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NONLINEAR FREE-SURFACE FLOWS EMERGING FROM VESSELS AND FLOWS UNDER A SLUICE GATE

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Abstract

Steady two-dimensional flows in a domain bounded below by an infinite horizontal wall and above by a semi-infinite horizontal wall, a vertical wall and a free surface are considered. The fluid is assumed to be inviscid and incompressible, and gravity is taken into account. The problem is solved numerically by series truncation. It is shown that for a given length of the vertical wall, there are two families of solutions. One family is characterized by a continuous slope at the separation point and a limiting configuration with a stagnation point and a 120° angle corner at the separation point. The other family is characterized by a stagnation point and a 90° angle corner at the separation point. Flows under a sluice gate with and without a rigid lid approximation upstream are also considered.

1. Introduction

We consider the steady two-dimensional irrotational flow of an inviscid incompressible fluid in a domain bounded below by the wall JI and above by the walls JB and BA and the free surface AI (see Figure 1). The problem models the free-surface flow emerging from a rectangular vessel. We restrict our attention to flows which approach downstream a uniform stream with velocity U and depth H. Therefore we assume that the flow is supercritical downstream, that is, that the Froude number

$$F = U/\sqrt{gH} \tag{1}$$

is greater than one. Here g denotes the acceleration of gravity.

The flow configuration of Figure 1 was considered by Gurevich [7], Budden and Norbury [4], Benjamin [1], Vanden-Broeck and Keller [12], Vanden-Broeck [11] and others. Gurevich [7] used free streamline theory to obtain solutions in the absence of gravity (that is, $F = \infty$). Budden and Norbury [4] obtained some uniqueness results

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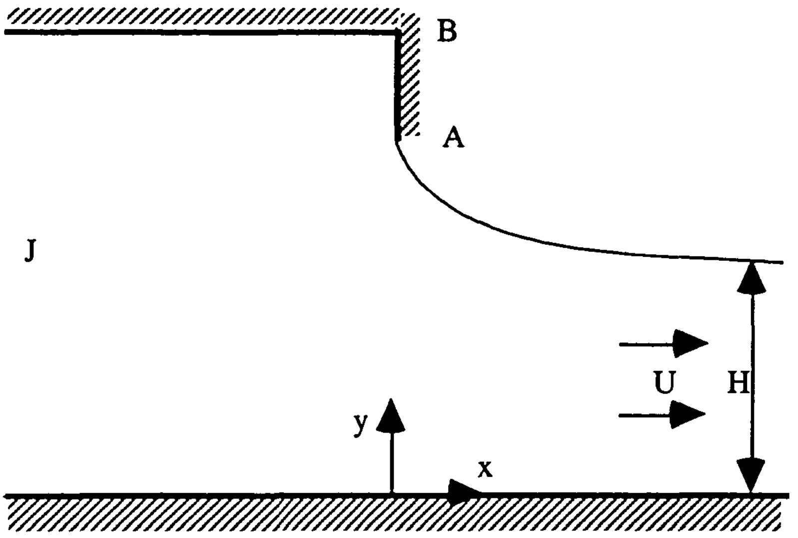

FIGURE 1. Sketch of the flow between two rigid boundaries past a vertical wall hanging from the upper boundary. The profile is a computed solution for $F^2 = 4$ and $b = -0.3$. The vertical scale is the same as the horizontal scale.

and derived asymptotic solutions for F large. Benjamin [2] and Vanden-Broeck and Keller [12] used conservation of mass and momentum to derive exact relations as the length of the vertical wall vanishes. Vanden-Broeck [11] obtained numerical solutions for arbitrary values of F when the vertical wall extends upwards to infinity.

In this paper we solve the problem numerically by series truncation for arbitrary length of the vertical wall and arbitrary values of the Froude number. The numerical procedure is similar to the one used by Vanden-Broeck and Keller [12] and Vanden-Broeck [11]. Our results include those of Vanden-Broeck [11] as a particular case. In addition, they satisfy the exact relations of Benjamin [2] and Vanden-Broeck and Keller [12] when the length of the vertical wall vanishes. We show also that for a given length of the vertical wall, there are two familes of solutions. One family is characterized by a continuous slope at the separation point and a limiting configuration with a stagnation point and a 120° angle corner at the separation point. The other family is characterized by a stagnation point and a 90° angle corner at the separation point.

The flow of Figure 1 represents also the flow under a sluice gate when the upstream free surface is replaced by a rigid lid. The only difference with the previous problem is that the distance between the upper horizontal wall and the lower horizontal wall needs to be adjusted to satisfy the free surface condition at $x = -\infty$. We show that there is a solution for each value of $F \geq 1$.

We also consider a complete sluice gate problem (that is, without the rigid lid approximation). The gate is inclined at an angle γ and both free surfaces are assumed to leave the gate tangentially. We present numerical solutions for various values of γ. Numerical solutions for large values of F were obtained before by Fangmeier and Strelkoff [6] and Masliyah, Nandakumar, Hemphill and Fung [9].

The problem is formulated in Section 2. The two families of solutions are described in Sections 3 and 4. The results for the sluice gate problem with and without the rigid lid approximation are presented in Sections 5 and 6 respectively.

2. Formulation

Let us consider the steady two-dimensional, irrotational flow of an inviscid, incompressible fluid in the region shown in Figure 1. We choose Cartesian coordinates with the x-axis on the bottom and y-axis directed vertically upwards. Gravity is acting in the negative y-direction. As $x \to \infty$, the flow is required to approach a uniform stream with constant velocity U and uniform depth H. The wall BA extends from the horizontal boundary JB. The free surface AI, the boundary JB and the vertical wall BA are parts of a streamline on which $\psi = UH$. The lower boundary JI is another streamline on which $\psi = 0$.

We introduce the complex potential $f = \phi + i\psi$, in which ϕ and ψ represent the potential function and the streamfunction respectively. Without loss of generality, we choose $\phi = 0$ at the separation point A. Let ϕ_b denote the value of the potential function at the point B. The pressure is assumed to be constant on the free surface. The Bernoulli equation yields

$$\tfrac{1}{2}q^2 + gy = \text{constant on the free surface AI.} \tag{2}$$

Here q denotes the magnitude of the velocity.

