BULL. AUSTRAL. MATH. SOC. 08A99, 18A23
VOL. 61 (2000) [217–239]

FORMATIONS, BIHOMOMORPHISMS AND NATURAL TRANSFORMATIONS

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Given a variety $\mathcal{V}$ and $\mathcal{V}$-algebras $\mathbf{A}$ and $\mathbf{B}$, an *algebraic formation* $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-homomorphism $F\colon \mathbf{R} \times \mathbf{A} \to \mathbf{B}$, for some $\mathcal{V}$-algebra $\mathbf{R}$, and the resulting functions $F(r, \_) : A \to B$ for $r \in R$ are termed *formable*. Firstly, as motivation for the study of algebraic formations, categorical formations and their relationship with natural transformations are explained. Then, formations and formable functions are described for some common varieties of algebras, including semilattices, lattices, groups, and implication algebras. Some of their general properties are investigated for congruence modular varieties, including the description of a uniform congruence which provides information on the structure of $\mathbf{B}$.

1. INTRODUCTION

Formations were first studied in [2] as a generalisation of natural transformations and functor categories to any mathematical structure that can be represented by a *template*, which includes categories and any variety of algebras, as well as relational and ordered structures. In particular for categories $\mathbf{A}$ and $\mathbf{B}$, a categorical formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ can be considered as a bifunctor $\mathbf{F}\colon \mathbf{R} \times \mathbf{A} \to \mathbf{B}$ for some category $\mathbf{R}$.

Section 2 of this work demonstrates that natural transformations can be encoded as the maps from the arrows of $\mathbf{A}$ to the arrows of $\mathbf{B}$ that are obtained from categorical formations $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$. Moreover, an equivalence is established between categorical formations and functor categories.

In Section 3 the attention is turned to an arbitrary variety $\mathcal{V}$ of algebras: categories are replaced by $\mathcal{V}$-algebras, functors and bifunctors by $\mathcal{V}$-homomorphisms and $\mathcal{V}$-bihomomorphisms, whereby natural transformations become *formable* functions, functor categories become $\mathcal{V}$-algebras $\mathbf{R}$ of *compatible* functions, with vertical composition replaced by the operations within $\mathbf{R}$. This is demonstrated in Section 4 with some results for diverse varieties to illustrate what is and is not possible for formations depending on the particular variety. For example, it is shown how to partition the homomorphisms

Received 19th May, 1999
The author gratefully acknowledges the support of the New Zealand Foundation for Research, Science, and Technology contract number UOS601.

between any two distributive lattices into disjoint classes that themselves become distributive lattices, in a way that is not possible for non-distributive lattices.

Section 5 then investigates applications of formations in universal algebra, specifically for the case when the variety is congruence modular. For such varieties there is a strong relationship between formations and congruences, and each formation gives rise to a uniform congruence.

2. Formations for Categories

Functors and natural transformations were first introduced in 1942 by Eilenberg and MacLane for their study of limits in cohomology. Since then the notion of a functor has been a central concept of category theory, a functor being considered a *morphism* of categories in the same sense that a $\mathcal{V}$-homomorphism is considered a *morphism* of $\mathcal{V}$-algebras for any variety $\mathcal{V}$ of algebras. Natural transformations are another basic concept and are often thought of as *morphisms* of functors. Let $\mathbf{A}$ and $\mathbf{B}$ be categories and $\mathbf{G}, \mathbf{H}\colon \mathbf{A} \to \mathbf{B}$ be functors. A natural transformation $\boldsymbol{\tau}\colon \mathbf{G} \to \mathbf{H}$ is usually considered to be a map that takes any object a in $\mathbf{A}$ to an arrow $G_{\mathrm{obj}}(a) \xrightarrow{\tau_a} H_{\mathrm{obj}}(a)$ in $\mathbf{B}$, where for any arrow $a_0 \xrightarrow{f} a_1$ in $\mathbf{A}$ one has that $\tau_{a_1} \circ G_{\mathrm{arr}}(f) = H_{\mathrm{arr}}(f) \circ \tau_{a_0}$ in $\mathbf{B}$.

$$\begin{array}{ccc} G_{\mathrm{obj}}(a_0) & \xrightarrow{G_{\mathrm{arr}}(f)} & G_{\mathrm{obj}}(a_1) \\ {\scriptstyle \tau_{a_0}}\downarrow & & \downarrow{\scriptstyle \tau_{a_1}} \\ H_{\mathrm{obj}}(a_0) & \xrightarrow{H_{\mathrm{arr}}(f)} & H_{\mathrm{obj}}(a_1) \end{array}$$

Define a map τ_{arr} from the arrows of $\mathbf{A}$ to the arrows of $\mathbf{B}$ by

$$\tau_{\mathrm{arr}}(f) = \tau_{a_1} \circ G_{\mathrm{arr}}(f) = H_{\mathrm{arr}}(f) \circ \tau_{a_0},$$

for any arrow $a_0 \xrightarrow{f} a_1$ of $\mathbf{A}$. Clearly then $\tau_a = \tau_{\mathrm{arr}}(\mathrm{id}_a)$, and $\tau_{\mathrm{arr}}(g \circ f) = \tau_{\mathrm{arr}}(g) \circ G_{\mathrm{arr}}(f) = H_{\mathrm{arr}}(g) \circ \tau_{\mathrm{arr}}(f)$, for any arrows $a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2$ of $\mathbf{A}$; conversely, any map of arrows τ_{arr} which obeys this equation defines a unique natural transformation $\boldsymbol{\tau}\colon \mathbf{G} \to \mathbf{H}$ with $\tau_a = \tau_{\mathrm{arr}}(\mathrm{id}_a)$ for each object a of $\mathbf{A}$ – this 'arrows only' description of a natural transformation is not new, see for example [**6**].

If $\boldsymbol{\tau}\colon \mathbf{G} \to \mathbf{H}$ and $\boldsymbol{\eta}\colon \mathbf{H} \to \mathbf{K}$ are natural transformations, then they have a vertical composition that is also a natural transformation $\boldsymbol{\eta} \circ \boldsymbol{\tau}\colon \mathbf{G} \to \mathbf{K}$. This allows the construction of functor categories: a collection of functors from $\mathbf{A}$ to $\mathbf{B}$ are the objects of a category whose arrows are natural transformations between these functors, with composition of arrows taken to be the vertical composition of natural transformations. Any natural transformation then gives a smallest functor category that contains it (with either one or two objects), and it also lies in the largest functor category $\mathbf{R} = \mathbf{B}^{\mathbf{A}}$, whose objects consist of all the functors from $\mathbf{A}$ to $\mathbf{B}$, and whose arrows are the natural

https://doi.org/10.1017/S000497270002222X Published online by Cambridge University Press

transformations between such functors. In particular, any two natural transformations are *compatible* in that they both lie in $\mathbf{B}^{\mathbf{A}}$.

For any arrows $a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2$ it is easily established that:

$$(\boldsymbol{\eta} \circ \boldsymbol{\tau})_{\mathrm{arr}}(g \circ f) = \eta_{\mathrm{arr}}(g) \circ \tau_{\mathrm{arr}}(f)$$
$$(\operatorname{dom} \boldsymbol{\eta})_{\mathrm{obj}}(\operatorname{dom} f) = \operatorname{dom} \eta_{\mathrm{arr}}(f)$$
$$(\operatorname{codom} \boldsymbol{\eta})_{\mathrm{obj}}(\operatorname{codom} f) = \operatorname{codom} \eta_{\mathrm{arr}}(f),$$

where dom and codom are respectively the domain and codomain operations. Incidentally, given three categories, the horizontal composition of natural transformations is simply the composition of the arrow maps.

The following theorem can be easily verified.

Theorem 2.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be categories, and* $\mathbf{R}$ *be the functor category* $\mathbf{B}^{\mathbf{A}}$*. Let* $F\colon R \times A \to B$ *be the evaluation functor defined by*

$$F_{\mathrm{obj}}(\mathbf{G}, a) = G_{\mathrm{obj}}(a)$$
$$F_{\mathrm{arr}}(\boldsymbol{\tau}, f) = \tau_{\mathrm{arr}}(f),$$

for any functor $\mathbf{G}\colon \mathbf{A} \to \mathbf{B}$*, object* a *of* $\mathbf{A}$*, natural transformation* $\boldsymbol{\tau}$*, and arrow* f *of* $\mathbf{A}$*. Then* $\mathbf{F}\colon \mathbf{R} \times \mathbf{A} \to \mathbf{B}$ *is a functor.*

A bit more can be said about the functor $\mathbf{F}\colon \mathbf{R} \times \mathbf{A} \to \mathbf{B}$ in Theorem 2.1. If one considers F as consisting of the two maps $F_{\mathrm{obj}}\colon R_{\mathrm{obj}} \to B_{\mathrm{obj}}^{A_{\mathrm{obj}}}$ and $F_{\mathrm{arr}}\colon R_{\mathrm{arr}} \to B_{\mathrm{arr}}^{A_{\mathrm{arr}}}$, then F is injective in the sense that given any functors $\mathbf{K}, \mathbf{K}'\colon \mathbf{S} \to \mathbf{R}$ for some category $\mathbf{S}$, if $FK = FK'$ (composed as maps) then $K = K'$. To see this note that for any object s of $\mathbf{S}$ one has

$$F_{\mathrm{obj}}\big(K_{\mathrm{obj}}(s)\big) = F_{\mathrm{obj}}\left(K'_{\mathrm{obj}}(s)\right)$$
$$F_{\mathrm{arr}}\left(\mathrm{id}_{K_{\mathrm{obj}}(s)}\right) = F_{\mathrm{arr}}\left(\mathrm{id}_{K'_{\mathrm{obj}}(s)}\right),$$

and so the functors $\mathbf{K}_{\mathrm{obj}}(s)$ and $\mathbf{K}'_{\mathrm{obj}}(s)$ have the same object and arrow maps, and hence are equal. Furthermore, for any arrow $s_0 \xrightarrow{l} s_1$ of $\mathbf{S}$ one has

$$F_{\mathrm{arr}}\big(K_{\mathrm{arr}}(l)\big) = F_{\mathrm{arr}}\big(K'_{\mathrm{arr}}(l)\big),$$

and so the natural transformations

$$\mathbf{K}_{\mathrm{arr}}(l)\colon \mathbf{K}_{\mathrm{obj}}(s_0) \to \mathbf{K}_{\mathrm{obj}}(s_1)$$
$$\mathbf{K}'_{\mathrm{arr}}(l)\colon \mathbf{K}'_{\mathrm{obj}}(s_0) \to \mathbf{K}'_{\mathrm{obj}}(s_1)$$

have the same arrow maps; but also $\mathbf{K}_{\mathrm{obj}}(s_0) = \mathbf{K}'_{\mathrm{obj}}(s_0)$ and $\mathbf{K}_{\mathrm{obj}}(s_1) = \mathbf{K}'_{\mathrm{obj}}(s_1)$ by the above, so these natural transformations are in fact equal.

