Can. J. Math., Vol. XXXII, No. 6, 1980, pp. 1288–1298

ON EXTENDING THE TRACE AS A LINEAR FUNCTIONAL, II

GEORGE A. ELLIOTT

1. Introduction. A positive bounded selfadjoint operator is in the trace class of von Neumann and Schatten ([**4**]) if the sum of its diagonal matrix elements with respect to some orthonormal basis is finite, and the trace is then defined to be this sum, which is independent of the basis. A bounded selfadjoint but not necessarily positive operator x is in the trace class if in the decomposition $x = x^+ - x^-$, with x^+ and x^- positive and $x^+x^- = 0$, both x^+ and x^- are in the trace class; the trace of x is then defined to be the difference of the finite traces of x^+ and x^-. The trace defined in this way is a linear functional on the trace class, and is unitarily invariant; if u is a unitary operator, the trace of uxu^{-1} is the same as the trace of x.

If the traces of x^+ and x^- are infinite, it still seems to be possible in some cases to assign a finite value to the difference of these infinite numbers, in such a way that it is natural to consider this as the trace of x.

In [**1**], there was constructed an extension of the trace, related to a fixed projection p, from the von Neumann–Schatten trace class to a certain algebra of Hilbert–Schmidt operators N_p, consisting of those Hilbert–Schmidt operators x such that the diagonal part with respect to p, $pxp + (1 - p)x(1 - p)$, belongs to the trace class. This extension, denoted by τ_p, is a linear functional on N_p which is unitarily invariant in the sense that if u is a unitary operator such that $uN_pu^{-1} = N_p$, then

$$\tau_p(uxu^{-1}) = \tau_p(x), \quad x \in N_p.$$

These properties determine the extension uniquely; necessarily

$$\tau_p(x) = \text{trace}\,(pxp + (1 - p)x(1 - p)), \, x \in N_p.$$

In the present paper it is shown that if p and q are two projections, then τ_p and τ_q agree on $N_p \cap N_q$, the intersection of their domains (3.1). In other words, τ_p may be written just as τ. It is shown by an example that it is not, however, possible to extend τ to be linear on the linear span

*Received May 17, 1978 and in revised form September 26, 1978. This work was done while the author was a Guest Scholar at the Research Institute for Mathematical Sciences, Kyoto University, partially supported by a grant from the Carlsberg Foundation. The final version was prepared while the author was a Visiting Professor in the Department of Pure Mathematics at the University of New South Wales supported by a special projects grant.

https://doi.org/10.4153/CJM-1980-098-x Published online by Cambridge University Press

of all the linear spaces N_p, p a projection (3.5). (Such an example, because of the above consistency, which allows a linear extension to $N_p + N_q$ for any two projections p and q, must involve at least three projections.)

There is not much difference among the linear extensions of the trace corresponding as above to different projections. Indeed, if the Hilbert space is separable, then any two such extensions, if proper, are unitarily equivalent, since the projections determining them are infinite and co-infinite, and so themselves unitarily equivalent. Somewhat greater variety is obtained by considering, for any operator $0 \leqq h \leqq 1$, the algebra N_h of those Hilbert–Schmidt operators x such that hxh and $\bar{h}x\bar{h}$ belong to the trace class, where $\bar{h} = (1 - h^2)^{1/2}$. Again there is a unique extension of the trace on the trace class to a unitarily invariant linear functional on N_h, given by

$$\tau_h(x) = \text{trace } (hxh + \bar{h}x\bar{h}),\ x \in N_h.$$

The proof of unitary invariance of τ_h (see Section 2) is more technical than in the case $h^2 = h$, which was handled by a simple algebraic computation in [**1**]; in particular the proof uses the closed graph theorem, applied to N_h with a suitable norm in which it is complete (see 2.7). The unitary invariance of τ_h also follows from the manifest unitary invariance of τ, the common extension of all the τ_p for different p (defined in the preceding paragraph), since τ in fact also extends τ_h (see Step 2 of 3.1). The very existence of τ, however, depends on a rather long argument, which makes up almost all of Section 3.

I am indebted to H. Araki and L.-E. Lundberg for helpful comments. I would like to emphasize that the origin of the ideas in this paper and its predecessor lies in the elegant theory of "second quantization" developed by Lundberg in [**2**] and [**3**], which, in particular, permitted him to make a precise mathematical analysis of vacuum polarization in quantum field theory, not requiring renormalization.

