing y , \overline{S} be the snake in \overline{T} containing x and $\overline{S'}$ be the snake in \overline{T} containing y . Set $B_S = \{S, S', \overline{S}, \overline{S'}\}$. The remaining discussion of this case is found with the discussion of \overline{S} in Subcase 3.1.1.

Subcase 3.2.3 ($i_{k+1} \in x^\downarrow, \bullet_{i_k} \in y, \textcircled{i_k} \in y$): Let $\overline{T} := \phi_{1,k}(T)$. Let S' be the snake in T containing y , \overline{S} be the snake in \overline{T} containing x and $\overline{S'}$ be the snake in \overline{T} containing y . Set $B_S = \{S, S', \overline{S}, \overline{S'}\}$. The remaining discussion of this case is found with the discussion of \overline{S} in Subcase 3.1.2.

Subcase 3.2.4 (Subcases 3.2.1, 3.2.2 and 3.2.3 do not apply): Set $\mathcal{B}_S = \{S\}$. Recall that, locally, at S , $\begin{smallmatrix} \bullet \\ \square \\ \bullet \end{smallmatrix}_{i_k} \mapsto \begin{smallmatrix} \bullet \\ \square \\ \bullet \end{smallmatrix}_{i_{k+1}}$. The swap affects no weight factors.

Case 4 ($\text{head}(S)$ is H4): We argue the case $S = \text{head}(S)$. When S is a multirow, weight preservation follows by combining the present argument with that of **Case 0.2**.

Finally, we assume $S = \{x, x^{\rightarrow}\}$. Let $\overline{T} = \phi_{3, \{x, x^{\rightarrow}\}}(T)$, and let \overline{S} be the snake containing x in \overline{T} . Set $\mathcal{B}_S = \{S, \overline{S}\}$. Locally, at S, \overline{S} , respectively, the swaps are

$$\begin{smallmatrix} \bullet \\ \mathcal{G} \\ \bullet \end{smallmatrix} \mapsto 0 \quad \text{and} \quad \begin{smallmatrix} \bullet \\ \square \\ \bullet \end{smallmatrix} \mapsto \hat{\beta}(x) \begin{smallmatrix} \mathcal{G} \\ \square \\ \bullet \end{smallmatrix}.$$

Since, by Proposition 12.17, $[T]P_{\mathcal{G}} = [\overline{T}]P_{\mathcal{G}}$, (12.3) is equivalent to the following.

CLAIM C.3. *We have $\text{wt } U + \text{wt } \overline{U} = \text{wt } \overline{T}_1$.*

Proof. In U and \overline{U} , x is not productive. However, x is productive in \overline{T}_1 . Let $\text{boxfactor}_{\overline{T}_1}(x) := a$. In \overline{T}_1 , x^{\rightarrow} is not productive. In U , x^{\rightarrow} is productive if and only if x^{\rightarrow} does not contain a label of family i . Further, x^{\rightarrow} is productive in U if and only if it is productive in \overline{U} and if and only if $(\mathcal{G}) \in \overline{x^{\rightarrow}}$ in \overline{T}_1 . In this case, $\text{boxfactor}_U(x^{\rightarrow}) = \text{boxfactor}_{\overline{U}}(x^{\rightarrow}) = \text{virtualfactor}_{\overline{x^{\rightarrow}} \in \overline{T}_1}(\mathcal{G}) := b$.

In T , $\text{edgefactor}(x) = 1 - \hat{\beta}(x)a$. Finally, $d(U) = d(\overline{U}) + 1 = d(\overline{T}_1) + 1$. If x^{\rightarrow} is productive in U , then the claim follows from

$$(-1)^{d(U)} \cdot (1 - \hat{\beta}(x)a) \cdot b + (-1)^{d(U)-1} \cdot b = \hat{\beta}(x)(-1)^{d(U)-1} \cdot ab.$$

Otherwise, we use the same identity with bs removed. □

Case 5 ($\text{head}(S)$ is H5): We only explicitly argue the case $S = \text{head}(S)$. The case where S is a multirow ribbon follows by combining the present arguments with those from **Case 0.2**. Hence, $S = \text{head}(S) = \{x, y := x^{\rightarrow}\}$.

