

GAP TAUBERIAN THEOREMS

JEFF CONNOR

In the first section we establish a connection between gap Tauberian conditions and isomorphic copies of c_0 for perfect coregular conservative BK spaces and in the second we give a characterisation of gap Tauberian conditions for strong summability with respect to a nonnegative regular summability matrix. These results are used to show that a gap Tauberian condition for strong weighted mean summability is also a gap Tauberian condition for ordinary weighted mean summability. We also make a remark regarding the support set of a matrix and give a Tauberian theorem for a class of conull spaces.

In this note we make some remarks regarding gap Tauberian theorems in the context of some settings associated with strong and ordinary summability. In the first part of this paper we provide an interpretation that leads to the construction of isomorphic copies of c_0 and establish a partial converse. In the second we characterise the gap Tauberian conditions for strong summability with respect to a nonnegative regular summability matrix. In a later section we restate this characterisation in terms of the support set of the matrix. We also apply these results to weighted mean summability methods and, as a bonus, we establish an analog of a gap Tauberian theorem for a large collection of conull spaces.

We use the standard notation of summability and Banach space theory as it is found in [4] and [13]. Let $\omega = \{\text{all scalar sequences}\}$, $\varphi = \{x \in \omega : x \text{ is finitely nonzero}\}$ and let c_0 , c , and l_∞ denote the vector subspaces of ω consisting of, respectively, the null sequences, the convergent sequences and the bounded sequences. We also let $e^k =$ the k th unit vector, $e = (1, 1, 1, \dots)$, and, if E is a Banach space, we let E' denote its continuous dual. If S is a subset of a Banach space E , we let $[S]_E$ denote the closed linear span of S in E . A *BK space* is a Banach space which is a vector subspace of ω that has the property the inclusion map from E into ω is continuous when ω is given the topology of coordinatewise convergence. A BK space E is said to be *conservative* if $c \subset E$ and is said to be *perfect* if $[c]_E = E$. We recall that if E and F are BK spaces and E is contained in F , then the inclusion map from E into F is continuous.

Received 7th May, 1992.

Research supported by an Ohio University Research Council Grant.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/93 \$A2.00+0.00.

An infinite array of scalars $T = (t_{n,k})_{n,k=1,\infty}$ is called a *regular summability matrix* provided $\lim_n \sum_k t_{n,k} x_k = \lim_k x_k$ whenever $x = (x_k) \in c$. The matrix T is called a *triangle* if $t_{n,n} \neq 0$ for all $n \in \mathbb{N}$ and $n < k$ implies $t_{n,k} = 0$. If $x \in \omega$ and $\sum_k t_{n,k} x_k$ exists for all $n \in \mathbb{N}$, we let Tx denote the sequence $(\sum_k t_{n,k} x_k : n \in \mathbb{N})$. Let $c_T = \{x \in \omega : Tx \in c\}$. c_T is called the *convergence domain* of T and if $x \in c_T$ we say that x is *T-summable* and set $T - \lim x = \lim_n \sum_k t_{n,k} x_k$. It is well-known that if T is a triangle, then c_T is a BK space with the norm $\|x\|_T = \sup_n \left| \sum_k t_{n,k} x_k \right| = \|Tx\|_\infty$.

Throughout this note we let $\gamma : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ denote an increasing function with $\gamma(0) = 0$ and set

$$G(\gamma) = \{x \in \omega : x_k - x_{k+1} \neq 0 \text{ implies there exists } \tau \in \mathbb{N} \text{ such that } k = \gamma(\tau)\}.$$

A gap Tauberian theorem for a matrix summability method T generally asserts that if γ satisfies some condition depending on T , $x \in G(\gamma)$ and $Tx \in c$ then x is convergent. With this in mind, we say that, for a sequence space E , $G(\gamma)$ is a *gap Tauberian condition for E* if $E \cap G(\gamma) \subset c$. For a given γ , we set $b^r(\gamma) = \sum_{i>\gamma(r)}^{\gamma(r+1)} e^i$.