2. The extension of the trace to N_h, $0 \leqq h \leqq 1$.

2.1. In this section h denotes a fixed selfadjoint operator in a Hilbert space such that $0 \leqq h \leqq 1$, and N_h denotes the algebra of Hilbert–Schmidt operators x such that $hxh + \bar{h}x\bar{h}$ belongs to the trace class, where $\bar{h} = (1 - h^2)^{1/2}$. We shall denote the trace class by L_1, the Hilbert–Schmidt class by L_2, and the algebra of all bounded operators by L_∞. We shall use that L_∞ is identified with the dual of the Banach space L_1 via the duality between L_1 and L_∞ defined by the trace, and also that L_2 is identified with its own dual (see [**4**]).

2.2. THEOREM. *There exists a unique linear functional τ_h on N_h extending the trace on $L_1 \subset N_h$ and such that if u is a unitary operator with*

$uN_hu^{-1} = N_h$ *then:*

$$\tau_h(uxu^{-1}) = \tau_h(x),\ x \in N_h.$$

Proof of uniqueness. Since multiplication by h or $\bar{h}$ on the left or on the right leaves N_h invariant, this also holds for the unitaries $u_+ = h + i\bar{h}$ and $u_- = h - i\bar{h} = u_+^{-1}$. In view of the polarization identity

$$\tfrac{1}{2}(u_+xu_+^{-1} + u_-xu_-^{-1}) = hxh + \bar{h}x\bar{h},\ x \in N_h,$$

uniqueness of a linear extension of the trace on N_h invariant under transformation by the unitaries u_+ and u_- is clear; the value at x must be just the trace of $hxh + \bar{h}x\bar{h}$.

2.3. The rest of this section is devoted to proving that the linear functional τ_h on N_h defined by

$$\tau_h(x) = \text{trace } (hxh + \bar{h}x\bar{h}), \quad x \in N_h,$$

has the unitary invariance property stated in 2.2. The first step is to establish a weaker property, involving only those unitary operators u such that $uN_h + N_hu \subset N_h$, that is, which multiply N_h. This is done in 2.4 to 2.10. (What is stated in 2.5 is a little more than this.) In 2.11, the invariance property of 2.2 is established, with τ_h as defined here.

Later, in 3.1 (cf. 3.4), a property even stronger than that of 2.2 will be established, by completely different methods.

2.4. *Definition.* Denote by M_h the algebra of operators multiplying N_h, that is, the set

$$\{y \in L_\infty |\ yN_h + N_hy \subset N_h\}.$$

2.5. THEOREM. τ_h *as defined in* 2.3 *satisfies:*

$$\tau_h(yx) = \tau_h(xy),\ x \in N_h,\ y \in M_h.$$

2.6. LEMMA. *The unit ball of* L_1 *is weak*-closed in* L_∞.

Proof. This follows immediately from the equation

$$\|x\|_1 = \sup_{y \in L_1,\, \|y\|_\infty \leqq 1} |\text{trace}(xy)|,\ x \in L_\infty,$$

where $\|x\|_1 < \infty$ if and only if $x \in L_1$. This equation in turn follows from the facts that L_∞ is identified isometrically with the dual of L_1 by the duality between L_1 and L_∞ determined by the trace, and that the intersection of L_1 with the unit ball of L_∞ is weak*-dense in this ball (if (p_n) is a net of projections of finite rank increasing to 1, then for any $y \in L_\infty$, each $p_ny \in L_1$ and p_ny converges weak* to y).

2.7. LEMMA. N_h *is complete in the norm*

$$\|x\|_h = \|x\|_2 + \|hxh\|_1 + \|\bar{h}x\bar{h}\|_1.$$

Proof. Let (x_n) be a Cauchy sequence in N_h with respect to the above norm. Then in particular (x_n) is Cauchy in L_2 and so converges in L_2 to $x \in L_2$. Since $L_1 \subset L_2 \subset L_\infty$, x_n converges weak* to x in L_∞, and hence hx_nh and $\bar{h}x_n\bar{h}$ converge weak* to hxh and $\bar{h}x\bar{h}$ respectively. Since the sequence (x_n) is bounded in the norm on N_h, the sequences (hx_nh) and $(\bar{h}x_n\bar{h})$ are bounded in L_1, whence by 2.6 hxh and $\bar{h}x\bar{h}$ belong to L_1. This shows that x belongs to N_h.