Subcase 5.1 ($\text{head}(S)$ is H5.1): Let $\mathcal{B}_S = \{S\}$. Locally, at S , the swap is $\begin{smallmatrix} \bullet \\ \mathcal{G} \\ \mathcal{H} \end{smallmatrix} \mapsto \begin{smallmatrix} \bullet \\ \mathcal{G}' \\ \mathcal{H} \end{smallmatrix}$, where $\mathcal{H} \in y$, $\text{family}(\mathcal{H}) = \text{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$. Since no labels move, (12.3) follows trivially.

Subcase 5.2 ($\text{head}(S)$ is H5.2): Locally, at S , the swap is $\begin{smallmatrix} \bullet \\ \mathcal{G} \\ \mathcal{H} \end{smallmatrix} \mapsto \begin{smallmatrix} \bullet \\ \mathcal{G}' \\ \mathcal{H} \end{smallmatrix} + \hat{\beta}(x) \cdot \begin{smallmatrix} \mathcal{G} \\ \square \\ \bullet \end{smallmatrix}$. By Lemma 5.6, y contains the westmost instance of \mathcal{G} and hence $(\mathcal{G}) \notin x$.

Subcase 5.2.1 ($\mathcal{G}^- \in x^\leftarrow$ with $\text{family}(\mathcal{G}^-) = i$): Let $\mathcal{B}_S = \{S\}$. Now, y is productive in U if and only if it is productive in T_1 and if and only if $(\mathcal{G}) \in \bar{y}$ in T_2 . In this case,

$$\text{boxfactor}_U(y) = \text{boxfactor}_{T_1}(y) = \text{virtualfactor}_{\bar{y} \in T_2}((\mathcal{G})) := a.$$

Let $\text{virtualfactor}_{y \in U}(\mathcal{H}) := b$. The box x is productive only in T_2 , with $\text{boxfactor}_{T_2}(x) = \hat{b}$. The box x^\leftarrow is productive in U and T_1 ; x^\leftarrow is not productive in T_2 . Furthermore, $\text{boxfactor}_U(x^\leftarrow) = \text{boxfactor}_{T_1}(x^\leftarrow) = \hat{\beta}(x)b$. Observe that $\text{virtualfactor}_{y \in T_1}(\mathcal{H}) = b - 1$. In this case, (12.3) is equivalent to $\text{wt } U = \text{wt } T_1 + \hat{\beta}(x) \text{wt } T_2$. If y is productive in U , this follows from the identity

$$b \cdot a \hat{\beta}(x)b = (b - 1) \cdot a \hat{\beta}(x)b + \hat{\beta}(x) \cdot a \cdot b.$$

If y is not productive in U , we use the same identity without a .

Subcase 5.2.2 ($\mathcal{G}^- \notin x^\leftarrow$ or $\text{family}(\mathcal{G}^-) \neq i$): Observe that $T \in S'_3$. Let $\bar{T} = \phi_3^{-1}(T)$. Note that $(\mathcal{H}) \notin \bar{y}$ in \bar{T} . Let \bar{S} be the snake of \bar{T} containing x . Set $\mathcal{B}_S = \{S, \bar{S}\}$. The swap at S is illustrated at the start of Case 5.2 above. Locally, at \bar{S} , the swap is $\begin{bmatrix} \bullet & \mathcal{G} \end{bmatrix} \mapsto 0$.

Observe that y is productive in U if and only if it is productive in \bar{U} and if and only if it is productive in T_1 ; y is not productive in T_2 . Moreover, if y is productive in U , then $\text{boxfactor}_U(y) = \text{boxfactor}_{\bar{U}}(y) = \text{boxfactor}_{T_1}(y) := a$. There is $(\mathcal{G}) \in \bar{y}$ in T_2 if and only if y is productive in U . In this case, $\text{virtualfactor}_{\bar{y} \in T_2}((\mathcal{G})) = a$.