Let us choose U as the unit velocity and H as the unit length. From the choice of our dimensionless variables, (2) becomes

$$q^2 + \frac{2}{F^2}(y-1) = 1, \tag{3}$$

where F^2 is the square of the Froude number defined in (1).

Next we introduce the complex velocity $\zeta = u - iv$, where u and v are the $x-$ and $y-$ components of the velocity. On AI, BA and JB, the streamfunction ψ equals 1.

The kinematic conditions on JB, BA, and IJ yield

$$v = 0, \quad \psi = 1, \quad -\infty < \phi < \phi_b, \tag{4}$$

$$u = 0, \quad \psi = 1, \quad \phi_b < \phi < 0, \tag{5}$$

$$v = 0, \quad \psi = 0, \quad -\infty < \phi < \infty. \tag{6}$$

As $\phi \to \infty$, the flow approaches a uniform stream with unit velocity. In addition, the flow is supercritical downstream. Since a supercritical flow is characterized by the presence of exponentially decaying terms, the complex velocity ζ can be described as

$$\zeta \sim 1 + Re^{-\pi\lambda\phi}, \quad \text{as } \phi \to \infty. \tag{7}$$

Here R is a constant to be determined as part of the solution, and λ is the smallest positive root of

$$F^2\pi\lambda - \tan\pi\lambda = 0. \tag{8}$$

The relations (7) and (8) can be derived by linearizing the flow around a uniform stream and solving the resulting equations by the method of separation of variables.

By analogy with other free-streamline problems, we expect ζ to behave at the separation point A like

$$\zeta \sim S + T(f - i)^{\frac{1}{2}}, \quad \text{as } f \to i. \tag{9}$$

Here S and T are constants to be found as part of the solution.

The problem is to find ζ as an analytic function of f in the strip $0 < \psi < 1$, satisfying the conditions (3) – (9).

Next we map the flow domain in the f-plane conformally into the interior of the upper half of the unit circle in the t-plane by the transformation

$$f = -\frac{1}{\pi}\log\frac{(1-t)^2}{4t}. \tag{10}$$

The bottom IJ is mapped onto the positive real diameter. The horizontal and the vertical walls JB, BA are mapped onto the negative real diameter. The free surface AI is mapped onto the circumference of the upper half unit circle. We use the notation $t = re^{i\sigma}$ so that the free surface is described by $r = 1$ and $0 \le \sigma \le \pi$. We denote by $t = b$ $(-1 \le b \le 0)$ the image of the corner B in the t-plane. There is a stagnation point at B and the appropriate singularity is

$$\zeta \sim W(t - b)^{\frac{1}{2}}, \quad \text{as } t \to b. \tag{11}$$

Here W is a constant to be found as part of the solution.

Considering (7) and (9) – (11), we represent the complex velocity ζ by the expansion

$$\zeta = \left(\frac{t-b}{1-b}\right)^{\frac{1}{2}} \exp\left\{A(1-t)^{2\lambda} + B(1+t)^{\frac{1}{2}} - B\sqrt{2} + \sum_{n=1}^{\infty} a_n(t^n - 1)\right\}. \tag{12}$$

The kinematic conditions (4) – (6) are satisfied by requiring the coefficients a_n of the power series to be real. The unknown constants A, B and the coefficients a_n must be determined so that the constant pressure condition (3) on the free surface is satisfied. It is convenient to eliminate y from (3) by differentiating (3) with respect to σ. Using the identity

$$\frac{\partial x}{\partial \phi} + i\frac{\partial y}{\partial \phi} = \frac{1}{\zeta}, \tag{13}$$

we obtain

$$[u(\sigma)u_\sigma(\sigma) + v(\sigma)v_\sigma(\sigma)] - \frac{1}{\pi F^2}\frac{v(\sigma)}{u(\sigma)^2 + v(\sigma)^2}\cot(\sigma/2) = 0. \tag{14}$$

We solve the problem numerically by truncating the infinite series in (12) after N terms. We find the $N+3$ unknowns λ, A, B and the coefficients a_n, $n = 1, \ldots, N$, by collocation. Thus we introduce the $N+2$ mesh points

$$\sigma_i = \frac{\pi}{N+2}\left(i - \frac{1}{2}\right), \quad i = 1, \ldots, N+2. \tag{15}$$

We obtain $N+2$ equations by satisfying (14) at these mesh points. Relation (8) provides another equation. These lead to a system of $N+3$ nonlinear algebraic equations for $N+3$ unknowns. For given values of b and F^2, this system of equations is solved by Newton's method.

Finally, the shape of the free surface is obtained by integrating numerically the relation

$$\frac{dx}{d\sigma} = -\frac{\cot(\sigma/2)}{\pi}\frac{u(\sigma)}{u(\sigma)^2 + v(\sigma)^2} \tag{16}$$

and

$$\frac{dy}{d\sigma} = -\frac{\cot(\sigma/2)}{\pi}\frac{v(\sigma)}{u(\sigma)^2 + v(\sigma)^2}. \tag{17}$$

Similarly, we obtain the length of the vertical wall BA by

$$z(b) = z(-1) + \int_{-1}^{b} \frac{df}{dt}\frac{1}{\zeta}dt, \tag{18}$$

where $z(t) = x + iy$ and $z(-1)$ is calculated from (16) and (17).