The following simple theorem shows that for any category $\mathbf{S}$, a functor $\mathbf{G}\colon \mathbf{S}\times\mathbf{A} \to \mathbf{B}$ is completely determined by the functor $\mathbf{F}\colon \mathbf{R}\times\mathbf{A} \to \mathbf{B}$ of Theorem 2.1 and a functor $\mathbf{K}\colon \mathbf{S} \to \mathbf{R}$. The proof consists of showing for any arrow l of $\mathbf{S}$ that the induced map $G_{\mathrm{arr}}(l, \_)\colon A_{\mathrm{arr}} \to B_{\mathrm{arr}}$ is the arrow map of some natural transformation, and so $\mathbf{G}$ can be factored through the functor category $\mathbf{R}$.

THEOREM 2.2. *Let $\mathbf{A}, \mathbf{B}, \mathbf{S}$ be categories, $\mathbf{G}\colon \mathbf{S}\times\mathbf{A} \to \mathbf{B}$ be a functor, and let $\mathbf{R}$, $\mathbf{F}\colon \mathbf{R}\times\mathbf{A} \to \mathbf{B}$ be as in Theorem 2.1. Then there is a unique functor $\mathbf{K}\colon \mathbf{S} \to \mathbf{R}$ for which*

$$G_{\mathrm{obj}}(s,a) = F_{\mathrm{obj}}\big(K_{\mathrm{obj}}(s), a\big)$$
$$G_{\mathrm{arr}}(l,f) = F_{\mathrm{arr}}\big(K_{\mathrm{arr}}(l), f\big),$$

for every object s of $\mathbf{S}$, a of $\mathbf{A}$, and arrow l of $\mathbf{S}$, f of $\mathbf{A}$.

PROOF: Firstly, for any object s of $\mathbf{S}$, define $G^s_{\mathrm{obj}}\colon A_{\mathrm{obj}} \to B_{\mathrm{obj}}$ and $G^s_{\mathrm{arr}}\colon A_{\mathrm{arr}} \to B_{\mathrm{arr}}$ by

$$G^s_{\mathrm{obj}}(a) = G_{\mathrm{obj}}(s,a)$$
$$G^s_{\mathrm{arr}}(f) = G_{\mathrm{arr}}(\mathrm{id}_s, f),$$

for any object a and arrow f of $\mathbf{A}$. For any arrows $a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2$ of $\mathbf{A}$ note that

$$\langle \mathrm{id}_s, g\circ f\rangle = \langle \mathrm{id}_s, g\rangle \circ \langle \mathrm{id}_s, f\rangle$$

in the category $\mathbf{S}\times\mathbf{A}$. Hence, $G_{\mathrm{arr}}(\mathrm{id}_s, g\circ f) = G_{\mathrm{arr}}(\mathrm{id}_s, g)\circ G_{\mathrm{arr}}(\mathrm{id}_s, f)$, and so $G^s_{\mathrm{arr}}(g\circ f) = G^s_{\mathrm{arr}}(g)\circ G^s_{\mathrm{arr}}(f)$. Moreover, for any object a of $\mathbf{A}$, $G^s_{\mathrm{arr}}(\mathrm{id}_a) = G_{\mathrm{arr}}(\mathrm{id}_s, \mathrm{id}_a) = \mathrm{id}_{G_{\mathrm{obj}}(s,a)} = \mathrm{id}_{G^s_{\mathrm{obj}}(a)}$, and so G^s_{arr} is the arrow map of a functor $\mathbf{G}^s\colon \mathbf{A}\to\mathbf{B}$, whose object map is G^s_{obj}.

Next, for any arrow $s_0 \xrightarrow{l} s_1$ of $\mathbf{S}$, define $G^l_{\mathrm{arr}}\colon A_{\mathrm{arr}} \to B_{\mathrm{arr}}$ by $G^l_{\mathrm{arr}}(f) = G_{\mathrm{arr}}(l,f)$. Note that

$$\langle l, g\circ f\rangle = \langle l, g\rangle \circ \langle \mathrm{id}_{s_0}, f\rangle$$

in $\mathbf{S}\times\mathbf{A}$. Hence one obtains $G^l_{\mathrm{arr}}(g\circ f) = G^l_{\mathrm{arr}}(g)\circ G^{s_0}_{\mathrm{arr}}(f)$. Similarly, $G^l_{\mathrm{arr}}(g\circ f) = G^{s_1}_{\mathrm{arr}}(g)\circ G^l_{\mathrm{arr}}(f)$. Thus G^l_{arr} is the arrow map of a natural transformation $\mathbf{G}^l\colon \mathbf{G}^{s_0} \to \mathbf{G}^{s_1}$.

Now define $K\colon S\to R$ by

$$K_{\mathrm{obj}}(s) = \mathbf{G}^s$$
$$K_{\mathrm{arr}}(l) = \mathbf{G}^l.$$

Note for any object s of $\mathbf{S}$ that $G^{\mathrm{id}_s}_{\mathrm{arr}} = G^s_{\mathrm{arr}}$, and so $K_{\mathrm{arr}}(\mathrm{id}_s) = \mathrm{id}_{K_{\mathrm{obj}}(s)}$. Also, for any arrows $s_0 \xrightarrow{l} s_1 \xrightarrow{m} s_2$ of $\mathbf{S}$ and arrow $a_0 \xrightarrow{f} a_1$ of $\mathbf{A}$ one has

$$\begin{aligned} G^{mol}_{\text{arr}}(f) &= G_{\text{arr}}\left(m \circ l, f \circ \mathrm{id}_{a_0}\right) \\ &= G_{\text{arr}}(m, f) \circ G_{\text{arr}}\left(l, \mathrm{id}_{a_0}\right) \\ &= G^{m}_{\text{arr}}(f) \circ G^{l}_{\text{arr}}\left(\mathrm{id}_{a_0}\right) \\ &= \left(\mathbf{G}^m \circ \mathbf{G}^l\right)_{\text{arr}}\left(f \circ \mathrm{id}_{a_0}\right) \\ &= \left(\mathbf{G}^m \circ \mathbf{G}^l\right)_{\text{arr}}(f). \end{aligned}$$

Hence the natural transformations $\mathbf{G}^{mol}\colon \mathbf{G}^{s_0} \to \mathbf{G}^{s_2}$ and $\left(\mathbf{G}^m \circ \mathbf{G}^l\right) : \mathbf{G}^{s_0} \to \mathbf{G}^{s_2}$ are the same. Thus, $K_{\text{arr}}(m \circ l) = K_{\text{arr}}(m) \circ K_{\text{arr}}(l)$, and so $\mathbf{K}\colon \mathbf{S} \to \mathbf{R}$ is a functor.

The uniqueness of $\mathbf{K}$ is then immediate from the comments proceeding Theorem 2.1. $\square$

The importance of Theorem 2.2 is that it demonstrates that the concepts of natural transformation and of functor category $\mathbf{B}^{\mathbf{A}}$ can be encoded in the functors $\mathbf{G}\colon \mathbf{S}\times\mathbf{A} \to \mathbf{B}$, which are termed *formations* and denoted by $\mathbf{G}\colon \mathbf{A} \rightrightarrows \mathbf{B}$. For each object s of $\mathbf{S}$, $\mathbf{G}^s\colon \mathbf{A} \to \mathbf{B}$ is a functor, and for each arrow $s_0 \xrightarrow{l} s_1$ of $\mathbf{S}$, $\mathbf{G}^l\colon \mathbf{G}^{s_0} \to \mathbf{G}^{s_1}$ is a natural transformation.

The motivation then for this work is that since there are algebraic analogues of the concepts of category and functor (namely, algebra and homomorphism), and hence too for functors $\mathbf{G}\colon \mathbf{S} \times \mathbf{A} \to \mathbf{B}$, natural transformations also have an algebraic analogue, which will be investigated in the following sections.

3. Formations for Varieties of Algebras

Definition 3.1: Let $\mathcal{V}$ be a variety. A *($\mathcal{V}$-)formation*, $\mathbf{F} = \langle \mathbf{A}, \mathbf{B}, \mathbf{R}, F\rangle$, consists of $\mathcal{V}$-algebras $\mathbf{A}, \mathbf{B}, \mathbf{R}$, and a function $F\colon R \to B^A$ which satisfies the condition for any n-ary term σ of $\mathrm{Clo}\,\mathcal{V}$, any $a_0, \dots, a_{n-1} \in A$, and any $r_0, \dots, r_{n-1} \in R$ that

$$F\left(\sigma^R(r_0, \dots, r_{n-1})\right)\left(\sigma^A(a_0, \dots, a_{n-1})\right) = \sigma^B\left(F(r_0)(a_0), \dots, F(r_{n-1})(a_{n-1})\right).$$

A formation $\mathbf{F} = \langle \mathbf{A}, \mathbf{B}, \mathbf{R}, F\rangle$ will be denoted by $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$, and $\mathbf{R}$ will be called the *underlying algebra* of $\mathbf{F}$.

An equivalent approach is to define a formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ to be a $\mathcal{V}$-homomorphism $\mathbf{F}\colon \mathbf{R} \times \mathbf{A} \to \mathbf{B}$ for some $\mathcal{V}$-algebra $\mathbf{R}$, and then for $r \in R$ let $F(r)$ denote the function $F(r, \_) : A \to B$. From this it is clear that the functions $F(r)$ are the algebraic equivalent in an algebraic formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ of the arrow maps of natural transformations in a categorical formation.

Definition 3.2: Let $\mathbf{F} = \langle \mathbf{A}, \mathbf{B}, \mathbf{R}, F\rangle$ be a $\mathcal{V}$-formation, $\mathbf{S}$ be a $\mathcal{V}$-algebra, and $\mathbf{K}\colon \mathbf{S} \to \mathbf{R}$ be a $\mathcal{V}$-homomorphism. Define $\mathbf{FK} = \langle \mathbf{A}, \mathbf{B}, \mathbf{S}, FK\rangle$, where $FK\colon S \to B^A$ is taken to be the composition of functions.

https://doi.org/10.1017/S000497270002222X Published online by Cambridge University Press

It is easily verified that if $\mathbf{F}$ is a $\mathcal{V}$-formation and $\mathbf{K}$ is a $\mathcal{V}$-homomorphism as above, then $\mathbf{FK}$ is also a $\mathcal{V}$-formation.