Again by the weak* closure in L_∞ of balls in L_1,

$$\|h(x_n - x_m)h\|_1 \leqq \epsilon \quad \text{for almost all } m$$

implies

$$\|h(x_n - x)h\|_1 \leqq \epsilon,$$

and similarly with $\bar{h}$ in place of h. This shows that

$$\|x_n - x\|_h = \|x_n - x\|_2 + \|h(x_n - x)h\|_1 + \|\bar{h}(x_n - x)\bar{h}\|_1$$

converges to 0.

2.8. Corollary. *For each $y \in M_h$, the map*

$$N_h \ni x \mapsto yx \in N_h$$

is continuous in the norm defined in 2.7.

Proof. Because N_h is complete in this norm, it is enough to show that the map is closed.

Suppose that (x_n) is a sequence in N_h such that $\|x_n\|_h$ converges to 0 and such that for some $y \in L_\infty$ and some $z \in N_h$, each $yx_n \in N_h$, and $\|yx_n - z\|_h$ converges to 0. Then in particular x_n and $yx_n - z$ converge to 0 in L_2, but since

$$\|yx_n\|_2 \leqq \|y\|_\infty \|x_n\|_2$$

(see [**4**]), so also does yx_n; this shows that $z = 0$.

2.9. Lemma. *L_1 is dense in N_h in the norm defined in* 2.7.

Proof. Denote by f_n the spectral projection of h corresponding to the interval $[n^{-1}, 1]$, $n = 1, 2, \ldots$. Choose an increasing sequence $g_1 \leqq g_2 \leqq \ldots$ of projections of finite rank inside the kernel of h with supremum equal to the projection onto this subspace. Set $f_n + g_n = e_n$; then e_n increases to 1, and $e_nh = he_n$ for each n. Since $hf_n \geqq n^{-1}f_n$, there is $y_n \in L_\infty$ with $y_nh = hy_n = f_n$. Hence $f_nN_hf_n \subset L_1$ (for $x \in N_h$, $f_nxf_n = y_nhxhy_n \in L_1$), and so $e_nN_he_n \subset L_1$.

For any $x \in L_1$, $\|x(1 - e_n)\|_1$ converges to 0 (since for any $\epsilon > 0$, $x = x_\epsilon + x_\epsilon'$ with x_ϵ of finite rank and $\|x_\epsilon'\|_1 \leqq \epsilon$), and hence $\|x - e_nxe_n\|_1$ converges to 0 ($\|x - exe\|_1 \leqq \|x(1 - e)\|_1 + \|(1 - e)xe\|_1$, and $\|(1 - e)xe\|_1 \leqq \|(1 - e)x\|_1\|e\|_\infty$). It is trivial that $\|x - e_nxe_n\|_2$

converges to 0 for any $x \in L_2$. It follows, by use of $e_n h = he_n$, that $x - e_n x e_n$ converges to 0 in the norm of N_h defined in 2.7.

2.10. *Proof of Theorem* 2.5. First note that for $x \in L_1$,

$$\text{trace}(hxh + \bar{h}x\bar{h}) = \text{trace}(x(h^2 + \bar{h}^2)) = \text{trace}\ (x),$$

so τ_h does extend the trace.

Hence for $y \in M_h$ and $x \in L_1$,

$$\tau_h(yx) = \text{trace}(yx) = \text{trace}(xy) = \tau_h(xy).$$

Fix $y \in M_h$. With respect to the norm in N_h defined in 2.7, τ_h is clearly continuous, the maps $x \mapsto yx$ and (by symmetry) $x \mapsto xy$ are continuous by 2.8, and L_1 is dense by 2.9. Therefore the equality $\tau_h(xy - yx) = 0$ for $x \in L_1$ persists for $x \in N_h$.