1. GAP TAUBERIAN CONDITIONS AND ISOMORPHIC COPIES OF c_0

The results in this section follow from the fundamental properties of BK spaces and the Bessaga-Pelczynski characterisation of Banach spaces containing isomorphic copies of c_0 .

THEOREM 1.1. *Let E be a conservative BK space and γ an increasing sequence of integers with $\gamma(0) = 0$. If $G(\gamma)$ is a gap Tauberian condition for E and $\langle b^r(\gamma) \rangle$ is basic in E , then $\langle b^r(\gamma) \rangle$ is equivalent to c_0 's unit vector basis and the norm of E and c_0 are equivalent on $Z = [\langle b^r(\gamma) \rangle]_E \subset c_0$.*

PROOF: Let $b^r = b^r(\gamma)$. First note that $\inf_r \|b^r\|_E > 0$. If not, then there is a subsequence $\langle b^{r_l} \rangle$ of $\langle b^r \rangle$ such that $\|b^{r_l}\|_E < 2^{-l}$ for each $l \in \mathbb{N}$. Now $x = \sum_l b^{r_l} \in E$ and it can be arranged for x to have both 0 and 1 as limit points. As $x \in G(\gamma) \cap E$ and $x \notin c$, $G(\gamma)$ is not a gap Tauberian condition for E .

Observe that, since $c_0 \subset E$ and the inclusion map is continuous, $\sum_r |f(b^r)| < \infty$ for all $f \in E'$. Since $\langle b^r \rangle$ is basic in E , $\langle b^r \rangle$ is equivalent to c_0 's unit vector basis [4, p.45].

The remainder of the claim follows from noting that $Z \subset c_0$ and that the inclusion maps from Z into c_0 and from c_0 into E are each continuous. □

The converse of the preceding proposition is obviously false: If $E = l_\infty$ and $\gamma(r) = r$ then $\langle b^r(\gamma) \rangle$ is basic, $Z = c_0$ and $G(\gamma) \cap l_\infty = l_\infty$. We can, however, establish a partial converse by making suitable restrictions on E . First, in order to avoid our counterexample, we assume that E is perfect. Also recall that if $G(\gamma)$ is a gap Tauberian condition for a conservative BK space E , then E must be coregular, that is, there must be an $f \in E'$ such that $f(e) \neq \sum_k f(e^k)$ [3].

A BK space E is said to have a *monotone norm* if, for all $x = \langle x_k \rangle \in E$ and $n \in \mathbb{N}$, $\left\| \sum_{k=1}^n x_k e^k \right\|_E \leq \left\| \sum_{k=1}^{n+1} x_k e^k \right\|_E$ and $\|x\|_E = \sup_n \left\| \sum_{k=1}^n x_k e^k \right\|_E$. Observe that if E has a monotone norm, then $\langle b^r(\gamma) \rangle$ is basic for any γ [4, p.36] and $[\varphi \cap G(\gamma)]_E = [\varphi]_E \cap G(\gamma)$.

THEOREM 1.2. *Let E be a perfect coregular BK space with a monotone norm. Then $G(\gamma)$ is a gap Tauberian condition for E if and only if $\langle b^r(\gamma) \rangle$ is equivalent to c_0 's unit vector basis.*

PROOF: Since E has a monotone norm, we have that $\langle b^r(\gamma) \rangle$ is basic in E and hence, if $G(\gamma)$ is a gap Tauberian condition for E , the preceding result yields that $\langle b^r(\gamma) \rangle$ is equivalent to c_0 's unit vector basis.

Now suppose that $\langle b^r(\gamma) \rangle$ is equivalent to c_0 's unit vector basis. Note that if $w \in G(\gamma) \cap [e^k]_E = [b^r(\gamma)]_E$, then there is a $z = \langle z_r \rangle \in c_0$ such that $w = \sum_r z_r b^r$ and hence $w \in c_0$.