For $\mathcal{V}$-algebras $\mathbf{A}$ and $\mathbf{B}$ several examples of formations $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ are apparent. Firstly, if $\mathbf{H}\colon \mathbf{A} \to \mathbf{B}$ is any $\mathcal{V}$-homomorphism, taking $\mathbf{R}$ to be a one-element algebra $\{1^R\}$, and defining $F\left(1^R\right) = H$ gives a $\mathcal{V}$-formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$. Conversely, if $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation and if $\{r\}$ is a one-element subalgebra of the underlying algebra $\mathbf{R}$ then it is easily seen that $F(r)$ must be a homomorphism $\mathbf{F}(r)\colon \mathbf{A} \to \mathbf{B}$.

Secondly, let $\mathbf{R}$ be any subalgebra of $\mathbf{B}$, and for $r \in R$ take $F(r)\colon A \to B$ to be the constant function giving the element r. Then $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation.

More-interesting examples can be obtained for the wide class of Jónsson-Tarski algebras. A variety of *Jónsson-Tarski algebras* is a variety $\mathcal{V}$ of algebras having a unary term 1 that is constant in each algebra of $\mathcal{V}$, and a binary term $\times$ for which 1 is a two-sided identity, so that for any term $\sigma \in \operatorname{Clo}\mathcal{V}$ one has the identity $\sigma(1, \ldots, 1) \approx 1$ in $\mathcal{V}$.

THEOREM 3.3. *Let $\mathcal{V}$ be a variety of Jónsson-Tarski algebras. Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathcal{V}$-algebras, and $\mathbf{H}\colon \mathbf{A} \to \mathbf{B}$ be a $\mathcal{V}$-homomorphism. Let $\mathbf{R}$ be a subalgebra of $\mathbf{B}$ that satisfies the condition for any n-ary term σ of $\operatorname{Clo}\mathcal{V}$, any $a_0, \ldots, a_{n-1} \in A$, and any $r_0, \ldots, r_{n-1} \in R$ that*

$$\begin{aligned}\sigma^B\left(r_0 \times^B H(a_0), \ldots, r_{n-1} \times^B H(a_{n-1})\right)& \\ = \sigma^B(r_0, \ldots, r_{n-1}) &\times^B \sigma^B\left(H(a_0), \ldots, H(a_{n-1})\right).\end{aligned}$$

Define $F\colon R \to B^A$ by $F(r)(a) = r \times^B H(a)$ (so in particular $F(1^R) = H$).

Then $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation.

The examples provided by Theorem 3.3 are of particular interest as they are in fact the only possible formations that can occur for Jónsson-Tarski algebras, in the following sense:

THEOREM 3.4. *Let $\mathcal{V}$ be a variety of Jónsson-Tarski algebras, and $\mathbf{A}$, $\mathbf{B}$ be $\mathcal{V}$-algebras. Let $\mathbf{G}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ be a $\mathcal{V}$-formation with underlying algebra $\mathbf{S}$. Define $K\colon S \to B$ by $K(s) = G(s)\left(1^A\right)$, and let R be the image of K in B.*

Then $\mathbf{R}$ is a subalgebra of $\mathbf{B}$ satisfying the condition of Theorem 3.3 where $\mathbf{H} = \mathbf{G}\left(1^S\right)$, and $\mathbf{K}\colon \mathbf{S} \to \mathbf{R}$ is a $\mathcal{V}$-homomorphism. Moreover, taking $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ to be the $\mathcal{V}$-formation with underlying algebra $\mathbf{R}$ as provided by Theorem 3.3, one obtains $\mathbf{G} = \mathbf{FK}$, and $\mathbf{K}$ is unique with this property.

In particular, for groups $\mathbf{A}$ and $\mathbf{B}$ a formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ essentially consists of a homomorphism $\mathbf{H}\colon \mathbf{A} \to \mathbf{B}$ where $H = F\left(1^R\right)$, the underlying algebra $\mathbf{R}$, which is a subalgebra of the centraliser of $H(A)$ in B, and for each $r \in R$ one has that $F(r)\colon A \to B$ is the function given by $F(r)(a) = r \cdot H(a) = H(a) \cdot r$.

A similar classification of formations is possible for Boolean algebras.

THEOREM 3.5. *Let $\mathcal{V}$ be the variety of Boolean algebras. Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be $\mathcal{V}$-algebras, and* $\mathbf{H}\colon \langle A, \wedge^A, \vee^A\rangle \to \langle B, \wedge^B, \vee^B\rangle$ *be a lattice-homomorphism. Let R be the interval* $\left[H\left(0^A\right)^{-1} \wedge^B H\left(1^A\right), 1^B\right]$ *of* $\mathbf{B}$*, and for any n-ary term σ of* $\operatorname{Clo}\mathcal{V}$ *define*

$$\sigma^R\left(r_0, \ldots, r_{n-1}\right) = \sigma^B\left(r_0, \ldots, r_{n-1}\right) \vee^B \left(H\left(0^A\right)^{-1} \wedge^B H\left(1^A\right)\right).$$

Define $F\colon R \to B^A$ *by* $F(r)(a) = r \wedge^B \left(H(a) \vee^B H\left(1^A\right)^{-1}\right)$.

Then $\mathbf{R}$ *is a $\mathcal{V}$-algebra and* $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ *is a $\mathcal{V}$-formation.*

THEOREM 3.6. *Let $\mathcal{V}$ be the variety of Boolean algebras, and* $\mathbf{A}$, $\mathbf{B}$ *be $\mathcal{V}$-algebras. Let* $\mathbf{G}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ *be a $\mathcal{V}$-formation with underlying algebra* $\mathbf{S}$*. Let R be the interval* $\left[G\left(0^S\right)\left(1^A\right), 1^B\right]$ *of* $\mathbf{B}$*,* $H = G\left(1^S\right)$*, and define* $K\colon S \to R$ *by* $K(s) = G(s)\left(1^A\right)$.

Then H and R satisfy the conditions of Theorem 3.5, and $\mathbf{K}\colon \mathbf{S} \to \mathbf{R}$ *is a $\mathcal{V}$-homomorphism. Moreover, taking* $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ *to be the $\mathcal{V}$-formation with underlying algebra* $\mathbf{R}$ *as provided by Theorem 3.5, one obtains* $\mathbf{G} = \mathbf{FK}$*, and* $\mathbf{K}$ *is unique with this property.*

Theorem 3.3 and Theorem 3.5 provide simple classifications for formations in the cases of Jónsson-Tarski algebras and Boolean algebras, and show that the functions $F(r)$ are related to one another for a given formation $\mathbf{F}$. As will be seen in Section 4, such simple classifications of formations are not possible in general for arbitrary varieties, but the images of these functions still possess some relatively nice properties. For a given formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$, consider the binary relation $\sim_{\mathbf{F}}$ defined on B by $b \sim_{\mathbf{F}} b'$ if there is an $r \in R$ with both b and b' in the image of $F(r)$. Then $\sim_{\mathbf{F}}$ is actually a tolerance (that is, a reflexive and symmetric binary relation that is respected by every term operation) on the subalgebra $\mathbf{im}\, F$ of $\mathbf{B}$, and so its transitive closure gives a congruence $\Theta_{\mathbf{F}}$ on $\mathbf{im}\, F$.

THEOREM 3.7. *Let $\mathcal{V}$ be a variety and* $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ *be a $\mathcal{V}$-formation. Define* $\Theta_{\mathbf{F}}$ *to be the binary relation defined on* $\operatorname{im} F$ *by* $\langle b, b'\rangle \in \Theta_{\mathbf{F}}$ *if and only if there is an* $n \geqslant 1$, $a_0, a_0', \ldots, a_{n-1}, a_{n-1}' \in A$, $r_0, \ldots, r_{n-1} \in R$ *for which*

$$\begin{aligned} b &= F\left(r_0\right)\left(a_0\right) \\ F\left(r_i\right)\left(a_i'\right) &= F\left(r_{i+1}\right)\left(a_{i+1}\right) \qquad \text{for } i < n-1 \\ F\left(r_{n-1}\right)\left(a_{n-1}'\right) &= b'. \end{aligned}$$

Then $\Theta_{\mathbf{F}}$ *is the least congruence on the subalgebra* $\mathbf{im}\, F$ *of* $\mathbf{B}$ *that identifies the elements within each* $\operatorname{im} F(r)$.

PROOF: It is clear from its definition that $\Theta_{\mathbf{F}}$ is reflexive, symmetric, and transitive. To see that $\Theta_{\mathbf{F}}$ is actually a congruence suppose that σ is an m-ary term of $\operatorname{Clo}\mathcal{V}$, and $\left\langle b_0, b_0'\right\rangle, \ldots, \left\langle b_{m-1}, b_{m-1}'\right\rangle \in \Theta_{\mathbf{F}}$. Without loss of generality one may assume that there is a single $n \geqslant 1$, $a_{ij}, a'_{ij} \in A$, $r_{ij} \in R$ for $i < m$, $j < n$ with for each $i < m$

$$\begin{aligned} b_i &= F(r_{i0})(a_{i0}) \\ F(r_{ij})\left(a'_{ij}\right) &= F(r_{ij+1})(a_{ij+1}) \qquad \text{for } j < n-1 \\ F(r_{in-1})\left(a'_{in-1}\right) &= b'_i. \end{aligned}$$

Clearly,

$$\begin{aligned} \sigma^B(b_0,\dots,b_{m-1}) &= F\left(\sigma^R(r_{00},\dots,r_{m-10})\right)\left(\sigma^A(a_{00},\dots,a_{m-10})\right), \\ \sigma^B\left(b'_0,\dots,b'_{m-1}\right) &= F\left(\sigma^R(r_{0n-1},\dots,r_{m-1n-1})\right)\left(\sigma^A\left(a'_{0n-1},\dots,a'_{m-1n-1}\right)\right), \end{aligned}$$

and for each $j < n-1$ it is easily seen that

$$\begin{aligned} &F\left(\sigma^R(r_{0j},\dots,r_{m-1j})\right)\left(\sigma^A\left(a'_{0j},\dots,a'_{m-1j}\right)\right) \\ &\qquad = F\left(\sigma^R(r_{0j+1},\dots,r_{m-1j+1})\right)\left(\sigma^A(a_{0j+1},\dots,a_{m-1j+1})\right). \end{aligned}$$

Hence $\left\langle\sigma^B(b_0,\dots,b_{m-1}),\sigma^B\left(b'_0,\dots,b'_{m-1}\right)\right\rangle \in \Theta_{\mathbf{F}}$. It is then clear that $\Theta_{\mathbf{F}}$ is the least congruence that identifies the elements within each $\operatorname{im} F(r)$. ∎

Under very mild assumptions more can be said about the congruence $\Theta_{\mathbf{F}}$. For example, if the variety $\mathcal{V}$ is subtractive (that is, there is a binary term $-$ in $\operatorname{Clo}\mathcal{V}$ and a constant 0 for which $x - x \approx 0$ and $x - 0 \approx x$ are identities in $\mathcal{V}$), then $\operatorname{im} F\left(0^R\right)$ is precisely one block of $\Theta_{\mathbf{F}}$, and any $\operatorname{im} F(r)$ that intersects with this block is identical with it. Moreover, if $\operatorname{im} F(r)$ and $\operatorname{im} F(r')$ lie in the same block of $\Theta_{\mathbf{F}}$ then $F(r)\left(0^A\right) \in \operatorname{im} F(r')$. If instead the variety is supposed to be congruence modular then, as will be shown in Section 5, each $\operatorname{im} F(r)$ for $r \in R$ is a block of $\Theta_{\mathbf{F}}$, and $\Theta_{\mathbf{F}}$ is actually a uniform congruence on $\operatorname{im} F$.