3. Discussion of the results

We use the numerical scheme described in the previous section to compute solutions for various values of b and F^2. The coefficients a_n are found to decrease very rapidly;

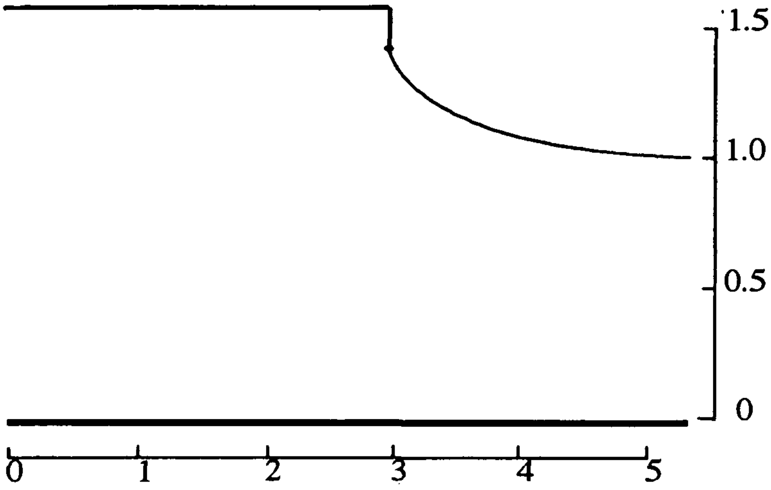

FIGURE 2. Computed profile for $F^2 = 2.5$ and $b = -0.5$. The flow has a continuous tangent at the separation. This solution corresponds to (*) in Figure 5.

for example, for $b = -0.5$, $F^2 = 4$, $|a_1| \approx 0.391$, $|a_{10}| \approx 0.28 \times 10^{-4}$, $|a_{60}| \approx 0.4 \times 10^{-9}$. All calculations were performed with $N = 60$.

Typical profiles are shown in Figures 1 – 3. We define the contraction coefficient C_c as the ratio of the thickness H of the jet at $x = \infty$ to the distance D of the separation point A from the bottom IJ. That is,

$$C_c = H/D. \tag{19}$$

We also introduce the dimensionless depth ratio

$$\alpha = H/\tilde{H}, \tag{20}$$

where $\tilde{H}$ is the upstream depth. Numerical values of α and C_c versus F^2 are presented in Figures 4 and 5 for various values of b. As $F^2 \to \infty$, α and C_c approach constant values and we recover the free streamline results of Gurevich [7]. For $b = 0$, the vertical wall extends vertically upwards to infinity and the problem reduces to the flow under a gate considered by Vanden-Broeck [11].

As α and C_c decrease, F^2 first decreases to a minimum value $F^2_{\min}$ and then increases up to a value F_b^2 (see Figures 4 and 5). When F^2 reaches F_b^2, a stagnation point occurs at A with a 120° angle corner. There are no solutions when $F^2 < F^2_{\min}$, two solutions when $F^2_{\min} < F^2 < F_b^2$, and a unique solution when $F^2 > F_b^2$. Table 1 shows numerical values of $F^2_{\min}$ and F_b^2 for some values of b.

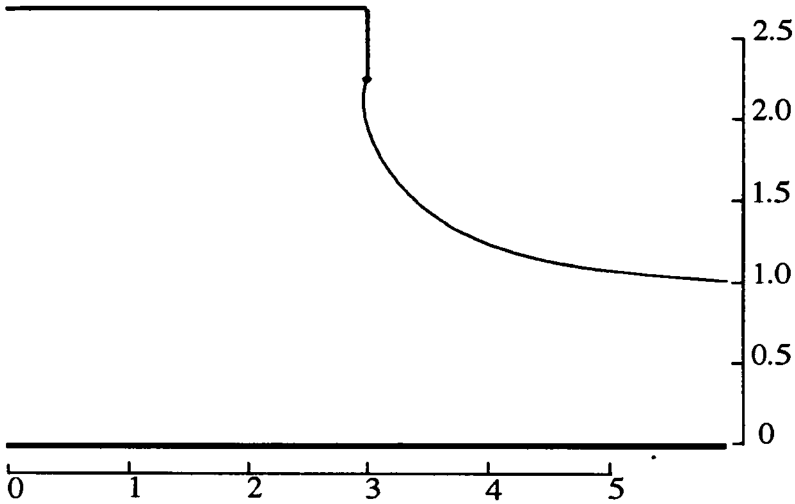

FIGURE 3. Compute profile for $F^2 = 2.5$ and $b = -0.5$. This solution corresponds to (○) in Figure 5. Observe that a portion of the free surface lies to the left of the vertical wall.

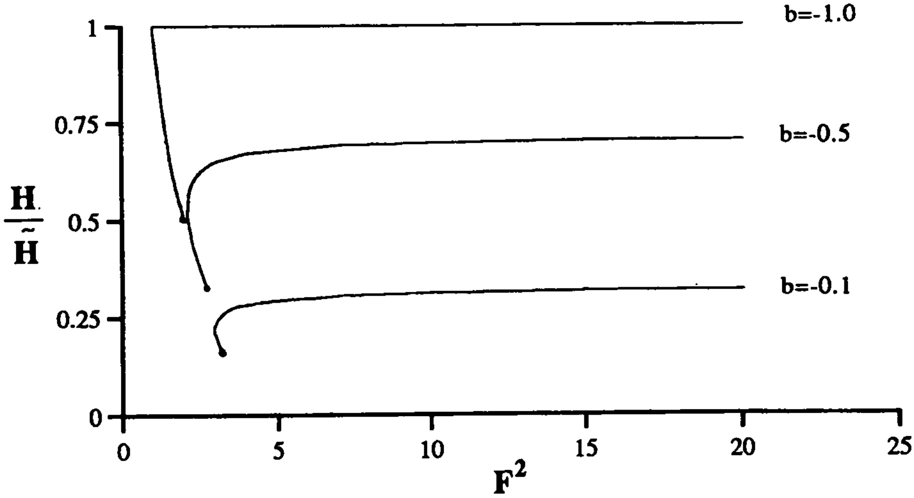

FIGURE 4. Relationship between $H/\tilde{H}$ and F^2 for three values of b. A dot (•) at the left end of each curve corresponds to a flow with a stagnation point and a 120° angle at the separation point.