Let $b := \text{virtualfactor}_{y \in U}(\mathcal{H})$ and $1 - c := \text{edgefactor}_{x \in \bar{U}}(\mathcal{G})$. Consequently, we have $\text{virtualfactor}_{y \in T_1}(\mathcal{H}) = b - 1$, while $\text{boxfactor}_{T_2}(x) = c/\hat{\beta}(x)$. Observe that $d(U) = d(T_1) = d(T_2) = d(\bar{U}) - 1$. By Proposition 12.17, $[T]P_{\mathcal{G}} = [\bar{T}]P_{\mathcal{G}}$, so (12.3) is equivalent to $\text{wt } U + \text{wt } \bar{U} = \text{wt } T_1 + \hat{\beta}(x) \text{wt } T_2$. If y is productive in U , this follows from

$$(-1)^{d(U)} \cdot b \cdot a + (-1)^{d(U)-1} \cdot (1 - c) \cdot a = (-1)^{d(U)} \cdot (b - 1) \cdot a + \hat{\beta}(x) \cdot (-1)^{d(U)} \cdot a \cdot \frac{c}{\hat{\beta}(x)}.$$

Otherwise, it follows from the same identity without as .

Case 5.3 ($\text{head}(S)$ is H5.3):

Subcase 5.3.1 ($(\mathcal{G}) \in x$): Set $\mathcal{B}_S = \{S\}$. Let $S = \{x, y := x^\rightarrow\}$. Locally, at S , the swap is $\begin{bmatrix} \bullet & \mathcal{G} \end{bmatrix} \mapsto \hat{\beta}(x) \begin{bmatrix} \mathcal{G} & \bullet \end{bmatrix}$. Let $a := \text{virtualfactor}_{x \in U}((\mathcal{G}))$. In U ,

x is not productive, while y is productive if and only if y^\rightarrow does not contain a label of family i . Further, y is productive in U if and only if $(\mathcal{G}) \in \bar{y}$ in T_1 . If y is productive in U , then $\text{boxfactor}_U(y) = \text{virtualfactor}_{\bar{y} \in T_1}(\mathcal{G}) := b$. In T_1 , x is productive, but y is not; $\text{boxfactor}_{T_1}(x) = a/\hat{\beta}(x)$. Here, (12.3) is equivalent to $\text{wt } U = \hat{\beta}(x) \text{wt } T_1$. If y is productive in U , this follows from $a \cdot b = \hat{\beta}(x) \cdot b \cdot a/\hat{\beta}(x)$. Otherwise, we use the same identity without b .

Subcase 5.3.2 ($\mathcal{G}^- \in x^\leftarrow$ with $\text{family}(\mathcal{G}^-) = i$): By (G.12) and Lemma 5.10, no label of family i appears in x 's column. Hence, the $\mathcal{G} \in y$ is the Westmost \mathcal{G} . In particular, $(\mathcal{G}) \notin x$, so this case is disjoint from Subcase 5.3.1. Set $B_S = \{S\}$.

Locally, at S , the swap is $\begin{bmatrix} \bullet & \mathcal{G} \end{bmatrix} \mapsto \hat{\beta}(x) \begin{bmatrix} \mathcal{G} & \bullet \end{bmatrix}$.

Let $a := \text{boxfactor}_U(x^\leftarrow)$. In U , x is not productive, while y is productive if and only if y^\rightarrow does not contain a label of family i if and only if $(\mathcal{G}) \in \bar{y}$ in T_1 . In this case, $\text{boxfactor}_U(y) = \text{virtualfactor}_{\bar{y} \in T_1}(\mathcal{G}) := b$. In T_1 , x^\leftarrow and y are not productive, while x is productive and $\text{boxfactor}_{T_1}(x) = a/\hat{\beta}(x)$. Here, (12.3) is equivalent to $\text{wt}(U) = \hat{\beta}(x)\text{wt}(T_1)$. If $y \in U$ is productive, then this follows from $ab = \hat{\beta}(x) \cdot b \cdot a/\hat{\beta}(x)$; otherwise, the same is true after removing the bs .