4. Formable Functions

In this section the individual functions that can comprise a formation, rather than the formation as a whole will be studied for some of the classical varieties in algebra. The aim is to present some of the similarities and differences that can occur from one variety to another. Although the results obtained in each case involve relatively simple characterisations, their diversity from one variety to another tends to suggest that they do not share a common generalisation. For ease of readability most superscripts adorning operation symbols will be dropped in this section.

Definition 4.1: Let $\mathcal{V}$ be a variety, $\mathbf{A}$, $\mathbf{B}$ be $\mathcal{V}$-algebras, and $f\colon A \to B$ be a function. If there is a $\mathcal{V}$-formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ and an element $r \in R$ in the underlying algebra $\mathbf{R}$ for which $f = F(r)$, then f is called *formable*.

A collection of formable functions $\{f_i\colon A \to B \mid i \in I\}$ is called *compatible* if there is a $\mathcal{V}$-formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ and elements $\{r_i \mid i \in I\}$ in the underlying algebra $\mathbf{R}$ for which each $f_i = F(r_i)$ for $i \in I$.

https://doi.org/10.1017/S000497270002222X Published online by Cambridge University Press

THEOREM 4.2. *Let $\mathcal{V}$ be the variety of meet-semilattices, and let* $\mathbf{A}$, $\mathbf{B}$ *be $\mathcal{V}$-algebras. A function $f\colon A \to B$ is formable if and only if f is a $\mathcal{V}$-homomorphism. Moreover, a collection of formable functions $\{f_i \mid i \in I\}$ is compatible if and only if for every $i, j \in I$ and every $a_0, a_1 \in A$ one has*

$$f_i(a_0) \wedge f_j(a_1) = f_i(a_1) \wedge f_j(a_0).$$

PROOF: Firstly, if a function $f\colon A \to B$ is formable then there is a $\mathcal{V}$-formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ with an underlying algebra $\mathbf{R}$ and $r \in R$ for which $f = F(r)$. Then clearly for any $a_0, a_1 \in A$ one has

$$\begin{aligned} f(a_0 \wedge a_1) &= F(r)(a_0 \wedge a_1) \\ &= F(r \wedge r)(a_0 \wedge a_1) \\ &= F(r)(a_0) \wedge F(r)(a_1) \\ &= f(a_0) \wedge f(a_1). \end{aligned}$$

Conversely, it is already known that any $\mathcal{V}$-homomorphism defines a $\mathcal{V}$-formation with a one-element underlying algebra.

Next, suppose that $\{f_i \mid i \in I\}$ is a compatible collection of formable functions from a $\mathcal{V}$-formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ with underlying algebra $\mathbf{R}$. Then for any $i, j \in I$ there are $r_i, r_j \in R$ for which $f_i = F(r_i)$ and $f_j = F(r_j)$. So for any $a_0, a_1 \in A$ one has

$$\begin{aligned} f_i(a_0) \wedge f_j(a_1) &= F(r_i)(a_0) \wedge F(r_j)(a_1) \\ &= F(r_i \wedge r_j)(a_1 \wedge a_0) \\ &= F(r_i)(a_1) \wedge F(r_j)(a_0) \\ &= f_i(a_1) \wedge f_j(a_0). \end{aligned}$$

More interesting is that in fact this condition is also sufficient to ensure compatibility. Indeed, suppose now that $\{f_i\colon A \to B \mid i \in I\}$ is a collection of formable functions that satisfy this condition. In the construction of a necessary $\mathcal{V}$-formation a bit more will be shown — this collection of formable functions can be taken as elements in an underlying algebra $\mathbf{R}$ for a formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$, given by $F(r)(a) = r(a)$, and they actually generate $\mathbf{R}$. Let $R \subseteq B^A$ consist of those functions $r\colon A \to B$ that can be expressed in the form

$$r(a) = f_{i_0}(a) \wedge \cdots \wedge f_{i_{n-1}}(a)$$

for each $a \in A$, for some $i_0, \ldots, i_{n-1} \in I$ and $n \geqslant 1$. Clearly, each $r \in R$ is formable (being a $\mathcal{V}$-homomorphism). It is not difficult to verify for any $j \in I$ that since each pair $\{f_{i_0}, f_j\}, \ldots, \{f_{i_{n-1}}, f_j\}$ satisfies the condition, then so will $\{r, f_j\}$, from which it follows that any two elements of R satisfy the condition. Define $\wedge$ on R by

$$(r \wedge r')(a) = r(a) \wedge r'(a),$$

https://doi.org/10.1017/S000497270002222X Published online by Cambridge University Press

for $r, r' \in R$. Then $\mathbf{R} = \langle R, \wedge \rangle$ is easily seen to be a $\mathcal{V}$-algebra. Finally, define $F\colon R \to B^A$ by $F(r) = r$, for $r \in R$. For $r_0, r_1 \in R$ and $a_0, a_1 \in A$, using the fact that

$$r_0\left(a_0\right) \wedge r_1\left(a_1\right) = r_0\left(a_1\right) \wedge r_1\left(a_0\right)$$

one obtains

$$\begin{aligned} F\left(r_0 \wedge r_1\right)\left(a_0 \wedge a_1\right) &= r_0\left(a_0 \wedge a_1\right) \wedge r_1\left(a_0 \wedge a_1\right) \\ &= r_0\left(a_0\right) \wedge r_0\left(a_1\right) \wedge r_1\left(a_0\right) \wedge r_1\left(a_1\right) \\ &= F\left(r_0\right)\left(a_0\right) \wedge F\left(r_1\right)\left(a_1\right). \end{aligned}$$

$\square$

Hence to check whether a collection of semilattice homomorphisms is compatible it suffices to check whether each pair $\{f_i, f_j\}$ for $i, j \in I$ is compatible.

As an example consider the case when $\mathbf{A}$ is the two-element meet-semilattice. There are three formable functions from $\mathbf{A}$ to $\mathbf{A}$, namely the identity function id, the constant function f_0 giving the bottom element 0 of A, and the constant function f_1 giving the top element 1 of A. Note that $\{f_0, \mathrm{id}\}$ is compatible, indeed one can for example take $\mathbf{R}$ to be the two-element semilattice $\{0,1\}$ and define the formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{A}$ by $F(0) = f_0$ and $F(1) = \mathrm{id}$. Similarly, $\{f_0, f_1\}$ is compatible, but note here that $\{\mathrm{id}, f_1\}$ is not compatible as $\mathrm{id}(1) \wedge f_1(0) \neq \mathrm{id}(0) \wedge f_1(1)$.

It is not difficult to see that for a variety $\mathcal{V}$ of lattices, a function is formable if and only if it is a $\mathcal{V}$-homomorphism. Let $\mathbf{A}$ be the two-element lattice, and $\mathbf{B}$ be the five-element non-modular lattice N_5, with elements $0, b_1, b_2 < b_3, 1$. Let $f, g, h\colon A \to B$ be the homomorphisms defined by

$$\begin{aligned} &f(0) = b_1, \qquad && g(0) = 0, \qquad && h(0) = 0, \\ &f(1) = 1, && g(1) = b_2, && h(1) = b_3. \end{aligned}$$

One can easily construct formations $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ that show $\{f, g\}$ and $\{f, h\}$ are each compatible. But g and h cannot belong to the same formation, and so $\{g, h\}$ is not compatible. A similar example can be constructed for the five-element modular, non-distributive lattice M_3.

However, the situation is much nicer for distributive lattices.

Theorem 4.3. *Let $\mathcal{V}$ be the variety of distributive lattices, and let $\mathbf{A}, \mathbf{B}$ be $\mathcal{V}$-algebras. A function $f\colon A \to B$ is formable if and only if f is a $\mathcal{V}$-homomorphism. Moreover, a collection of formable functions $\{f_i \mid i \in I\}$ is compatible if and only if for every $i, j \in I$ and every $a_0, a_1 \in A$ one has*

$$\begin{aligned} f_i\left(a_0\right) \wedge f_j\left(a_1\right) &= f_i\left(a_1\right) \wedge f_j\left(a_0\right) \\ f_i\left(a_0\right) \vee f_j\left(a_1\right) &= f_i\left(a_1\right) \vee f_j\left(a_0\right). \end{aligned}$$

Furthermore, if $\{f_i, f_j\}$ and $\{f_j, f_k\}$ are each compatible then $\{f_i, f_k\}$ is also compatible (and so also $\{f_i, f_j, f_k\}$ is compatible).

PROOF: The statement concerning the equivalence of formable functions with $\mathcal{V}$-homomorphisms follows as in the proof of Theorem 4.2, as does the implication that compatible formable functions satisfy the two stated conditions.