We note that the free surface leaves the vertical wall tangentially except for the limiting configuration corresponding to $F^2 = F_b^2$ (dots (•) in Figures 4 and 5). The velocity q_A at the separation point is different from zero except for the limiting

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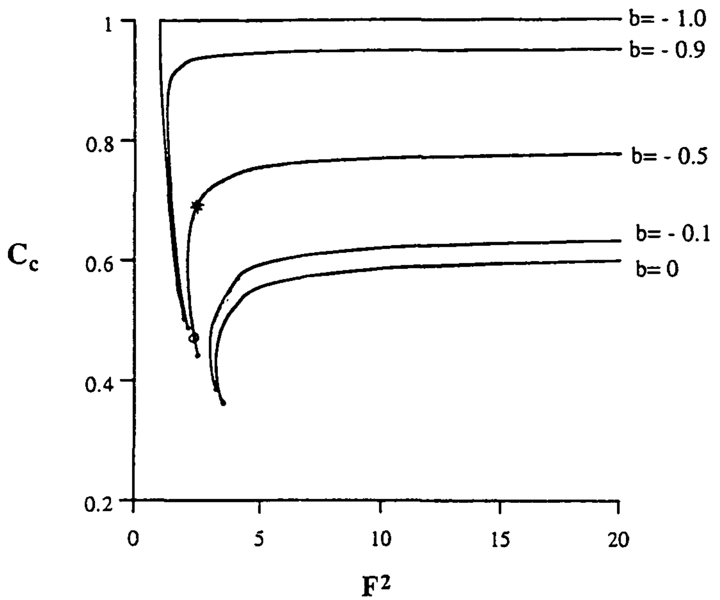

FIGURE 5. Relationship between the contraction ratio C_c and F^2.

TABLE 1. Values of $F^2_{\min}$ and F_b^2 for various values of b.

b	$F^2_{\min}$	F_b^2
-0.1	2.987	3.2
-0.3	2.532	2.8
-0.5	2.106	2.5
-0.75	1.598	2.25

configuration with the 120° angle corner at $F^2 = F_b^2$. As α and C_c decrease in Figures 4 and 5, the velocity q_A at the separation point decreases monotonically from one to zero. For most values of q_A, the free surface is completely on the right of the wall BA as shown in Figure 2. However when q_A is small, a portion of free surface is on the left of the wall BA (see Figure 3).

The problem reduces to the flow past a semi-infinite horizontal plate when $b = -1$. The complex velocity ζ expressed in (12) is no longer valid since the point B coincides with the separation point A. The appropriate representation of ζ is obtained

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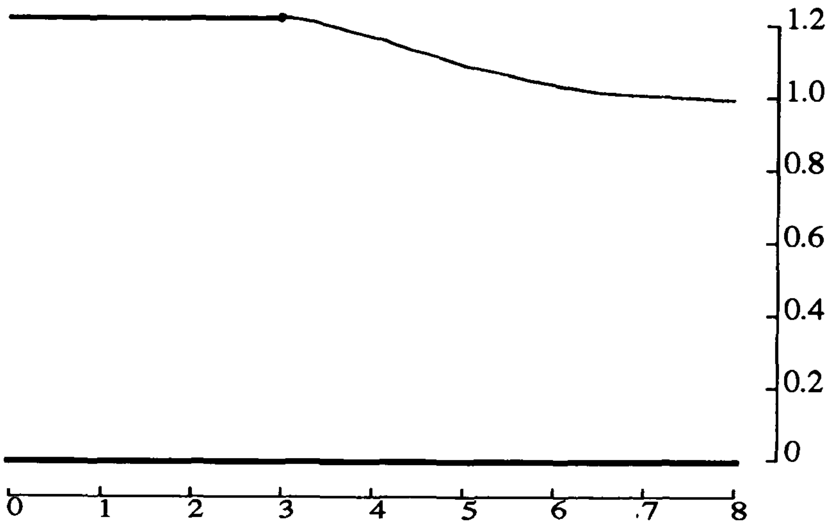

FIGURE 6. Computed profile for $F^2 = 1.25$ and $b = -1$. The length of the vertical wall is zero.

by removing the singularity at B (flow inside a corner). That is,

$$\zeta = \exp\left\{A(1-t)^{2\lambda} + B(1+t)^{\frac{1}{2}} - B\sqrt{2} + \sum_{n=1}^{\infty} a_n(t^n - 1)\right\}.$$

The rest of the calculations follow closely the one described earlier. A flow profile for $F^2 = 1.25$ and $b = -1$ is shown in Figure 6. Benjamin [2] and Vanden-Broeck and Keller [12] used conservation of mass and momentum to derive the exact relations

$$F^2 = 1/C_c = 1/\alpha. \tag{21}$$

The corresponding solutions form a branch which bifurcates from the uniform stream at $F^2 = 1$ (see Figures 4 and 5). Our numerical results agree with (21) to within 0.5%. When $F^2 = 2$, the flow ultimately reaches a limiting configuration with a 120° angle corner at the separation point (see Figure 7).

4. Flow with a stagnation point

The results in the previous section provide numerical evidence that there are flows with a stagnation point at which separation occurs with a 120° angle corner. It was shown by Lee and Vanden-Broeck [8] that there are additional solutions with a stagnation point at which separation occurs with a 90° angle corner for the flow with

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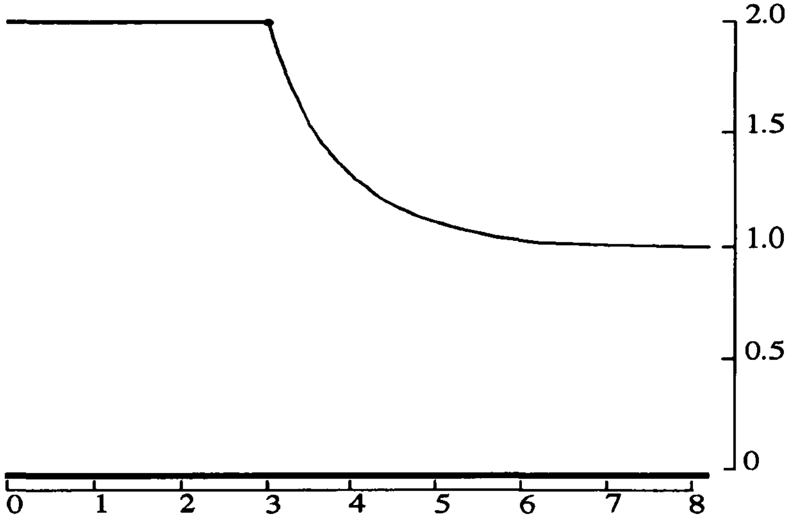

FIGURE 7. Computed profile for $F^2 = 2.0$ and $b = -1$. The length of the vertical wall is zero. The flow separates with a 120° angle corner at which a stagnation point occurs.

an infinite vertical wall (that is, $b = 0$). We shall use the series truncation technique adopted earlier to find such solutions for $b \neq 0$.