Next note that, since E is coregular and $\langle e^k \rangle$ is basic in E , $e \notin [\varphi]_E$. Since E is perfect, it follows that if $x \in E$, then there is a scalar λ_x and a $w_x \in [\varphi]_E$ such that $x = \lambda_x e + w_x$. If $x \in G(\gamma) \cap E$, then $w_x \in G(\gamma) \cap [\varphi]_E$ and hence x is convergent to λ_x . □

COROLLARY 1.3. *Every perfect coregular conservative BK space with a monotone norm has a gap Tauberian condition.*

PROOF: Observe that, since E is coregular, $\sum_k e^k$ does not converge in E . It follows that there is a γ and a $\delta > 0$ such that

$$\left\| \sum_{k>\gamma(r)}^{\gamma(r+1)} e^k \right\| > \delta$$

for all $r \in \mathbb{N}$. Now, since E is conservative, $\sum_r |f(b^r(\gamma))| < \infty$ for all $f \in E'$. This is enough to insure that $\langle b^r(\gamma) \rangle$ is equivalent to c_0 's unit vector basis and hence $G(\gamma)$ is a gap Tauberian condition for E . □

COROLLARY 1.4. *Let T be a triangular regular matrix summability matrix such that c_T is a BK space with a monotone norm. Then there is a subspace W of c_0 such that T restricted to W is an isomorphism onto its range.*

PROOF: Since T is a regular triangle, Fridy [6] has shown that there is a γ such that $G(\gamma)$ is a gap Tauberian condition for c_T . Now, since $\langle b^r(\gamma) \rangle$ is equivalent to c_0 's unit vector basis in c_T , $\langle Tb^r(\gamma) \rangle$ is equivalent to c_0 's unit vector basis in c_0 . Let $W = [\langle b^r(\gamma) \rangle]_{c_0}$. Since $\langle b^r(\gamma) \rangle$ and $\langle Tb^r(\gamma) \rangle$ are both equivalent to c_0 's unit vector basis, T acts as an isomorphism from W onto $T(W)$. □

2. GAP TAUBERIAN THEOREMS FOR STRONG SUMMABILITY

In this section we characterise gap Tauberian conditions for strong summability with respect to nonnegative regular summability matrices. Before giving our characterisation, we recall a few definitions and facts.

If T is a nonnegative regular summability method, $x \in \omega$ and λ is a scalar, we say that x is *strongly T -summable to λ* if $\lim_n \sum_k t_{n,k} |x_k - \lambda| = 0$. If $A \subset \mathbb{N}$, we let I_A denote the characteristic function of A . Finally we recall the following result [10, 4.1, 5].

THEOREM 2.1. *Let T be a nonnegative regular summability method. If x is strongly T -summable to λ , then there is a subset A of \mathbb{N} such that $T - \lim I_A = 1$ and $\lim_k (x_k - \lambda)I_A(k) = 0$.*

THEOREM 2.2. *Let T be a nonnegative regular summability method. Then the following statements are equivalent.*

- (1) $G(\gamma)$ is a gap Tauberian condition for strong T -summability.
- (2) For all $A \subset \mathbb{N}$ such that $T - \lim I_A = 1$, $(\gamma(r), \gamma(r + 1)) \cap A \neq \emptyset$ for all but finitely many $r \in \mathbb{N}$.
- (3) $\limsup_n \sum_r \sum_{k > \gamma(n_r)}^{\gamma(n_{r+1})} t_{n,k} > 0$ for all increasing subsequences $\langle n_r \rangle$ of natural numbers.

PROOF: (1) implies (2): Suppose that (2) fails and that we have found $A \subset \mathbb{N}$ such that $T - \lim I_A = 1$ and an increasing sequence $\langle n_r \rangle$ of natural numbers such that $A \cap (\gamma(n_r), \gamma(n_r + 1)) = \emptyset$ for all $r \in \mathbb{N}$. Set $B = \bigcup_{r \in \mathbb{N}} (\gamma(n_r), \gamma(n_r + 1))$ and let $x = I_B$. Since $B \subset \mathbb{N} - A$, x is strongly T -summable to 0 and has both 0 and 1 as limit points. Since $x \in G(\gamma)$, $G(\gamma)$ is not a gap-Tauberian condition for strong T -summability.