Starting with a collection $\{f_i \mid i \in I\}$ of formable functions that satisfy the two conditions one can construct an algebra $\mathbf{R}$ as follows. Let $R \subseteq B^A$ consist of those functions $r\colon A \to B$ that can be expressed in the form

$$r(a) = \sigma^B \left(f_{i_0}(a), \ldots, f_{i_{n-1}}(a)\right)$$

for each $a \in A$, for some $i_0, \ldots, i_{n-1} \in I$ and n-ary term σ of $\operatorname{Clo}\mathcal{V}$. Define $\wedge$ and $\vee$ on R by

$$\begin{aligned}(r \wedge r')\,(a) &= r(a) \wedge r'(a)\\ (r \vee r')\,(a) &= r(a) \vee r'(a),\end{aligned}$$

for $r, r' \in R$. By induction on the complexity of a term σ one sees that

$$\sigma^R\,(r_0, \ldots, r_{n-1})\,(a) = \sigma^B\big(r_0(a), \ldots, r_{n-1}(a)\big),$$

from which it is clear that $\mathbf{R} = \langle R, \sigma^R \mid \sigma \in \operatorname{Clo}\mathcal{V}\rangle$ satisfies the equational theory of $\mathbf{B}$, and hence is a $\mathcal{V}$-algebra. Next, using distributivity in $\mathbf{B}$ it is easily seen for any $i, j \in I$ that $f_i \wedge f_j$ and $f_i \vee f_j$ are each $\mathcal{V}$-homomorphisms. For any $i, j, k \in I$ and $a_0, a_1 \in A$ one has

$$\begin{aligned}(f_i \wedge f_j)\,(a_0) \wedge f_k\,(a_1) &= f_i\,(a_0) \wedge f_j\,(a_0) \wedge f_k\,(a_1) \wedge f_k\,(a_1)\\ &= f_i\,(a_1) \wedge f_j\,(a_1) \wedge f_k\,(a_0) \wedge f_k\,(a_0)\\ &= (f_i \wedge f_j)\,(a_1) \wedge f_k\,(a_0)\,.\end{aligned}$$

Using distributivity in $\mathbf{B}$ one also obtains

$$\begin{aligned}(f_i \wedge f_j)\,(a_0) \vee f_k\,(a_1) &= \big(f_i\,(a_0) \wedge f_j\,(a_0)\big) \vee f_k\,(a_1)\\ &= \big(f_i\,(a_0) \vee f_k\,(a_1)\big) \wedge \big(f_j\,(a_0) \vee f_k\,(a_1)\big)\\ &= \big(f_i\,(a_1) \vee f_k\,(a_0)\big) \wedge \big(f_j\,(a_1) \vee f_k\,(a_0)\big)\\ &= \big(f_i\,(a_1) \wedge f_j\,(a_1)\big) \vee f_k\,(a_0)\\ &= (f_i \wedge f_j)\,(a_1) \vee f_k\,(a_0)\,.\end{aligned}$$

Hence $\{f_i \wedge f_j, f_k\}$ satisfies the conditions, similarly for $\{f_i \vee f_j, f_k\}$. Using induction on the complexity of σ, one then obtains that any two elements of $\mathbf{R}$ are $\mathcal{V}$-homomorphisms and satisfy the conditions. Defining $F\colon R \to B^A$ by $F(r) = r$ for $r \in R$, one can see as in the proof of Theorem 4.2 that $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation.

Finally, suppose that $\{f_i, f_j\}$ and $\{f_j, f_k\}$ are each compatible. From this one can verify directly for any $a_0, a_1 \in A$ that

$$\left(f_i\left(a_0\right) \wedge f_k\left(a_1\right)\right) \wedge f_j\left(a_0 \vee a_1\right) = \left(f_i\left(a_1\right) \wedge f_k\left(a_0\right)\right) \wedge f_j\left(a_0 \vee a_1\right).$$

Also, one can check that

$$\begin{aligned} f_i\left(a_0\right) \vee f_j\left(a_0 \vee a_1\right) &= f_i\left(a_1\right) \vee f_j\left(a_0 \vee a_1\right) \\ f_k\left(a_1\right) \vee f_j\left(a_0 \vee a_1\right) &= f_k\left(a_0\right) \vee f_j\left(a_0 \vee a_1\right). \end{aligned}$$

Using distributivity it follows that

$$\left(f_i\left(a_0\right) \wedge f_k\left(a_1\right)\right) \vee f_j\left(a_0 \vee a_1\right) = \left(f_i\left(a_1\right) \wedge f_k\left(a_0\right)\right) \vee f_j\left(a_0 \vee a_1\right).$$

As the meet, as well as the join of $f_i\left(a_0\right) \wedge f_k\left(a_1\right)$ with $f_j\left(a_0 \vee a_1\right)$ is the same as that of $f_i\left(a_1\right) \wedge f_k\left(a_0\right)$, distributivity in $\mathbf{B}$ gives that they must be equal. Similarly, one can verify that

$$f_i\left(a_0\right) \vee f_k\left(a_1\right) = f_i\left(a_1\right) \vee f_k\left(a_0\right).$$

Hence $\{f_i, f_k\}$ is compatible. $\square$

Hence, once again a collection of formable functions $\{f_i \mid i \in I\}$ is compatible if and only if each pair $\{f_i, f_j\}$ is compatible. Also note that for distributive lattices $\mathbf{A}$ and $\mathbf{B}$ Theorem 4.3 gives that the relation of compatibility of two formable functions $f_i, f_j \colon A \to B$ is transitive. Combined with the comments proceeding Theorem 4.2 one sees that this transitivity holds for varieties of lattices if and only if the variety consists only of distributive lattices. The $\mathcal{V}$-homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ are partitioned into disjoint classes, each consisting of those homomorphisms that together are compatible and itself having the structure of a distributive lattice $\mathbf{R}$ when $\wedge$ and $\vee$ are defined by

$$\begin{aligned} \left(f_i \wedge f_j\right)(a) &= f_i(a) \wedge f_j(a) \\ \left(f_i \vee f_j\right)(a) &= f_i(a) \vee f_j(a) \end{aligned}$$

for $a \in A$ and $\{f_i, f_j\}$ compatible.

Similarly, for the variety $\mathcal{V}$ of boolean algebras a function $f \colon A \to B$ is formable if and only if $f \colon \langle A, \wedge, \vee\rangle \to \langle B, \wedge, \vee\rangle$ is a lattice-homomorphism. Moreover, a collection of formable functions $\{f_i \mid i \in I\}$ is compatible if and only if for every $i, j \in I$ and every $a_0, a_1 \in A$ one has that f_i, f_j satisfy the conditions stated in Theorem 4.3.

Theorem 4.4. *Let $\mathcal{V}$ be a variety of groups, and let $\mathbf{A}, \mathbf{B}$ be $\mathcal{V}$-algebras. A function $f \colon A \to B$ is formable if and only if for every $a_0, a_1, a_2 \in A$ one has*

$$f\left(a_0 a_1\right) f\left(a_2\right) = f\left(a_0\right) f\left(a_1 a_2\right).$$

Moreover, a collection of formable functions $\{f_i \mid i \in I\}$ *is compatible if and only if for every* $i, j \in I$ *and every* $a \in A$ *one has*

$$f_i(1)^{-1} f_i(a) = f_j(1)^{-1} f_j(a).$$

PROOF: If $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation with underlying algebra $\mathbf{R}$ then for any $r \in R$ and $a_0, a_1, a_2 \in A$ one has

$$F(r)(a_0 a_1) F(r)(a_2) = F(rr)(a_0 a_1 a_2) = F(r)(a_0) F(r)(a_1 a_2).$$

Hence any formable function satisfies the stated condition. Furthermore, for any $r_0, r_1 \in R$ and $a \in A$ one obtains

$$\begin{aligned}
\left(F(r_0)(1)\right)^{-1} F(r_0)(a) &= F\left(r_0^{-1}\right)\left(1^{-1}\right) F(r_0)(a) \\
&= F(1)(a) \\
&= F\left(r_1^{-1}\right)\left(1^{-1}\right) F(r_1)(a) \\
&= \left(F(r_1)(1)\right)^{-1} F(r_1)(a).
\end{aligned}$$

Now, suppose $f\colon A \to B$ is a function satisfying the stated condition. One then has in particular for any $a \in A$ that

$$f(a)f(1) = f(1a)f(1) = f(1)f(a1) = f(1)f(a),$$

and hence $f(1)^{-1}f(a) = f(a)f(1)^{-1}$. Define $h\colon A \to B$ by $h(a) = f(1)^{-1}f(a)$, for $a \in A$. Then for any $a_0, a_1 \in A$ one has

$$\begin{aligned}
h(a_0 a_1) &= f(1)^{-1} f(a_0 a_1) \\
&= f(1)^{-1} f(a_0 a_1) f(1) f(1)^{-1} \\
&= f(1)^{-1} f(a_0) f(a_1) f(1)^{-1} \\
&= h(a_0) h(a_1),
\end{aligned}$$

so that $h\colon \mathbf{A} \to \mathbf{B}$ is a $\mathcal{V}$-homomorphism. Let

$$R = \{r \in B \mid rh(a) = h(a)r \text{ for every } a \in A\},$$

the centraliser of $h(A)$ in B, which is a subgroup of $\mathbf{B}$. Note that for any $a \in A$

$$f(1)h(a) = f(1)f(1)^{-1}f(a) = f(1)^{-1}f(1)f(a) = f(1)^{-1}f(a)f(1) = h(a)f(1),$$

so that $f(1) \in R$. Define $F\colon R \to B^A$ by $F(r)(a) = rh(a)$, for $r \in R$ and $a \in A$. Clearly, $F\big(f(1)\big) = f$. For any $r_0, r_1 \in R$ and $a_0, a_1 \in A$ one has

$$\begin{aligned}
F(r_0 r_1)(a_0 a_1) &= r_0 r_1 h(a_0 a_1) \\
&= r_0 r_1 h(a_0) h(a_1) \\
&= r_0 h(a_0) r_1 h(a_1) \\
&= F(r_0)(a_0) F(r_1)(a_1),
\end{aligned}$$

and hence $F\colon \mathbf{R}\times\mathbf{A} \to \mathbf{B}$ is a $\mathcal{V}$-homomorphism, giving that $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation. Thus f is formable.

Note that $F\colon R \to B^A$ is injective, so that its image, consisting of compatible formable functions itself inherits a group structure. Moreover, if $\{f_i \mid i \in I\}$ is a collection of formable functions for which

$$f_i(1)^{-1}f_i(a) = f_j(1)^{-1}f_j(a),$$

for every $i, j \in I$ and $a \in A$, then the above gives the same homomorphism h and the same formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ starting with any choice of f_i for $i \in I$, and so $\{f_i \mid i \in I\}$ is compatible. $\square$

Clearly then for a variety of groups the relation of compatibility of two formable functions $f_i, f_j\colon A \to B$ is transitive. Each class has the structure of a group $\mathbf{R}$ in the variety whose identity $1\colon A \to B$ is the unique $\mathcal{V}$-homomorphism in the class, which is given by $1(a) = f(1)^{-1}f(a)$, for $a \in A$ and any formable $f\colon A \to B$ in the class (and hence $f(a) = f(1)1(a)$). The multiplication $\cdot$ and inverse ${}^{-1}$ operations in $\mathbf{R}$ are given by

$$(f_i \cdot f_j)(a) = f_i(1)f_j(a)$$
$$\left(f^{-1}\right)(a) = f\left(a^{-1}\right)^{-1}$$

for $a \in A$.