Locally at the separation point A, we require the flow to be inside a right angle. Therefore, the local behavior of the complex velocity ζ at this point is

$$\zeta \sim K(f - i)^{\frac{1}{2}}, \quad \text{as } f \to i. \tag{22}$$

Here K is a constant to be found as part of the solution. Following the formulation in Section 2, after replacing (9) by (22), the new expression for ζ is

$$\zeta(t) = (t + 1)\left(\frac{t - b}{1 - b}\right)^{\frac{1}{2}} \exp\left\{A(1 - t)^{2\lambda} + \sum_{n=1}^{\infty} a_n(t^n - 1)\right\}. \tag{23}$$

By truncating the infinite series in (23) after N terms, there are $N + 2$ unknowns to be determined. They are λ, A and $\{a_n\}_{n=1}^{N}$. We solve the problem numerically by introducing the $N + 1$ collocation points

$$\sigma_i = \frac{\pi}{N + 1}\left(i - \frac{1}{2}\right), \quad i = 1, \ldots, N + 1.$$

We obtain a system of $N + 2$ equations by satisfying (14) at these mesh points and imposing the relation (8). The rest of the computations follows closely the calculation of Section 2. It is found that the coefficients a_n decrease very rapidly; for example,

$$|a_1| \approx 0.88, \quad |a_{10}| \approx 10^{-4}, \quad |a_{60}| \approx 10^{-9} \quad \text{when } b = -0.5 \quad \text{and } F^2 = 2.8.$$

All calculations were performed with $N = 60$.

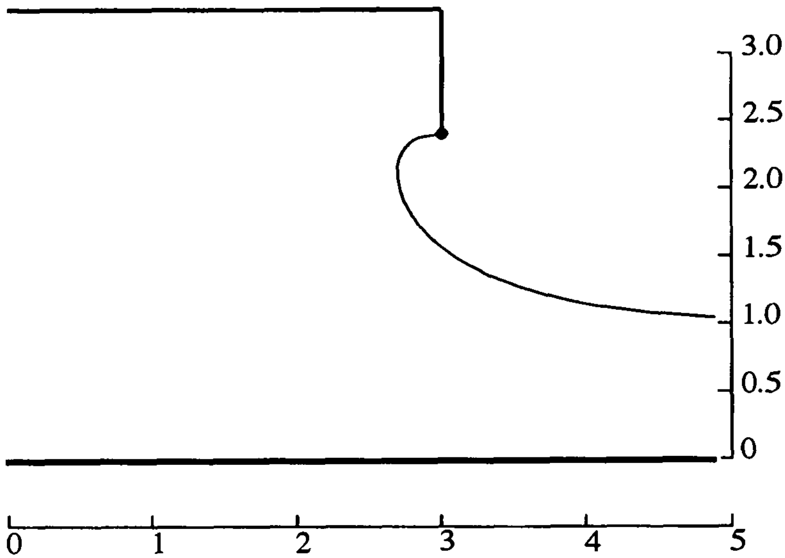

FIGURE 8. Computed profile of the flow which separates with a 90° angle, for $F^2 = 2.8$ and $b = -0.5$.

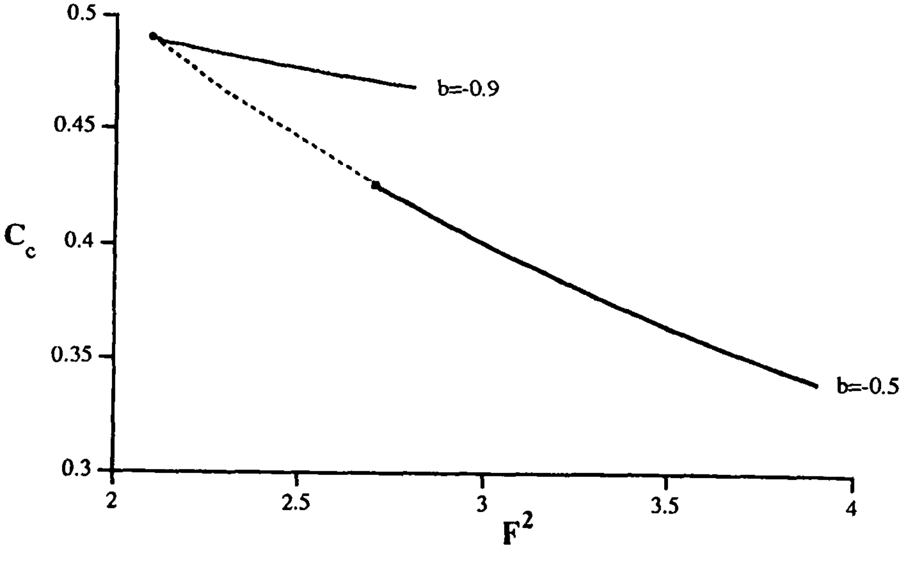

FIGURE 9. Relationship between the contraction ratio C_c and F^2. The broken line corresponds to the solutions which separate at a 120° angle corner (that is, $F = F_b$).