(2) implies (1): Now suppose that γ satisfies (2), x is strongly T -summable and $x \in G(\gamma)$. Without loss of generality, suppose that x is strongly T -summable to 0. Select $A \subset \mathbb{N}$ such that $T - \lim I_A = 1$ and $\lim_k x_k I_A(k) = 0$.

Let $\beta : \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary increasing function. Select an increasing sequence of integers $\langle n_l \rangle$ such that $(\gamma(n_l), \gamma(n_l + 1)) \cap \{\beta(r) : r \in \mathbb{N}\} \neq \emptyset$ and

$(\gamma(n_l), \gamma(n_l + 1)) \cap A \neq \emptyset$ for all $l \in \mathbb{N}$. Now select natural numbers $\theta(l)$ and $\rho(l)$ such that $\beta(\theta(l)) \in (\gamma(n_l), \gamma(n_l + 1))$ and $\rho(l) \in A \cap (\gamma(n_l), \gamma(n_l + 1))$ for all $l \in \mathbb{N}$. Note that, since $x \in G(\gamma)$, $x_{\beta(\theta(l))} = x_{\rho(l)}$ for all l and, since $\rho(l) \in A$, $\lim_l x_{\rho(l)} = 0$. Hence every subsequence of x has a subsequence which converges to 0 and thus x converges to 0.

The equivalence of (2) and (3) follows immediately from noting that, if $B = \bigcup_{r \in \mathbb{N}} (\gamma(n_r), \gamma(n_r + 1))$, then $T\text{-}\lim I_B = 0$ if and only if $\limsup_n \sum_r \sum_{k > \gamma(n_r)}^{\gamma(n_r + 1)} t_{n,k} = 0$. \square

3. AN APPLICATION TO WEIGHTED MEAN SUMMABILITY

In this section we apply the results of the preceding sections to weighted mean summability methods generated by positive sequences. Let $\langle p_k \rangle$ be a sequence of non-negative real numbers with $p_1 > 0$ and set $P_n = \sum_{k=1}^n p_k$. The weighted mean summability matrix (R, p) is defined by $(R, p)_{n,k} = p_k/P_n$ for $k \leq n$ and $(R, p)_{n,k} = 0$ otherwise.

Suppose that $p_k > 0$ for all k . Then (R, p) is a triangle and hence its convergence domain is can be regarded as a BK space where, if x is in the convergence domain of (R, p) , $\|x\| = \sup_n P_n^{-1} \left| \sum_{k=1}^n x_k p_k \right|$. This is a monotone norm: Since $\langle P_n \rangle$ is an increasing sequence, there is an $l \leq n$ such that $\left\| \sum_{k=1}^n x_k e^k \right\| = P_l^{-1} \left| \sum_{k=1}^l x_k p_k \right|$. It now follows from the definition of the norm that

$$\left\| \sum_{k=1}^{n+1} x_k e^k \right\| \geq P_l^{-1} \left| \sum_{k=1}^l x_k p_k \right| = \left\| \sum_{k=1}^n x_k e^k \right\|$$

and that $\|x\| = \sup_n \left\| \sum_{k=1}^n x_k e^k \right\|$. It is also straightforward to verify that (R, p) is a type-M summability method and hence, if (R, p) is regular, its convergence domain is a perfect coregular BK space [13, p.42].

THEOREM 3.1. *Let $p = \langle p_k \rangle$ be a sequence of positive numbers and suppose that (R, p) is a regular weighted mean summability matrix. The following statements are equivalent.*

- (1) $G(\gamma)$ is a gap Tauberian condition for (R, p) summability
- (2) $G(\gamma)$ is a gap Tauberian condition for strong (R, p) summability
- (3) $\inf_r P_{\gamma(r)}^{-1} \sum_{k > \gamma(r-1)}^{\gamma(r)} p_k > 0$.