2.11. *Proof of Theorem* 2.2 (*existence*). In 2.10 it was shown that τ_h defined as in 2.3 extends the trace. It remains to show that τ_h has the invariance property stated in Theorem 2.2. In other words, we must show that if u is a unitary in L_∞ such that $uN_hu^{-1} = N_h$, then the functional θ on N_h defined by

$$\theta(x) = \tau_h(uxu^{-1}), \quad x \in N_h$$

is equal to τ_h.

Clearly θ is linear and extends the trace on L_1. Since $uN_hu^{-1} = N_h$ implies $uM_hu^{-1} = M_h$, θ also has the property established for τ_h in 2.5. Let ϕ be any functional on N_h with these properties, and let us show that ϕ must be equal to τ_h. To use the proof of uniqueness given in 2.2 (exactly as given there), it is enough to show that ϕ is invariant under transformation by the unitaries u_+ and u_- defined there, or, more generally, by any unitary u such that $u, u^{-1} \in M_h$. Let $a \in N_h$. Then $ua \in N_h$, and by the assumption that $\phi(xy) = \phi(yx)$ if $x \in N_h$ and $y \in M_h$, applied to $x = ua$ and $y = u^{-1}$, $\phi(uau^{-1}) = \phi(xy) = \phi(yx) = \phi(a)$. It follows by the uniqueness argument in 2.2 that ϕ is uniquely determined, and equal to τ_h defined as in 2.3. In particular, θ is equal to τ_h. This shows that τ_h defined in this way satisfies the invariance condition of Theorem 2.2, and the proof of Theorem 2.2 is thus complete.

3. Consistency of the extensions τ_h, $0 \leqq h \leqq 1$.

3.1. Theorem. *For any selfadjoint operators* $0 \leqq h \leqq 1$, $0 \leqq k \leqq 1$, *the extensions* τ_h *and* τ_k *of the trace to* N_h *and* N_k *coincide on* $N_h \cap N_k$.

Proof. Step 1. We can dispose immediately of the case that $hk = kh$.

https://doi.org/10.4153/CJM-1980-098-x Published online by Cambridge University Press

In this case, if $x \in N_h \cap N_k$, then

$$\begin{aligned}\tau_h(x) &= \text{trace }(hxh + \bar{h}x\bar{h}) \\ &= \text{trace }(k(hxh + \bar{h}x\bar{h})k + \bar{k}(hxh + \bar{h}x\bar{h})\bar{k}) \\ &= \text{trace }(khxhk + k\bar{h}x\bar{h}k + \bar{k}hxh\bar{k} + \bar{k}\bar{h}x\bar{h}\bar{k}) \\ &= \text{trace }(h(kxk + \bar{k}x\bar{k})h + \bar{h}(kxk + \bar{k}x\bar{k})\bar{h}) \\ &= \text{trace }(kxk + \bar{k}x\bar{k}) \\ &= \tau_k(x).\end{aligned}$$

Step 2. With p the spectral projection of h corresponding to the interval $[1/2, 1]$, there exist $s, t \in L_\infty$ with $p = sh = hs$ and $1 - p = t\bar{h} = \bar{h}t$. Then $N_h \subset N_p$.

By Step 1, with $k = p$, it follows that τ_p extends τ_h. Therefore, to prove the theorem, it is sufficient to consider p in place of h, and an analogous projection q in place of k.

Step 3. We may suppose that the Hilbert space in which the projections p and q act is separable, since for any $x \in N_p \cap N_q$ there is a direct sum decomposition of the space into separable subspaces, each one invariant under p, q, x and x^*.

Step 4. We may suppose that both p and $1 - p$ have infinite rank, since otherwise $N_p = L_1$, and so we may write p and q as operator matrices:

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} r & s \\ s^* & t \end{pmatrix},$$

where

$$r \geqq 0, t \geqq 0, rs = s(1 - t), r - r^2 = ss^*, \text{ and } t - t^2 = s^*s.$$

Step 5. Extending the Hilbert space if necessary, we may suppose that ker s and coker s are both infinite, so that there exists a unitary operator u with $us \geqq 0$.