Theorem 4.5. *Let $\mathcal{V}$ be a variety of rings with unity, and let $\mathbf{A}, \mathbf{B}$ be $\mathcal{V}$-algebras. A function $f\colon A \to B$ is formable if and only if for every $a_0, a_1, a_2 \in A$ one has*

$$f(a_0 + a_1) + f(a_2) = f(a_0) + f(a_1 + a_2)$$
$$f(a_0a_1)\,f(a_2) = f(a_0)\,f(a_1a_2)$$
$$f(1)\,f(a_0) + f(a_1) = f(a_0) + f(1)\,f(a_1)\,.$$

Moreover, a collection of formable functions $\{f_i \mid i \in I\}$ is compatible if and only if for every $i, j \in I$ and every $a \in A$ one has

$$f_i(a) - f_i(0) = f_j(a) - f_j(0).$$

Proof: This theorem can be verified using a similar argument to that in Theorem 4.4. In particular, if $f\colon A \to B$ is a function that satisfies the three stated conditions, one can define a function $h\colon A \to B$ by $h(a) = f(a) - f(0)$, for $a \in A$. One can then verify for any $a_0, a_1 \in A$ that $h(a_0 + a_1) = h(a_0) + h(a_1)$. Moreover, using the fact that

$$f(a_0a_1) - f(0) = f(1)f(a_0a_1) - f(1)f(0),$$

one also obtains $h(a_0a_1) = h(a_0)\,h(a_1)$. Let

$$R = \left\{r \in B \mid rh(a) = 0 = h(a)r \text{ for every } a \in A\right\},$$

the annihilator of $h(A)$ in B. Define addition, subtraction, and multiplication operations on R to be the restrictions to R of those of $\mathbf{B}$, and unity taken to be $1 - h(1)$. By induction on the complexity of a term σ of $\mathrm{Clo}\,\mathcal{V}$ it can be verified that

$$\sigma^R(r_0,\ldots,r_{n-1}) = \sigma^B(r_0,\ldots,r_{n-1}) - h\left(\sigma^A(0,\ldots,0)\right).$$

Hence $\mathbf{R} = \langle R, \sigma^R \mid \sigma \in \mathrm{Clo}\,\mathcal{V}\rangle$ is a $\mathcal{V}$-algebra (as it satisfies any equation that is satisfied by both $\mathbf{A}$ and $\mathbf{B}$). Define $F\colon R \to B^A$ by $F(r)(a) = r + h(a)$, for $r \in R$ and $a \in A$. Clearly, $F\colon R \to B^A$ is injective, $f(0) \in R$ with $F\big(f(0)\big) = f$, and it is not difficult to verify that $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation. ∎

Here, each class of compatible functions has the structure of a ring $\mathbf{R}$ in the variety $\mathcal{V}$, where $0, 1\colon A \to B$ are given by

$$\begin{aligned} 0(a) &= f(a) - f(0) \\ 1(a) &= 0\,(a-1) + 1, \end{aligned}$$

for $a \in A$ and any formable $f\colon A \to B$ in the class. Note that 0 is the unique 'ring-homomorphism' in the class, in the sense that it respects the addition and multiplication operations; it is not necessarily a $\mathcal{V}$-homomorphism as it is not required that $0(1) = 1$. Indeed, one has $0(1) = 1$ if and only if $1 = 0$ in $\mathbf{R}$ if and only if $\mathbf{R}$ is the zero ring $\{0\}$.

Contrast the result for rings with the earlier results. For semilattices every formation consists solely of $\mathcal{V}$-homomorphisms, for groups every formation contains a unique $\mathcal{V}$-homomorphism, whereas for rings with unity a formation can contain at most one $\mathcal{V}$-homomorphism.

THEOREM 4.6. *Let $\mathbf{D}$ be a ring with unity, $\mathcal{V}$ be a variety of $\mathbf{D}$-modules, and let $\mathbf{A}, \mathbf{B}$ be $\mathcal{V}$-algebras. A function $f\colon A \to B$ is formable if and only if for every $a_0, a_1, a_2 \in A$, and every $d \in D$ one has*

$$\begin{aligned} f(a_0 + a_1) + f(a_2) &= f(a_0) + f(a_1 + a_2) \\ f(d \cdot a_0) + d \cdot f(a_1) &= d \cdot f(a_0) + f(d \cdot a_1). \end{aligned}$$

Moreover, a collection of formable functions $\{f_i \mid i \in I\}$ is compatible if and only if for every $i, j \in I$ and every $a \in A$ one has

$$f_i(a) - f_i(0) = f_j(a) - f_j(0).$$

Theorem 4.6 can be proved in a similar manner as for the previous theorems. In fact the situation in this case is even simpler: for a variety of $\mathbf{D}$-modules each class of compatible functions has the structure of a $\mathbf{D}$-module $\mathbf{R}$ isomorphic to $\mathbf{B}$. The zero 0, and scalar multiplication $d \cdot f$ for any $d \in D$ are given by

$$\begin{aligned} 0(a) &= f(a) - f(0) \\ (d \cdot f)(a) &= 0(a) + d \cdot f(0), \end{aligned}$$

for $a \in A$ and $f\colon A \to B$ in the class.

A variety of *implication algebras* has one basic binary operation $\to$ that obeys the following identities:

$$\begin{aligned}(x \to y) &\to x \approx x\\ (x \to y) &\to y \approx (y \to x) \to x\\ x \to (y \to z) &\approx y \to (x \to z)\,.\end{aligned}$$

Such varieties are important algebraic tools in logic, and are extensively discussed in for example [7]. From these identities the following can be deduced:

$$\begin{gathered}x \to x \approx y \to y\\ (x \to y) \to (x \to z) \approx x \to (y \to z)\\ x \to \big((x \to y) \to z\big) \approx y \to (x \to z)\\ x \to (x \to y) \approx x \to y.\end{gathered}$$

Hence, denoting the constant $x \to x$ by 1, one has $x \to 1 \approx 1$ and $1 \to x \approx x$.

THEOREM 4.7. *Let $\mathcal{V}$ be a variety of implication algebras, and let $\mathbf{A}, \mathbf{B}$ be $\mathcal{V}$-algebras. A function $f\colon A \to B$ is formable if and only if for every $a_0, a_1, a_2 \in A$ one has*

$$f\big((a_0 \to a_1) \to a_2\big) = \big(f(a_0) \to f(a_1)\big) \to f(a_2)\,.$$

Moreover, a collection of formable functions $\{f_i \mid i \in I\}$ is compatible if and only if for every $i, j \in I$ and every $a \in A$ one has

$$f_i(1) \to f_i(a) = f_j(1) \to f_j(a).$$

PROOF: Starting with a function $f\colon A \to B$ that satisfies the stated condition, define a function $h\colon A \to B$ by $h(a) = f(1) \to f(a)$, for $a \in A$. Let $R \subseteq B^A$ consist of those functions $g\colon A \to B$ for which

$$g\big((a_0 \to a_1) \to a_2\big) = \big(g(a_0) \to g(a_1)\big) \to g(a_2)\,,$$

for every $a_0, a_1, a_2 \in A$, and for which

$$g(1) \to g(a) = h(a),$$

for every $a \in A$. Note in particular that $f \in R$. Define $\to$ on R by

$$(g_0 \to g_1)(a) = g_0(1) \to g_1(a),$$

for $g_0, g_1 \in R$ and $a \in A$. To see that $g_0 \to g_1 \in R$ one has that

$$\begin{aligned}(g_0 \to g_1)((a_0 \to a_1) \to a_2) &= g_0(1) \to g_1((a_0 \to a_1) \to a_2) \\ &= g_0(1) \to \big((g_1(a_0) \to g_1(a_1)) \to g_1(a_2)\big) \\ &= \big(g_0(1) \to (g_1(a_0) \to g_1(a_1))\big) \to (g_0(1) \to g_1(a_2)) \\ &= \big((g_0 \to g_1)(a_0) \to (g_0 \to g_1)(a_1)\big) \to (g_0 \to g_1)(a_2).\end{aligned}$$

Also, one has that

$$\begin{aligned}(g_0 \to g_1)(1) \to (g_0 \to g_1)(a) &= (g_0(1) \to g_1(1)) \to (g_0(1) \to g_1(a)) \\ &= g_0(1) \to (g_1(1) \to g_1(a)) \\ &= g_0(1) \to (g_0(1) \to g_0(a)) \\ &= h(a).\end{aligned}$$

Hence $g_0 \to g_1 \in R$. Furthermore, as

$$\begin{aligned}g_0(a_0) \to g_0(1) &= \big((g_0(a_0) \to g_0(1)) \to g_0(1)\big) \to (g_0(a_0) \to g_0(1)) \\ &= g_0(1) \to (g_0(a_0) \to g_0(1)) \\ &= 1,\end{aligned}$$

one has that

$$\begin{aligned}(g_0 \to g_1)(a_0 \to a_1) &= g_0(1) \to g_1(a_0 \to a_1) \\ &= g_0(1) \to \big((g_1(1) \to g_1(a_0)) \to g_1(a_1)\big) \\ &= g_0(1) \to \big((g_0(1) \to g_0(a_0)) \to g_1(a_1)\big) \\ &= g_0(a_0) \to (g_0(1) \to g_1(a_1)) \\ &= g_0(a_0) \to g_1(a_1).\end{aligned}$$

Next, note that if $g_0(1) = g_1(1)$ then for any $a \in A$ one has

$$\begin{aligned}g_0(a) &= (g_1(1) \to g_0(1)) \to g_0(a) \\ &= (g_1(a) \to g_0(a)) \to g_0(a) \\ &= (g_0(a) \to g_1(a)) \to g_1(a) \\ &= (g_0(1) \to g_1(1)) \to g_1(a) \\ &= g_1(a),\end{aligned}$$

and so $g_0 = g_1$. Moreover, for any term σ of $\operatorname{Clo}\mathcal{V}$ and any $g_0, \ldots, g_{n-1} \in R$ it is straight-forward to verify that

$$\sigma^R(g_0, \ldots, g_{n-1})(1) = \sigma^B(g_0(1), \ldots, g_{n-1}(1)).$$

Hence, $\mathbf{R} = \langle R, \sigma^R \mid \sigma \in \operatorname{Clo}\mathcal{V}\rangle$ satisfies the equational theory of $\mathbf{B}$, and so is a $\mathcal{V}$-algebra. Defining $F\colon R \to B^A$ to be the inclusion map, one easily sees that $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a $\mathcal{V}$-formation.