PROOF: (1) implies (2) follows immediately from the observation that, for any nonnegative regular summability method T , if a sequence x is strongly T -summable to λ then $T - \lim x = \lambda$.

(2) implies (3): Suppose that $\inf_r P_{\gamma(r)}^{-1} \sum_{k>\gamma(r-1)}^{\gamma(r)} p_k = 0$. Let $\langle \delta_n \rangle$ be a sequence of positive numbers decreasing monotonically to 0. Select r_1 such that $P_{\gamma(r_1)}^{-1} \sum_{k>\gamma(r_1-1)}^{\gamma(r_1)} p_k < \delta_1$. Now suppose that $n > 1$ and $r_1 < \dots < r_{n-1}$ have been selected. First, using the hypothesis that (R, p) is regular, select N such that $l > N$ implies that

$$P_l^{-1} \sum_{i=1}^{n-1} \sum_{k>\gamma(r_i-1)}^{\gamma(r_i)} p_k < \delta_n/2.$$

Now select r_n such that $\gamma(r_n - 1) > N$ and $P_{\gamma(r_n)}^{-1} \sum_{k>\gamma(r_n-1)}^{\gamma(r_n)} p_k < \delta_n/2$. Note that

$$P_{\gamma(r_n)}^{-1} \sum_{i=1}^n \sum_{k>\gamma(r_i-1)}^{\gamma(r_i)} p_k < \delta_n.$$

An elementary computation using the definition of (R, p) yields that if $l \in [\gamma(r_n - 1), \gamma(r_{n+1}))$, then

$$\sum_{i=1}^{\infty} \sum_{k>\gamma(r_i)}^{\gamma(r_i+1)} (R, p)_{l,k} \leq P_{\gamma(r_n)}^{-1} \sum_{i=1}^n \sum_{k>\gamma(r_i-1)}^{\gamma(r_i)} p_k < \delta_n.$$

It follows that

$$\limsup_n \sum_{i=1}^{\infty} \sum_{k>\gamma(r_i)}^{\gamma(r_i+1)} (R, p)_{n,k} = 0$$

and hence, by 2.2(3), $G(\gamma)$ is not a gap Tauberian condition for strong (R, p) -summability.

(3) implies (1): Note that

$$\inf_r \|b^{r-1}(\gamma)\| \geq \inf_r P_{\gamma(r)}^{-1} \sum_{k>\gamma(r-1)}^{\gamma(r)} p_k > 0$$

and, since (R, p) is regular, it follows that $\langle b^r(\gamma) \rangle$ is equivalent to e_0 's unit vector basis. The result now follows from 1.2. □

COROLLARY 3.2. *$G(\gamma)$ is a gap Tauberian condition for (R, p) if there is a $\delta > 0$ such that*

$$\frac{\sum_{k>\gamma(r-1)}^{\gamma(r)} p_k}{\sum_{k=1}^{\gamma(r-1)} p_k} > \delta$$

for all $r \in \mathbb{N}$.

REMARKS. 1. Recall that a conservative BK space E is *conull* if it is not coregular, that is, $f(e) = \sum_k f(e^k)$ for all $f \in E'$. It has been shown in [3] that, for any γ and any conull BK space E , $G(\gamma)$ is never a gap Tauberian condition for E . It turns out, however, that one can establish an analog of a gap Tauberian theorem for a large class of conservative conull BK spaces.

Let $r = \langle r_n \rangle$ be an increasing sequence of natural numbers with $r_1 = 1$. For $x \in \omega$, let

$$O_n^r(x) = \max\{|x_u - x_v| : r_n \leq u < v \leq r_{n+1}\}$$

and set

$$\Omega(r) = \{x \in \omega : \lim_n O_n^r(x) = 0\}.$$

If we define $\|x\|_r = x_1 + \sup_n O_n^r(x)$, then $(\Omega(r), \|\cdot\|_r)$ is a conservative conull BK space [13, p.95]. Also, given an arbitrary conservative conull BK space E , there is an r such that $\Omega(r) \subset E$.