A typical profile is shown in Figure 8. For a fixed value of b, the solutions with 90° angle corner exist for all values of F^2 greater than F_b^2. Here F_b is the value of the Froude number corresponding to the solution with a 120° angle corner (see Section 3). It was observed that the solutions with a 90° angle corner approach the solution with a 120° angle corner at A as $F^2 \downarrow F_b^2$.

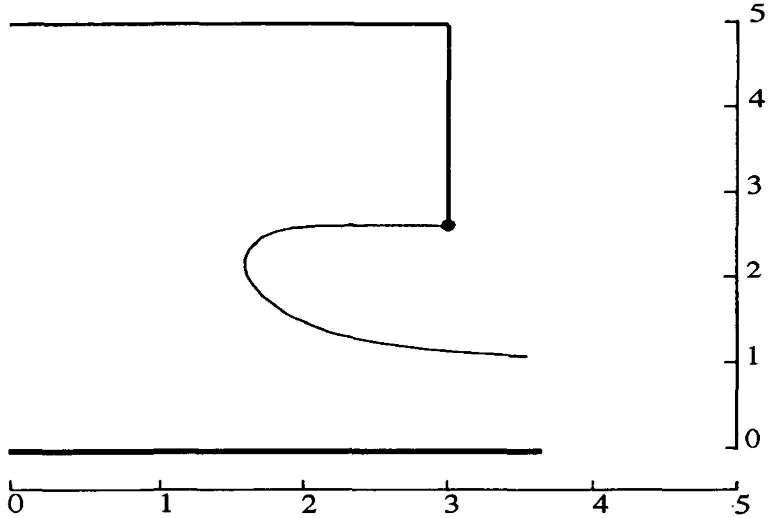

FIGURE 10. Same as Figure 8, for $F^2 = 3.2$ and $b = -0.5$. Observe that the tail of the free surface is not far from the separation point. This indicates a significant loss of accuracy in the numerical procedure.

The numerical results show that there is a family of solutions depending continuously on two parameters, F^2 and b. For $-1 < b \leq 0$, the contraction coefficient C_c decreases as F^2 increases as shown in Figure 9. Computational difficulties arise inevitably as the distribution of collocation points tends to concentrate near the vertical wall BA when F^2 increases. The resulting free surface profile is illustrated in Figure 10. As F^2 increases further, the numerical scheme no longer converges. We conjecture that the free surface extends infinitely to the left of the wall BA as $F^2 \to \infty$. Such a flow can be expected to be unstable because the heavy fluid lies on top of the light fluid.

5. Flow under a sluice gate with the the rigid lid approximation

The flow of Figure 1 represents also the flow under a sluice gate when the upstream free surface is approximated by the rigid lid. With this approximation, the kinematic condition on JB remains as in (4). Far upstream the flow is uniform with velocity $\tilde{U}$ and constant depth $\tilde{H}$ (see Figure 11). The wall BA is now called the sluice gate. We

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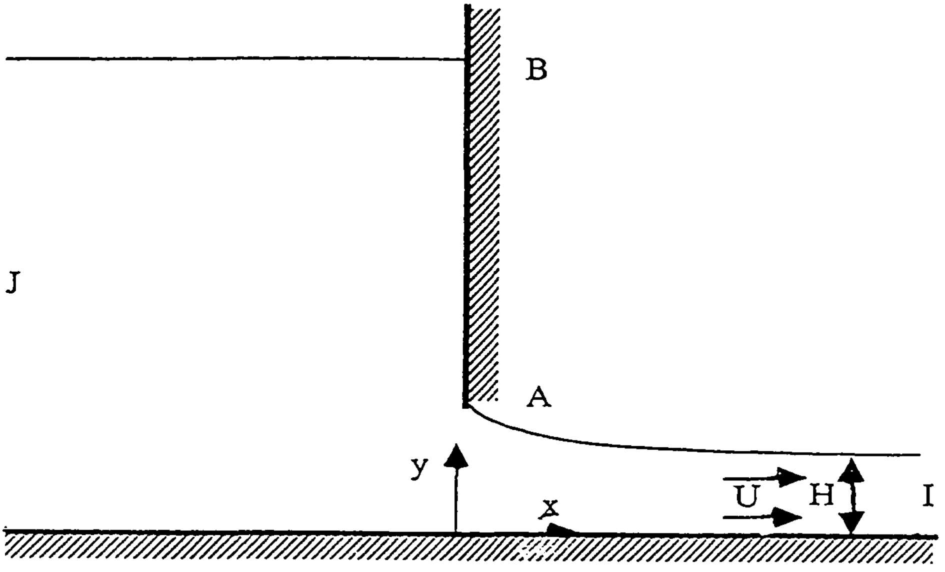

FIGURE 11. Sketch of the flow and of the system of coordinates. The free surface profile is a computed solution for $F^2 = 10$. The vertical scale is the same as the horizontal scale.

define the upstream Froude number as

$$\tilde{F} = \tilde{U}/\sqrt{g\tilde{H}}, \tag{24}$$

and the downstream Froude number F is already defined in (1).

The principle of conservation of mass implies that

$$\tilde{U}\tilde{H} = UH. \tag{25}$$

Evaluating (2) at $x = \pm\infty$, yields

$$\frac{1}{2}\tilde{U}^2 + g\tilde{H} = \frac{1}{2}U^2 + gH. \tag{26}$$

Following Binnie [3] and using (1), (25) and (26) we find that there are two types of solutions, namely

$$F^2 = 2/[\alpha(\alpha + 1)] \tag{27}$$

and

$$\alpha = 1, \tag{28}$$

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where α is defined in (20). The first solution shows that $F > 1$ when $\alpha < 1$ and $F < 1$ when $\alpha > 1$. Here we assume, without loss of generality, that $\alpha < 1$ since the free surface usually falls to a lower level. It follows that the approaching stream must be subcritical ($\tilde{F} < 1$) and the receding stream supercritical ($F > 1$). The second solution, (28), corresponds to a flow with the same velocity and the same depth both upstream and downstream. Vanden-Broeck and Keller [12] calculated such solutions for an inclined gate. These solutions will not be considered further in this paper.