Subcase 5.3.3 (Subcases 5.3.1 and 5.3.2 do not apply): There is no $\mathcal{G}^- \in x^\leftarrow$ because we are not in Subcase 5.3.2. Since we are not in Case 5.1, there is not $\mathcal{H} \in \bar{y}$ with $\text{family}(\mathcal{H}) = i + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$. By the assumption that we are in Case 5, there can be no label of $\text{family}(\mathcal{G})$ in the column of x . Thus, if the $\mathcal{G} \in y$ were not Westmost, then $(\mathcal{G}) \in x$, and we would be in Subcase 5.3.1, which would be a contradiction. Therefore, we conclude that $T \in \mathcal{S}'_3$. Let $\bar{T} := \phi_3^{-1}(T)$. Let \bar{S} be the snake in \bar{T} containing x . Then, set $B_S = \{S, \bar{S}\}$.

Locally, at S , we have that the swap is $\begin{bmatrix} \bullet & \mathcal{G} \end{bmatrix} \mapsto \hat{\beta}(x) \begin{bmatrix} \mathcal{G} & \bullet \end{bmatrix}$. Locally, at \bar{S} , the swap is $\begin{bmatrix} \bullet & \mathcal{G} \end{bmatrix} \mapsto 0$. By Proposition 12.17, $[T]P_k = [\bar{T}]P_k$. Therefore, (12.3) is equivalent to $\text{wt}(U) + \text{wt}(\bar{U}) = \hat{\beta}(x)\text{wt}(T_1)$. This is exactly proved (up to renaming of tableaux) in Case 4.

Case 6 ($\text{head}(S)$ is H6): Here, $S = \{x, x^\rightarrow\}$.

Subcase 6.1 ($\bullet_{\mathcal{G}} \notin x^{\rightarrow\uparrow}$): Let $B_S = \{S\}$. Locally, at S , $\begin{bmatrix} \bullet_{\mathcal{G}} & \mathcal{G}^+ \end{bmatrix} \mapsto \beta(x) \cdot \begin{bmatrix} \mathcal{G} & \mathcal{G}^+ \end{bmatrix} + \hat{\beta}(x) \cdot \begin{bmatrix} \mathcal{G} & \bullet_{\mathcal{G}^+} \end{bmatrix}$.

In U , T_1 and T_2 , x is not productive. Now, x^\rightarrow is productive in T_1 if and only if it is productive in T_2 if and only if it is productive in U . In the case of productivity,

$$a := \text{boxfactor}_U(x^\rightarrow) = \text{boxfactor}_{T_1}(x^\rightarrow) = \text{boxfactor}_{T_2}(x^\rightarrow).$$

Next, let $\text{edgefactor}_{x \in U}(\mathcal{G}) := 1 - b$. Thus, $\text{edgefactor}_{x^\rightarrow \in T_2}(\mathcal{G}^+) = 1 - b/\hat{\beta}(x)$.

Finally, (12.3) is equivalent to $\text{wt}(U) = \beta(x)\text{wt}(T_1) + \hat{\beta}(x)\text{wt}(T_2)$. If x^\rightarrow is productive in U , then this follows from

$$(1 - b) \cdot a = \beta(x) \cdot a + \hat{\beta}(x) \cdot (1 - b/\hat{\beta}(x)) \cdot a;$$

otherwise, we are done by the same expression without the as .