Now define $Z(r) \subset \omega$ by

$$Z(r) = \{x : \text{there is a scalar } \lambda \text{ such that } x_k = \lambda \text{ if } r_{2n} < j \leq r_{2n+1} \text{ for all } n \in \mathbb{N}\}.$$

An elementary $\epsilon - N$ argument yields that $Z(r) \cap \Omega(r) \subset c$, which we record via the following proposition.

PROPOSITION 4.1. *$Z(r)$ is a Tauberian condition for $\Omega(r)$.*

This is analogous to a gap Tauberian theorem in that $\langle \sum_{j>r_{2n}}^{r_{2n+1}} e^j : n \in \mathbb{N} \rangle$ is equivalent to c_0 's unit vector basis in $\Omega(r)$.

2. Let T be a nonnegative regular summability matrix and $x = \langle x_k \rangle \in \omega$. For each $\epsilon > 0$ and $\lambda \in \mathbb{R}$, let $A(\epsilon, \lambda) = \{k : |x_k - \lambda| \geq \epsilon\}$. The sequence x is said to be *T-statistically convergent to λ* if $T - \lim I_{A(\epsilon, \lambda)} = 0$ for all $\epsilon > 0$. It is known that if x is strongly T -summable to λ then x is T -statistically convergent to λ and, if x is bounded and T -statistically convergent to λ , then x is strongly T -summable to λ [2].

The above observations allow us to conclude that $G(\gamma)$ is a gap Tauberian condition for T -statistical convergence if and only if $G(\gamma)$ is a gap Tauberian condition for strong T -summability. Since strong T -summability implies T -statistical convergence, a gap Tauberian condition for T -statistical convergence must also be a gap Tauberian condition for strong T -summability. To establish the converse, suppose $G(\gamma)$ is a gap Tauberian condition for strong T -summability and that $x \in G(\gamma)$ is T -statistically convergent to λ . Define \tilde{x} by $\tilde{x}_k = x_k$ if $|x_k - \lambda| < 1$ and $\tilde{x}_k = \lambda + 1$ otherwise. Note that \tilde{x} is bounded and statistically convergent to λ and hence strongly T -summable to λ . Now, since $\tilde{x} \in G(\gamma)$, \tilde{x} , and hence x , is convergent to λ .

When we restrict our attention to (R, p) where $p_k = 1$ for all k , (R, p) is the Cesaro mean (order 1) and (R, p) -statistical convergence is the usual statistical convergence as discussed in [7] and [11]. Theorem 3.1 yields Fridy's result that $G(\gamma)$ is a gap Tauberian condition for statistical convergence if $\liminf_n \gamma(n + 1)/\gamma(n) > 1$ and its converse.

3. If we recall a few elementary facts about the Stone-Ćech compactification of \mathbb{N} , the result of section 2 can be given an interpretation in the setting of $\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$. Recall that $\beta\mathbb{N}$ can be identified with the set of all ultrafilters on \mathbb{N} [12, 8]. If p is an ultrafilter on \mathbb{N} , we say that p is a free ultrafilter if p does not contain a bounded subset of \mathbb{N} and recall that \mathbb{N}^* can be identified with the collection of all free ultrafilters on \mathbb{N} . If $A \subset \mathbb{N}$, we let $A^* = \{p \in \mathbb{N}^* \mid A \in p\}$ and recall that $\{A^* \mid A \subset \mathbb{N}\}$ is a base of clopen sets for the relative topology on \mathbb{N}^* . Also recall that if $x = \langle x_n \rangle \in l_\infty$, then x can be extended to $x^\beta \in C(\beta\mathbb{N})$ by defining $x^\beta(p) = \bigcap \{cl_{\mathbb{R}}(x^{-1}(A)) \mid A \in p\}$ for an ultrafilter p on \mathbb{N} and that, for a sequence $x \in l_\infty$, $\lim_k x_k = \lambda$ if and only if $x^\beta \upharpoonright \mathbb{N}^* = \lambda$.