Step 6. Denote by v the operator matrix

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$vqv^{-1} = \begin{pmatrix} uru^{-1} & us \\ (us)^* & t \end{pmatrix}.$$

From $vp = pv$ it follows immediately that $vN_pv^{-1} = N_p$ and

$$\tau_p(x) = \tau_p(vxv^{-1}), \quad x \in N_p$$

(this does not use 2.2). Moreover, for $x \in N_q$,

$$\begin{aligned}\tau_q(x) &= \text{trace }(qxq + (1 - q)x(1 - q)) \\ &= \text{trace }(v(qxq + (1 - q)x(1 - q))v^{-1}) \\ &= \tau_{vqv^{-1}}(vxv^{-1}).\end{aligned}$$

Hence the desired identity

$$\tau_p(x) = \tau_q(x), \quad x \in N_p \cap N_q,$$

that is,

$$\tau_p(vxv^{-1}) = \tau_{vqv^{-1}}(vxv^{-1}), \quad x \in N_p \cap N_q,$$

is equivalent to the identity

$$\tau_p(x) = \tau_{vqv^{-1}}(x), \quad x \in N_p \cap N_{vqv^{-1}}$$

(recall that $vpv^{-1} = p$).

Therefore, replacing q by vqv^{-1}, so that r is replaced by uru^{-1} and s by $us \geqq 0$, we may suppose that $s \geqq 0$.

Step 7. It is enough to show that if the operator matrix

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = y$$

is such that $y \in N_q$ then $\tau_q(y) = 0$.

Indeed, this is certainly necessary, since any such y belongs to N_p and satisfies $\tau_p(y) = 0$. If x is an arbitrary element of $N_p \cap N_q$, say

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then with

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = y, \qquad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = z,$$

$x = y + z$ and $z \in L_1$. In particular, as $y = x - z \in N_q$, y has the form above. Moreover,

$$\tau_p(x) = \tau_p(z) = \tau_q(z),$$

so to show that $\tau_p(x) = \tau_q(x)$ it is enough to show that $\tau_q(y) = 0$.

Step 8. Denote by e the support projection of s. Then e commutes with r and t, and $er = e(1 - t)$.

To show this it is enough to show that s commutes with r and t, since $rs = s(1 - t)$ (Step 4). From $s = s^*$ (Step 6), and $r - r^2 = t - t^2 = s^2$ (Step 4) it follows that r and t commute with s^2, and therefore also with s (as $s \geqq 0$; see Step 6).

Step 9. Suppose that the operator matrix

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = y$$

belongs to N_q. To show that $\tau_q(y) = 0$ (cf. Step 7) it is enough to consider the case that $b = ebe$ and $c = ece$; in other words, we may suppose that $e = 1$.

Indeed, if f denotes the operator matrix

$$\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

then the case that we must reduce the given one to is that $y = fyf$. Since e commutes with r, s and t (Step 8), f commutes with q, and in particular multiplies N_q. Using both properties of f, we have

$$\begin{aligned} \tau_q(y) &= \tau_q(fyf + (1-f)y(1-f)) \\ &= \tau_q(fyf) + \tau_q((1-f)y(1-f)). \end{aligned}$$

We shall show that $\tau_q((1-f)y(1-f)) = 0$; this will reduce the consideration of $\tau_q(y)$ to that of $\tau_q(fyf)$, and fyf is just y with b and c replaced by ebe and ece. Since $s(1-e) = 0$, $q(1-f)$ commutes with p. Hence by Step 1,

$$\begin{aligned} \tau_q((1-f)y(1-f)) &= \tau_{q(1-f)}((1-f)y(1-f)) \\ &= \tau_p((1-f)y(1-f)) = 0. \end{aligned}$$

Step 10. Suppose that the operator matrix

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = y$$

belongs to N_q. We shall now show that $\tau_q(y) = 0$. (Cf. Step 7.)

First, calculation yields (in view of Steps 8 and 9):

$$qyq = \begin{pmatrix} scr + rbs & scs + rb(1-r) \\ (1-r)cr + sbs & (1-r)cs + sb(1-r) \end{pmatrix};$$

$$(1-q)y(1-q) = \begin{pmatrix} -sc(1-r) - (1-r)bs & scs + (1-r)br \\ rc(1-r) + sbs & -rcs - sbr \end{pmatrix}.$$

Since $y \in N_q$, both qyq and $(1-q)y(1-q)$ and hence also their difference belong to L_1. We have then

$$qyq - (1-q)y(1-q) = \begin{pmatrix} sc + bs & rb - br \\ cr - rc & cs + sb \end{pmatrix} \in L_1,$$

and we deduce that the commutators $[s, b-c]$, $[r, b-c]$ belong to L_1.