Subcase 6.2 ($\bullet_{\mathcal{G}} \in x^{\rightarrow\uparrow}$): Thus, $T \in \mathcal{S}_2$. Let $\bar{T} := \phi_2(T)$. Let \bar{S} be the snake of \bar{T} containing x . Let S' and \bar{S}' be the snakes of T and \bar{T} containing $x^{\rightarrow\uparrow}$, respectively. Set $\mathcal{B}_S = \{S, S', \bar{S}, \bar{S}'\}$.

Notice that S' falls into **Case 3** and, in fact, the $\mathcal{B}_{S'}$ defined there equals the current \mathcal{B}_S . Hence, (12.3) holds by **Case 3**.

Case 7 (head(S) is H7): Here, $S = \{x, x^{\rightarrow\uparrow}\}$.

Subcase 7.1 ($\bullet_{\mathcal{G}} \notin x^{\rightarrow\uparrow}$): Let $\mathcal{B}_S = \{S\}$. Locally, at S ,

$$\begin{array}{|c|c|} \hline \bullet & \mathcal{G}^+ \\ \hline \oplus & \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline \bullet & \mathcal{G}^+ \\ \hline \end{array} + \beta(x) \cdot \begin{array}{|c|c|} \hline \mathcal{G} & \mathcal{G}^+ \\ \hline \end{array} + \hat{\beta}(x) \cdot \begin{array}{|c|c|} \hline \mathcal{G} & \bullet \\ \hline \mathcal{G}^+ & \\ \hline \end{array}.$$

First, x is not productive in U , T_1 , T_2 or T_3 , whereas x^\rightarrow is productive in each of U , T_1 , T_2 and T_3 or otherwise not productive in any of these tableaux. If x^\rightarrow is productive, then

$$a := \text{boxfactor}_U(x^\rightarrow) = \text{boxfactor}_{T_j}(x^\rightarrow) \quad \text{for } j = 1, 2, 3.$$

Second, let $b := \text{virtualfactor}_{x \in U}(\mathcal{G})$. Third, $\text{edgefactor}_{x^\rightarrow \in T_3}(\mathcal{G}^+) = 1 - b/\hat{\beta}(x)$. Fourth, $d(T_1) = d(U)$, $d(T_2) = d(T_3) = d(U) + 1$.

Here, (12.3) is equivalent to $\text{wt}(U) = \text{wt}(T_1) + \beta(x)\text{wt}(T_2) + \hat{\beta}(x)\text{wt}(T_3)$. If x^\rightarrow is productive in U , then this follows from

$$(-1)^{d(U)} b \cdot a = (-1)^{d(U)} a + \beta(x)(-1)^{d(U)+1} a + \hat{\beta}(x)(-1)^{d(U)+1} \left(1 - \frac{b}{\hat{\beta}(x)}\right) \cdot a;$$

otherwise, we are done by the same identity without the as .

Subcase 7.2 ($\bullet_{\mathcal{G}} \in x^{\rightarrow\uparrow}$): Thus, $T \in \mathcal{S}_1(k)$. Let $\bar{T} := \phi_{1,k}(T)$. Let \bar{S} be the snake of \bar{T} containing x . Let S' and \bar{S}' be the snakes of T and \bar{T} containing $x^{\rightarrow\uparrow}$, respectively. Set $\mathcal{B}_S = \{S, S', \bar{S}, \bar{S}'\}$.

Notice that S' falls into **Case 3** and, in fact, the $\mathcal{B}_{S'}$ defined there equals the current \mathcal{B}_S . Hence, (12.3) holds by **Case 3**.

Case 8 ($\text{head}(S)$ is H8): Here, $\text{head}(S) = S$. The definitions of \mathcal{B}_S and subsequent analysis are exactly the same as in **Case 3**.

Case 9 ($\text{head}(S)$ is H9): Let $\mathcal{B}_S = \{S\}$. If $\text{head}(S) = S$, then, since swapping at S does nothing (including no change to any \bullet indices), (12.3) is trivially true. If $\text{head}(S) \neq S$, then we use an argument similar to **Subcase 0.2**.

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