If T is a nonnegative regular summability matrix, the support set K_T of T is defined by

$$K_T = \bigcap \{A^* \mid A \subset \mathbb{N}, \lim_n \sum_k t_{n,k} I_A(k) = 1\}.$$

The support set of a matrix was introduced in [9] and has been discussed by a number of authors. Note that $p \in K_T$ if and only if $p \in A^*$ whenever $T - \lim I_A = 1$. As was noted in [1], Theorem 2.1 can be reformulated to yield that K_T is a P -set in \mathbb{N}^* (that is, K_T is interior to any G_δ that contains K_T). It can also be used to show that if $x \in l_\infty$, then x is strongly T -summable to λ if and only if $x^\beta \upharpoonright K_T = \lambda$. It follows that $G(\gamma)$ is a gap Tauberian condition for bounded strong T -summability if and only if $x \in G(\gamma) \cap l_\infty$ and $x^\beta \upharpoonright K_T = \lambda$ implies that $x^\beta \upharpoonright \mathbb{N}^* = \lambda$.

THEOREM 4.2. *$G(\gamma)$ is a gap Tauberian condition for strong T -summability if*

and only if

$$cl_{N^*} \left(\bigcup_{r \in \mathbb{N}} (\gamma(n_r), \gamma(n_r + 1)) \right) \cap K_T \neq \emptyset$$

for any increasing sequence of integers $\langle n_r \rangle$.

PROOF: Observe that $cl_{N^*} \left(\bigcup_{k \in \mathbb{N}} (\gamma(n_r), \gamma(n_r + 1)) \right) \cap A^* \neq \emptyset$ for all $A \subset \mathbb{N}$ such

that $T - \lim I_A = 1$ if and only if $\limsup_n \sum_r \sum_{k > \gamma(n_r)}^{\gamma(n_r + 1)} t_{n,k} > 0$. The result now follows from 2.2(3). \square

REFERENCES

- [1] R. Atalla, 'On the multiplicative behaviour of regular matrices', *Proc. Amer. Math. Soc.* **26** (1970), 437–446.
- [2] J. Connor, 'Two valued measures and summability', *Analysis* **10** (1990), 373–385.
- [3] J. Connor and A.K. Snyder, 'Tauberian theorems for conull spaces', *Internat. J. Math. Math. Sci.* **8** (1985), 689–692.
- [4] J. Diestel, *Sequences and series in Banach spaces* (Springer-Verlag, Berlin, Heidelberg, New York, 1984).
- [5] A.R. Freedman and J.J. Sember, 'Densities and summability', *Pacific J. Math.* **95** (1981), 293–305.
- [6] J. Fridy, 'Tauberian theorems via block dominated matrices', *Pacific J. Math.* **81** (1979), 81–91.
- [7] J. Fridy, 'On statistical convergence', *Analysis* **5** (1985), 301–313.
- [8] L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math. (Van Nostrand, Princeton, 1960).
- [9] M. Henriksen, 'Multiplicative summability methods and the Stone-Čech compactification', *Math. Z.* **71** (1959), 427–435.
- [10] J.D. Hill and W.T. Sledd, 'Approximation in bounded summability fields', *Canad. J. Math.* **20** (1968), 410–415.
- [11] I.J. Maddox, 'A Tauberian theorem for statistical convergence', *Math. Proc. Cambridge Philos. Soc.* **106** (1989), 277–280.
- [12] W. Rudin, 'Homogeneity problems in the theory of Cech compactifications', *Duke Math. J.* **23** (1956), 409–420.
- [13] A. Wilansky, *Summability through functional analysis* (North-Holland, Amsterdam, 1984).

Department of Mathematics
Ohio University
Athens OH 45701
United States of America