We must show that $\tau_q(y) = 0$, that is, that

$$\text{trace } (qyq + (1-q)y(1-q)) = 0.$$

Inspection of the matrices for qyq and $(1-q)y(1-q)$ reveals that for this it is sufficient to show that:

$$\text{trace } (r(b-c)s - s(b-c)r) = 0,$$

$$\text{trace } [s, b-c] = 0.$$

Since

$$r(b-c)s - s(b-c)r = [r(b-c), s] + [r, s(b-c)]$$

(this uses $rs = sr$; see Step 8), both trace equalities follow from 3.2 below, applied three times, once to each of the three commutators involved.

To verify that the hypotheses of 3.2 are satisfied for each of the commutators

$$[r(b-c), s], [r, s(b-c)] \text{ and } [s, b-c],$$

recall that both s (see Step 6) and r are selfadjoint, and that b, c and therefore also $b-c$, $r(b-c)$ and $s(b-c)$ belong to L_2. Moreover, since $[r, s] = 0$ (see Step 8), and both $[s, b-c]$ and $[r, b-c]$ belong to L_1 (see the third paragraph of this step), it follows by the derivation law that also

$$[r(b-c), s] = r[b-c, s] \in L_1,$$
$$[r, s(b-c)] = s[r, b-c] \in L_1,$$

and so the third hypothesis of 3.2 is satisfied in each of the three cases under consideration.

3.2. LEMMA. *Let $a = a^* \in L_\infty$ and $b \in L_2$ be such that $ab - ba \in L_1$. Then*

$$\text{trace } (ab - ba) = 0.$$

Proof. By [**5**], $a = a_1 + a_2$ where a_1 has an orthonormal basis of eigenvectors and $a_2 \in L_2$. Since $a_2 \in L_2$, $a_2b - ba_2 \in L_1$ and

$$\text{trace } (a_2b - ba_2) = 0$$

(see [**4**]). On the other hand, $a_1b - ba_1$ is equal to $(ab - ba) - (a_2b - ba_2)$ and so belongs to L_1. Moreover, $a_1b - ba_1$ has zero diagonal with respect to the basis diagonalizing a_1. It follows that

$$\text{trace } (a_1b - ba_1) = 0.$$

Adding these two displayed equations gives the conclusion of the lemma.

3.3. *Remark.* In 3.2, a need only be assumed to be normal, although the proof is then much more difficult; see [**6**].

3.4. *Remark.* 3.1 is equivalent to the following statement: For each selfadjoint operator $0 \leqq h \leqq 1$, if $x \in N_h$ and if $u \in L_\infty$ is a unitary operator such that also $uxu^{-1} \in N_h$, then

$$\tau_h(uxu^{-1}) = \tau_h(x).$$

To see this, just note that $uxu^{-1} \in N_h$ is equivalent to $x \in N_{u^{-1}hu}$, and

$$\tau_h(uxu^{-1}) = \text{trace}(huxu^{-1}h + \bar{h}uxu^{-1}\bar{h}) = \tau_{u^{-1}hu}(x).$$

This is a much stronger invariance property than that stated in 2.2. The invariance property stated in 2.5, however, does not seem to follow from 3.1 (except for multipliers which are unitary).

3.5. *Remark.* While the different τ_h, $0 \leqq h \leqq 1$ are consistent, so that they have a common extension, say τ, to the union of all N_h, $0 \leqq h \leqq 1$ (this, by 3.1.1, is equal to the union just over $h = h^2$), it is not possible to extend τ so as to be linear on the linear span of this union.