The remaining statements in the theorem can be easily verified as before. ▯

The interest in this result is that varieties of implication algebras, although congruence three-permutable, are not congruence permutable. Theorem 4.7 however shows that they behave much like varieties of groups, rings, and modules with respect to formations, in that the formable functions are partitioned into disjoint classes, whereas such a partitioning is not obtained for non-distributive lattices. Each class has the structure of an implication algebra $\mathbf{R}$ in the variety, where $\to$ is given by

$$(f_i \to f_j)\,(a) = f_i(1) \to f_j(a),$$

for $a \in A$ and $\{f_i, f_j\}$ compatible. Furthermore, $1\colon \mathbf{A} \to \mathbf{B}$ is the unique homomorphism in the class.

One final point should be noted for any variety $\mathcal{V}$ that has been considered in this section. Starting with a $\mathcal{V}$-formation $\mathbf{G}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ with underlying algebra $\mathbf{S}$, it has been shown for the compatible functions $\{G(s) \mid s \in S\}$ how to construct a $\mathcal{V}$-algebra $\mathbf{R}$ using these functions as the elements. Then defining $F\colon R \to B^A$ to be the inclusion map, one obtains the $\mathcal{V}$-formation $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ with underlying algebra $\mathbf{R}$. The relationship between $\mathbf{G}$ and $\mathbf{F}$ is as follows: there exists a unique $\mathcal{V}$-homomorphism $\mathbf{K}\colon \mathbf{S} \to \mathbf{R}$ (namely $K(s) = G(s)$ for $s \in S$) for which $\mathbf{G} = \mathbf{F}\mathbf{K}$. Hence in studying $\mathcal{V}$-formations $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ one can restrict one's attention to cases when the underlying algebra $\mathbf{R}$ has the compatible functions as elements, and F is simply the inclusion map. This can in fact be done if and only if the kernel of the function $G\colon S \to B^A$ is a congruence of $\mathbf{S}$ – such is the case for example if for every basic operation σ of $\mathcal{V}$, either σ^S is constant or σ^A is surjective.

5. Congruence Modular Varieties

A variety $\mathcal{V}$ of algebras, that is, a class of similar algebras defined by equations is termed *congruence modular* if the congruence lattice of each algebra in the variety is modular. The theory of congruence modular varieties has been extensively developed in universal algebra, see for example [**3**], and they are considered one of the principally-studied branches of universal algebra. Examples include varieties of groups, rings, modules, lattices, boolean algebras, and implication algebras. Although this includes a wide assortment of varieties, formations for congruence modular varieties will be shown to have some rather rich properties.

Definition 5.1: The *TC-centre* of an algebra $\mathbf{A}$ is the congruence relation $Z(\mathbf{A})$ defined on $\mathbf{A}$ by

$$\langle a, b\rangle \in Z(\mathbf{A})$$

if and only if for every $n \geqslant 1$, for all $c_1, d_1, \ldots, c_{n-1}, d_{n-1} \in A$, for every n-ary term σ

$$\begin{aligned} & \sigma^A(a, c_1, \ldots, c_{n-1}) = \sigma^A(a, d_1, \ldots, d_{n-1}) \\ \leftrightarrow \quad & \sigma^A(b, c_1, \ldots, c_{n-1}) = \sigma^A(b, d_1, \ldots, d_{n-1}). \end{aligned}$$

Firstly, two preliminary results are necessary. Both are well-known results that are easy consequences of work due to Herrmann [5] and Gumm [4].

THEOREM 5.2. *Let $\mathcal{V}$ be a congruence modular variety. There is a ternary term d in* Clo $\mathcal{V}$, *called a* Gumm difference term, *so that for any $\mathcal{V}$-algebra $\mathbf{A}$ and any $a, b \in A$ one has $d^A(a, a, b) = b$. If $\langle a, b\rangle \in Z(\mathbf{A})$ then also $d^A(a, b, b) = a$.*

Moreover, for any n-ary term σ in Clo $\mathcal{V}$, *any $\mathcal{V}$-algebra $\mathbf{A}$, and any $a_i, b_i, c_i \in A$ for $i = 0, \ldots, n-1$ with each $\langle a_i, b_i\rangle \in Z(\mathbf{A})$, one has*

$$\begin{aligned} & d^A\left(\sigma^A(a_0, \ldots, a_{n-1}), \sigma^A(b_0, \ldots, b_{n-1}), \sigma^A(c_0, \ldots, c_{n-1})\right) \\ & \qquad = \sigma^A\left(d^A(a_0, b_0, c_0), \ldots, d^A(a_{n-1}, b_{n-1}, c_{n-1})\right). \end{aligned}$$

THEOREM 5.3. *A variety $\mathcal{V}$ is congruence modular if and only if for some $n \geqslant 0$ there are ternary terms $d_1, \ldots, d_n, d$ in* Clo $\mathcal{V}$ *such that for any $\mathcal{V}$-algebra $\mathbf{A}$ and any $a, b \in A$ the following hold:*

$$\begin{aligned} a &= d_1^A(a, b, b) && \\ d_i^A(a, b, a) &= a && \text{for } 1 \leqslant i \leqslant n \\ d_i^A(a, b, b) &= d_{i+1}^A(a, b, b) && \text{for even } i < n \\ d_i^A(a, a, b) &= d_{i+1}^A(a, a, b) && \text{for odd } i < n \\ d_n^A(a, b, b) &= d^A(a, b, b) && \\ d^A(a, a, b) &= b. && \end{aligned}$$

Moreover, the term d in Theorem 5.3 is in fact a Gumm difference term. Conversely, for congruence modular varieties any Gumm difference term can be taken as the term d.

The following lemma is key to the results obtained in this section. It displays a strong relationship between compatible functions and the congruences on an algebra.

LEMMA 5.4. *Let $\mathcal{V}$ be a congruence modular variety with a Gumm difference term d, and let $\mathbf{A}$, $\mathbf{B}$ be $\mathcal{V}$-algebras. Suppose that $f_0, f_1 \colon A \to B$ are each formable, and that $\{f_0, f_1\}$ is compatible.*

For any congruence relation φ on $\mathbf{B}$, and $a_0, a_1 \in A$ one has

$$\langle f_0(a_0), f_0(a_1)\rangle \in \varphi \quad \leftrightarrow \quad \langle f_1(a_0), f_1(a_1)\rangle \in \varphi.$$

Furthermore, for any congruence relation φ on $\mathbf{B}$, if there exist $a_0, a_1 \in A$ for which $\langle f_0(a_0), f_1(a_1)\rangle \in \varphi$, then for any $a \in A$ one also has

$$\langle f_0(a), f_1(d^A(a_1, a_0, a))\rangle \in \varphi.$$

PROOF: Let $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ be a $\mathcal{V}$-formation with underlying algebra $\mathbf{R}$, for which there are $r_0, r_1 \in R$ with $F(r_0) = f_0$ and $F(r_1) = f_1$.

Firstly, suppose that there are $a_0, a_1 \in A$ for which $\langle f_0(a_0), f_0(a_1) \rangle \in \varphi$. By induction it can be seen that $\langle f_1(a_0), f_1(d_i^A(a_0, a_1, a_1)) \rangle \in \varphi$ for all $i \leqslant n$. Indeed $d_1^A(a_0, a_1, a_1) = a_0$. If $\langle f_1(a_0), f_1(d_i^A(a_0, a_1, a_1)) \rangle \in \varphi$ for some $i < n$ then if i is even one has $d_i^A(a_0, a_1, a_1) = d_{i+1}^A(a_0, a_1, a_1)$; whereas if i is odd then

$$\begin{aligned} f_1(a_0)\ \varphi\ f_1\left(d_i^A(a_0, a_1, a_1)\right) &= F\left(d_i^R(r_1, r_0, r_1)\right)\left(d_i^A(a_0, a_1, a_1)\right) \\ &= d_i^B\left(F(r_1)(a_0), F(r_0)(a_1), F(r_1)(a_1)\right) \\ &\varphi\ d_i^B\left(F(r_1)(a_0), F(r_0)(a_0), F(r_1)(a_1)\right) \\ &= F\left(d_i^R(r_1, r_0, r_1)\right)\left(d_i^A(a_0, a_0, a_1)\right) \\ &= F\left(d_{i+1}^R(r_1, r_0, r_1)\right)\left(d_{i+1}^A(a_0, a_0, a_1)\right) \\ &= d_{i+1}^B\left(F(r_1)(a_0), F(r_0)(a_0), F(r_1)(a_1)\right) \\ &\varphi\ d_{i+1}^B\left(F(r_1)(a_0), F(r_0)(a_1), F(r_1)(a_1)\right) \\ &= F\left(d_{i+1}^R(r_1, r_0, r_1)\right)\left(d_{i+1}^A(a_0, a_1, a_1)\right) \\ &= f_1\left(d_{i+1}^A(a_0, a_1, a_1)\right). \end{aligned}$$

In particular,

$$\begin{aligned} f_1(a_0)\ \varphi\ f_1\left(d_n^A(a_0, a_1, a_1)\right) &= F\left(d^R(r_0, r_0, r_1)\right)\left(d^A(a_0, a_1, a_1)\right) \\ &= d^B\left(F(r_0)(a_0), F(r_0)(a_1), F(r_1)(a_1)\right) \\ &\varphi\ d^B\left(F(r_0)(a_0), F(r_0)(a_0), F(r_1)(a_1)\right) \\ &= f_1(a_1). \end{aligned}$$

Next, suppose instead that there exist $a_0, a_1 \in A$ for which $\langle f_0(a_0), f_1(a_1) \rangle \in \varphi$, and let $a \in A$. By a similar induction argument to that above, it can be seen that $\langle f_0(a), F\left(d_n^R(r_0, r_1, r_1)\right)(a) \rangle \in \varphi$. Hence, one obtains

$$\begin{aligned} & f_0(a)\ \varphi F\left(d^R(r_0, r_1, r_1)\right)(a) \\ &\quad = d^B\left(F(r_0)(a_0), F(r_1)(a_0), F(r_1)(a)\right) \\ &\quad \varphi\, d^B\left(F(r_1)(a_1), F(r_1)(a_0), F(r_1)(a)\right) \\ &\quad = f_1\left(d^A(a_1, a_0, a)\right). \end{aligned}$$

$\square$

It is clear that in general a formable function $f\colon A \to B$ need not be a $\mathcal{V}$-homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ (indeed in some cases there might not even exist a $\mathcal{V}$-homomorphism that is compatible with f). However, if $\mathcal{V}$ is congruence modular then, by using the kernel $\ker f$ of f, the image $\operatorname{im} f$ of f inherits a quotient structure from $\mathbf{A}$, giving the following weaker result:

THEOREM 5.5. *Let $\mathcal{V}$ be a congruence modular variety, and let $\mathbf{A}$, $\mathbf{B}$ be $\mathcal{V}$-algebras. Suppose that $f: A \to B$ is formable. Then $\ker f$ is a congruence on $\mathbf{A}$.*

For every σ of $\operatorname{Clo}\mathcal{V}$ let $\sigma^{\operatorname{im} f}$ be the operation on $\operatorname{im} f$ defined by

$$\sigma^{\operatorname{im} f}\left(f\left(a_0\right), \ldots, f\left(a_{n-1}\right)\right) = f\left(\sigma^A\left(a_0, \ldots, a_{n-1}\right)\right)$$

for any $a_0, \ldots, a_{n-1} \in A$.