The mathematical formulation of Section 2 applies directly to the flow of Figure 11. That is the complex potential function f and the complex velocity function ζ are represented by (10) and (12) respectively.

We now introduce the $N+2$ mesh points

$$\sigma_i = \frac{\pi}{N+2}\left(i - \frac{1}{2}\right), \quad i = 1, \dots, N+2.$$

We determine the function $u(\sigma)$ and $v(\sigma)$ and their derivatives in terms of $N+4$ unknowns λ, A, B, F^2 and $\{a_n\}_{n=1}^{N}$ by substituting $t = e^{i\sigma}$ into (12). We find the $N+4$ unknowns by satisfying (14) at these $N+2$ mesh points. Thus we obtain $N+2$ nonlinear algebraic equations. An extra equation is obtained by imposing relation (8). By rewriting (27) in terms of ζ, we have

$$F^2 = \frac{2}{|\zeta(0)|(|\zeta(0)|+1)}. \tag{29}$$

Relation (29) imposes the dynamic boundary condition at $x = -\infty$. This provides the last equation. This system of $N+4$ nonlinear equations with $N+4$ unknowns is solved by Newton's method. The coefficients a_n decrease rapidly; for example, $|a_{60}/a_1| \sim 10^{-7}$ for $b = -0.5$. The numerical results show that there is a one-parameter family of solutions. The parameter is chosen to be the length b of the gate BA. From (27), the upstream depth can be expressed as a function of the Froude number F as

$$\alpha = \frac{1}{2}\left(\sqrt{1+8/F^2} - 1\right). \tag{30}$$

Typical profiles are shown in Figures 11–13. As $b \to -1$, $F^2 \to 1$ and the length BA approaches zero whereas the flow reduces to the uniform stream. As $b \to 0$, $F^2 \to \infty$ and the length of the vertical wall approaches infinity. The corresponding value of C_c can be obtained by using the free-streamline theory. For $F^2 = \infty$, (14) reduces to $u^2 + v^2 = 1$. It follows from (12) that the solution is

$$u_\infty - i v_\infty = t^{\frac{1}{2}}. \tag{31}$$

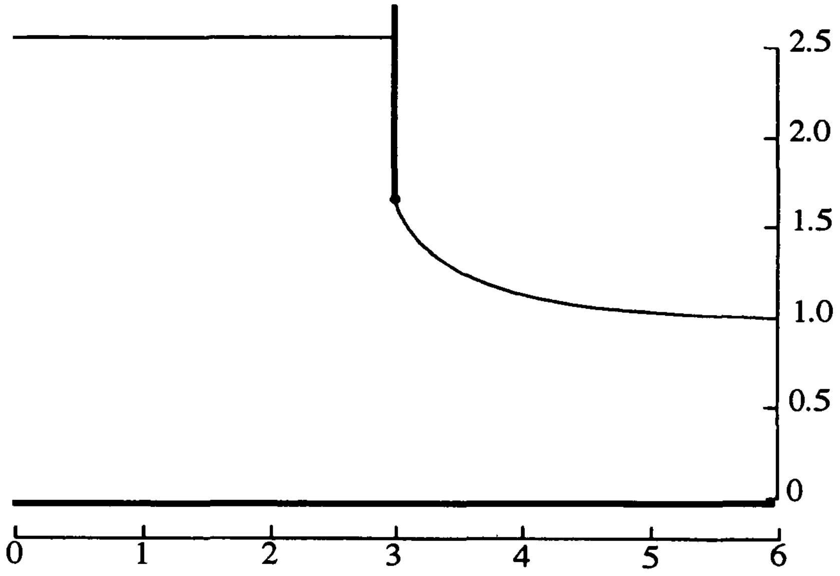

FIGURE 12. Computed free surface profile for $b = -0.2 (F^2 = 3.68)$.

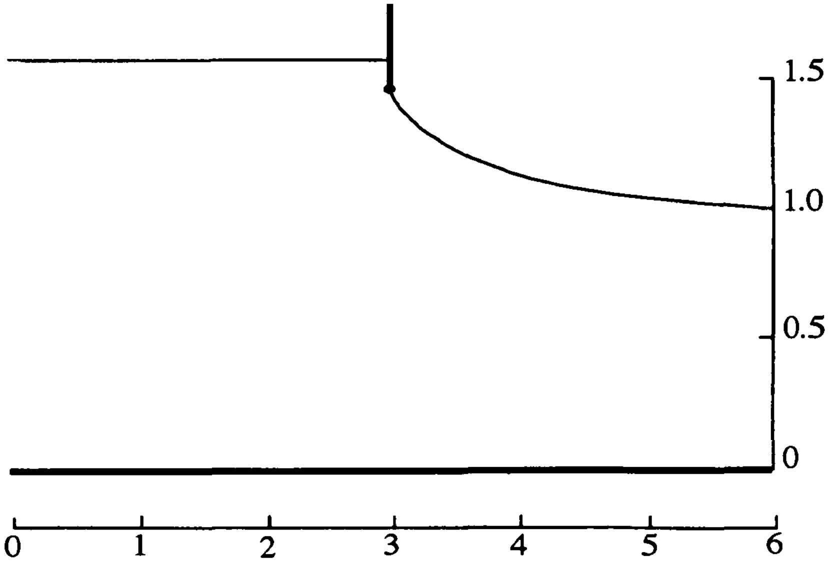

FIGURE 13. Computed free surface profile for $b = -0.6 (F^2 = 1.931)$.

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The distance D of the separation point A from the bottom JI can be obtained by integrating (17), using (31), with respect to σ from $\sigma = 0$ to $\sigma = \pi$. This gives

$$C_c = \pi/(\pi + 2) \approx 0.61.$$

For each $-1 \leq b \leq 0$, there is a unique solution. We note that for all solutions, the free surface is completely on the right of the gate (that is, there are no solutions similar to the ones in Figures 3 and 8 for the sluice gate problem).