To see this, fix an orthonormal basis, and consider the three diagonal operators a, b and c with the following eigenvalue sequences:

$$\begin{array}{rrrrrrrrrrl} 1, & -1, & 2, & -2, & 3, & -3, & 4, & -4, & 5, & -5, & \ldots, \\ 2, & 1, & -1, & 3, & 4, & 5, & -2, & 7, & 6, & 9, & \ldots, \\ -1, & 2, & 1, & 4, & -2, & 6, & 3, & 8, & -3, & 10, & \ldots. \end{array}$$

Here, for each $n = 1, 2, \ldots$ the column beginning with $-n$ is $(-n, 2n - 1, 2n)$, and for each $m = 1, 2, \ldots$ the column beginning with m is a cyclic permutation of (a unique) one of the columns beginning with $-n$, $n = 1, 2, \ldots$. Clearly, a, b and c are unitarily equivalent to one another, and also to the operator matrix

$$\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} = \tilde{d}$$

where d is diagonal with eigenvalue sequence

$$1, 2, 3, 4, 5, \ldots.$$

Since $d^{-1} \in L_2$, the operator matrix $\tilde{d}^{-1}$ belongs to the domain of τ. Therefore so also do a^{-1}, b^{-1}, c^{-1}, and, by unitary invariance of τ,

$$\tau(a^{-1}) = \tau(b^{-1}) = \tau(c^{-1}) = \tau(\tilde{d}^{-1}) = 0.$$

On the other hand, $a^{-1} + b^{-1} + c^{-1}$ has eigenvalues

$$-n^{-1} + (2n - 1)^{-1} + (2n)^{-1}, \quad n = 1, 2, \ldots,$$

each with multiplicity three, so $a^{-1} + b^{-1} + c^{-1}$ is positive, nonzero, and belongs to L_1. Thus:

$$\tau(a^{-1} + b^{-1} + c^{-1}) \neq \tau(a^{-1}) + \tau(b^{-1}) + \tau(c^{-1}).$$

3.6. *Application.* The criterion of G. Stamatopoulos (thesis, University of Pennsylvania, 1974) for the equivalence of quasifree pure states of the gauge invariant subalgebra of the canonical anticommutation relation C^*-algebra (or Fermion C^*-algebra) determined by projections e and f in the one-particle Hilbert space to be equivalent, namely, that $e - f$ be Hilbert–Schmidt and that trace$(1 - e)f(1 - e)$ and trace$(1 - f)e(1 - f)$, which are then finite, be equal, may be simplified in terms of the present extension τ of the trace as follows: $e - f$ should be Hilbert–Schmidt, so

https://doi.org/10.4153/CJM-1980-098-x Published online by Cambridge University Press

that it belongs to the domain of τ (since $e(e - f)e = e(e - f)^2e$), and $\tau(e - f)$ should be equal to 0.

3.7. *Application.* Theorem 3.1 has the following rather striking consequence for square-summable sequences of real numbers, obtained by restricting τ to operators diagonal with respect to a fixed basis. (Note that the proof given above that τ is well defined does not become any simpler for this restriction of τ.) If two sequences $\lambda, \mu \in l_2$ have the property that their terms occur in equal and opposite pairs, i.e., if each of λ, μ has the form $(a, -a, b, -b, c, -c, \ldots)$ up to a permutation, and if moreover $\lambda', \mu' \in l_1$ are such that $\lambda + \lambda' = \mu + \mu'$, then $\sum \lambda_n' = \sum \mu_n'$. (Indeed, $\lambda + \lambda'$ is the eigenvalue sequence of a selfadjoint element of the domain of τ, which with respect to a suitable projection has diagonal part $\in L_1$ and of trace $\sum \lambda_n'$. It can be shown that in fact $\lambda + \lambda'$ is a completely general such eigenvalue sequence.) It would be desirable to have a proof of this fact using only sequences.

References

1. G. A. Elliott, *On extending the trace as a linear functional*, Rep. Math. Phys. *15* (1979), 199–203.
2. L.-E. Lundberg, *Quasi-free "second quantization"*, Comm. Math. Phys. *50* (1976), 103–112.
3. ——— *Vacuum polarization due to potential perturbations*, preprint, University of Copenhagen (1977).
4. R. Schatten and J. von Neumann, *The cross-space of linear transformations II*, Ann. of Math. *47* (1946), 608–630.
5. J. von Neumann, *Charakterisierung des Spektrums eines Integraloperators*, Actualités Sci. Indust., no. 229 (Hermann, Paris, 1935).
6. G. Weiss, *The Fuglede commutativity theorem modulo the Hilbert–Schmidt class and generating functions for matrix operators I*, Trans. Amer. Math. Soc. *246* (1978), 193–209.

University of Copenhagen,
Copenhagen, Denmark