Then $\mathbf{im}\, f = \langle \operatorname{im} f, \sigma^{\operatorname{im} f} \mid \sigma \in \operatorname{Clo}\mathcal{V}\rangle$ is a $\mathcal{V}$-algebra, and $f: \mathbf{A} \to \mathbf{im}\, f$ is a surjective $\mathcal{V}$-homomorphism. Furthermore, $\mathbf{im}\, f$ is a subalgebra of $\mathbf{B}$ if and only if $f: \mathbf{A} \to \mathbf{B}$ is a $\mathcal{V}$-homomorphism.

PROOF: Let $\mathbf{F}: \mathbf{A} \rightrightarrows \mathbf{B}$ be a $\mathcal{V}$-formation with an element $r \in R$ in the underlying algebra for which $f = F(r)$. Let σ be an n-ary term of $\operatorname{Clo}\mathcal{V}$, and $\langle a_0, a_0'\rangle, \ldots, \langle a_{n-1}, a_{n-1}'\rangle \in \ker f$, so that $F(r)\left(a_i\right) = F(r)\left(a_i'\right)$ for $i < n$. Then

$$\begin{aligned} F\left(\sigma^R(r, \ldots, r)\right)\left(\sigma^A\left(a_0, \ldots, a_{n-1}\right)\right) &= \sigma^B\left(F(r)\left(a_0\right), \ldots, F(r)\left(a_{n-1}\right)\right) \\ &= \sigma^B\left(F(r)\left(a_0'\right), \ldots, F(r)\left(a_{n-1}'\right)\right) \\ &= F\left(\sigma^R(r, \ldots, r)\right)\left(\sigma^A\left(a_0', \ldots, a_{n-1}'\right)\right). \end{aligned}$$

Hence by Lemma 5.4 one has $F(r)\left(\sigma^A\left(a_0, \ldots, a_{n-1}\right)\right) = F(r)\left(\sigma^A\left(a_0', \ldots, a_{n-1}'\right)\right)$, and so $\ker f$ is a congruence on $\mathbf{A}$. It follows immediately that each $\sigma^{\operatorname{im} f}$ is a well-defined operation on $\operatorname{im} f$, that $f: \mathbf{A} \to \mathbf{im}\, f$ is a surjective homomorphism, and that $\mathbf{im}\, f$ is isomorphic to the quotient $\mathbf{A}/\ker f$, a $\mathcal{V}$-algebra.

Finally, it is clear that $\sigma^{\operatorname{im} f}$ is the restriction of σ^B to $\operatorname{im} f$ if and only if $f: \mathbf{A} \to \mathbf{B}$ is a $\mathcal{V}$-homomorphism. ∎

The next result displays a rather surprising relationship between compatible functions for congruence modular varieties.

THEOREM 5.6. *Let $\mathcal{V}$ be a congruence modular variety, and let $\mathbf{A}$, $\mathbf{B}$ be $\mathcal{V}$-algebras. Suppose that $f_0, f_1: A \to B$ are each formable, and that $\{f_0, f_1\}$ is compatible.*

Then $\ker f_0 = \ker f_1$, so that the algebras $\mathbf{im}\, f_0$ and $\mathbf{im}\, f_1$ are isomorphic. Also, if $\operatorname{im} f_0 \cap \operatorname{im} f_1 \neq \emptyset$ then as sets $\operatorname{im} f_0 = \operatorname{im} f_1$.

PROOF: The fact that $\ker f_0 = \ker f_1$ follows from Lemma 5.4 by considering the trivial congruence $0_{\operatorname{Con}\mathbf{B}}$ of $\mathbf{B}$. So then $\mathbf{im}\, f_0$ is isomorphic to $\mathbf{A}/\ker f_0 = \mathbf{A}/\ker f_1$, which is isomorphic to $\mathbf{im}\, f_1$.

Next, suppose that there exist $a_0, a_1 \in A$ with $f_0\left(a_0\right) = f_1\left(a_1\right)$. Again considering the trivial congruence $0_{\operatorname{Con}\mathbf{B}}$ of $\mathbf{B}$ and applying Lemma 5.4 one obtains for any $a \in A$ that $f_0(a) = f_1\left(d^A\left(a_1, a_0, a\right)\right)$. Hence, $\operatorname{im} f_0 \subseteq \operatorname{im} f_1$; likewise one can obtain $\operatorname{im} f_1 \subseteq \operatorname{im} f_0$, and so these sets are equal. ∎

COROLLARY 5.7. *Let $\mathcal{V}$ be a congruence modular variety, and let $\mathbf{F}: \mathbf{A} \rightrightarrows \mathbf{B}$ be a $\mathcal{V}$-formation. Then $\Theta_{\mathbf{F}}$ is a uniform congruence on the subalgebra $\mathbf{im}\, F$ of $\mathbf{B}$, and for any $r \in R$, $\operatorname{im} F(r)$ is a block of $\Theta_{\mathbf{F}}$.*

PROOF: From Theorem 5.6 it is clear for any $r, r' \in R$ that $\operatorname{im} F(r)$ and $\operatorname{im} F(r')$ are either equal or disjoint, and they have the same cardinality. Hence the tolerance on $\mathbf{im}\, F$ formed by identifying the elements within each $\operatorname{im} F(r)$ is actually the congruence $\Theta_{\mathbf{F}}$. ∎

Besides enabling counting arguments on $\mathbf{im}\, F$, $\Theta_{\mathbf{F}}$ is also useful in the analysis of an arbitrary congruence relation φ of $\mathbf{B}$. For example, the behaviour of φ on the individual blocks of $\Theta_{\mathbf{F}}$ is determined by its behaviour on any chosen block, as shown by the following theorem, which is an immediate consequence of Lemma 5.4.

COROLLARY 5.8. *Let $\mathcal{V}$ be a congruence modular variety, and let $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ be a $\mathcal{V}$-formation with underlying algebra $\mathbf{R}$.*

For any congruence relation φ on $\mathbf{B}$ if there are $a_0, a_1 \in A$ and $r_0 \in R$ for which $\left\langle F(r_0)(a_0), F(r_0)(a_1) \right\rangle \in \varphi$ then for any $r \in R$ one has $\left\langle F(r)(a_0), F(r)(a_1) \right\rangle \in \varphi$. Also, if there are $r_0, r_1 \in R$ and $a_0 \in A$ for which $\left\langle F(r_0)(a_0), F(r_1)(a_0) \right\rangle \in \varphi$ then for any $a \in A$ one has $\left\langle F(r_0)(a), F(r_1)(a) \right\rangle \in \varphi$.

To illustrate these ideas consider a congruence modular variety $\mathcal{V}$ with Gumm difference term d, and let $\mathbf{B}$ be a $\mathcal{V}$-algebra with $\mathbf{A}$ a subalgebra of $\mathbf{B}$. Take $\Delta(\mathbf{B})$ to be the congruence on $Z(\mathbf{B})$ generated by the pairs $\langle\langle b_0, b_0\rangle, \langle b_1, b_1\rangle\rangle$ for $b_0, b_1 \in B$, and take $\mathbf{R} = Z(\mathbf{B})/\Delta(\mathbf{B})$. Define $F\colon R \times A \to B$ by $F\big(\langle b, b'\rangle/\Delta(\mathbf{B}), a\big) = d^B(b, b', a)$, for $\langle b, b'\rangle \in Z(\mathbf{B})$. Then it is not difficult to verify that $\mathbf{F}\colon \mathbf{A} \rightrightarrows \mathbf{B}$ is a well-defined formation (indeed, for example this is an immediate consequence of Theorem 2.4 of [**1**]). Clearly, for any $b \in B$, $F\big(\langle b, b\rangle/\Delta(\mathbf{B})\big)\colon A \to B$ is the inclusion map, so by Theorem 5.6 each $F(r)\colon A \to B$ is also injective, and by Corollary 5.7 A is a block of the uniform congruence $\Theta_{\mathbf{F}}$ on $\mathbf{im}\, F$. Now, one can easily verify that $\mathbf{im}\, F$ is the subalgebra of $\mathbf{B}$ consisting of the union of those blocks of $Z(\mathbf{B})$ that have non-empty intersection with $\mathbf{A}$. Taking any block $a/Z(\mathbf{B})$ of $Z(\mathbf{B})$ inside $\operatorname{im} F$, as $\big\langle F(r)(a'), a'\big\rangle \in Z(\mathbf{B})$ for any $r \in R$ and $a' \in A$, it is easily seen that $F(r)$ restricted to the block $a/Z(\mathbf{B})$ gives a bijection of $A \cap a/Z(\mathbf{B})$ with $F(r)(A) \cap a/Z(\mathbf{B})$. Hence one obtains the following result:

THEOREM 5.9. *Let $\mathcal{V}$ be a congruence modular variety, $\mathbf{B}$ be a $\mathcal{V}$-algebra, and $\mathbf{A}$ be a subalgebra of $\mathbf{B}$. Let $\mathbf{C}$ be the subalgebra of $\mathbf{B}$ consisting of the union of the blocks $a/Z(\mathbf{B})$ for $a \in A$.*

Then A is a block of a uniform congruence θ on $\mathbf{C}$ of index α say. Moreover, inside any block $a/Z(\mathbf{B})$, $A \cap a/Z(\mathbf{B})$ is a block of the uniform equivalence relation $\theta \cap (a/Z(\mathbf{B}) \times a/Z(\mathbf{B}))$ also of index α.

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