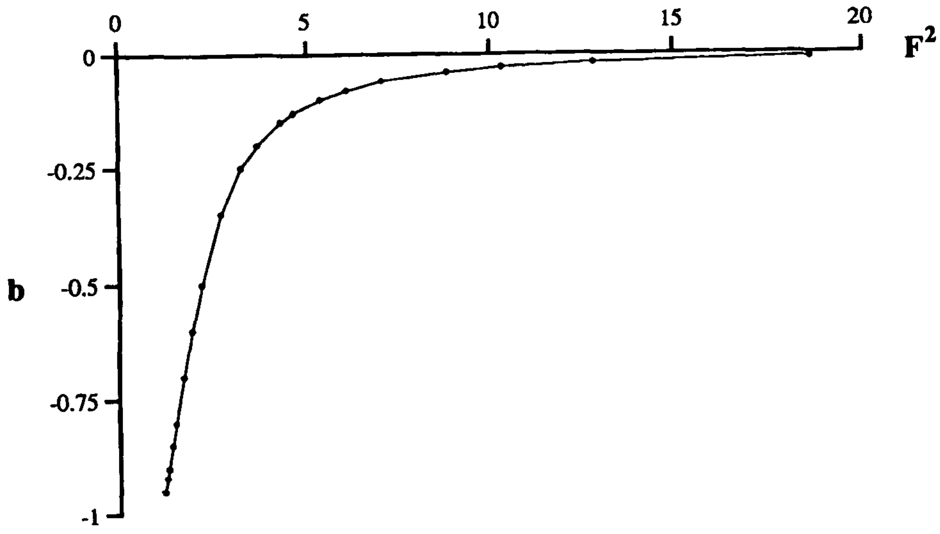

FIGURE 14. The length b of the gate is shown here as a function of F^2.

Figure 14 shows that the relationship between b and F^2 is a one-to-one correspondence. As $b \to 0$, $F^2 \to \infty$. The critical value $F^2 = 1$ corresponds to $b = -1$.

In Figure 15 we present the numerical values of C_c versus F^2. Here C_c is the contraction coefficient defined in (19). Figure 16 gives the contraction coefficient C_c as a function of $D/\tilde{H}$, where D is the gate opening and $\tilde{H}$ is the upstream depth (see Figure 11). Many previous results on sluice gates were limited to large values of F, that is, $D/\tilde{H} < 0.5$ (see [6]). Our results extend their findings to all values of $F \geq 1$. We note that on the range $D/\tilde{H} < 0.5$, C_c is a monotonic decreasing function. This is also the case in the results presented in Figure 13 of [6].

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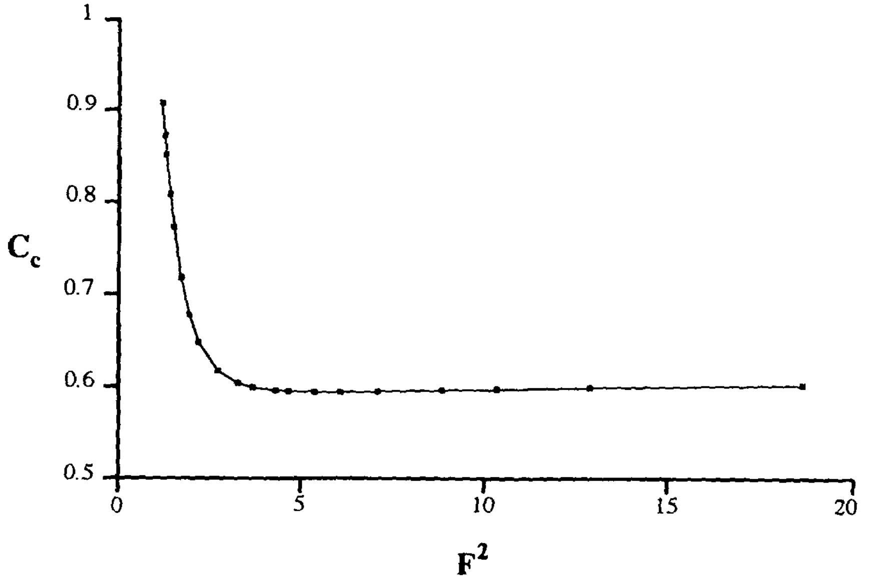

FIGURE 15. The contraction ratio C_c is shown here as a function of F^2.

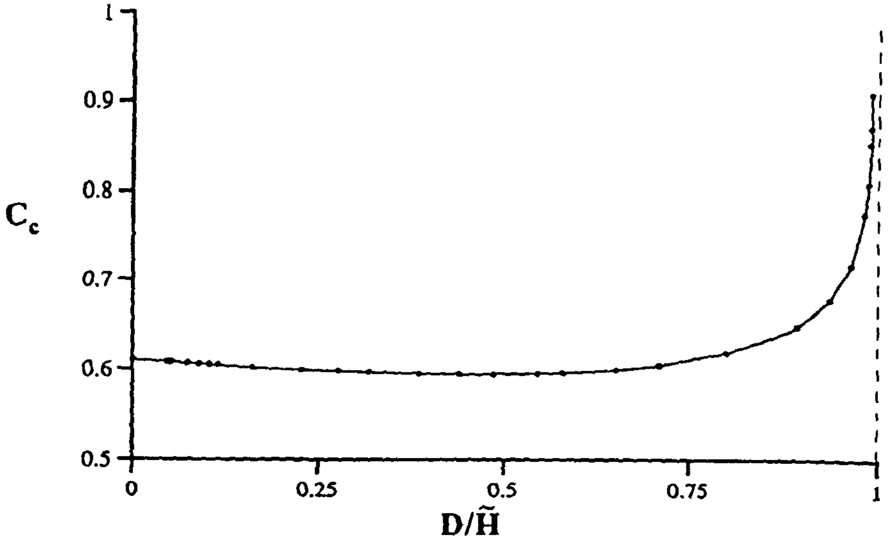

FIGURE 16. Relationship between C_c and the ratio of the gate opening to the upstream depth $D/\tilde{H}$.

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6. Flow under a sluice gate without the rigid lid approximation

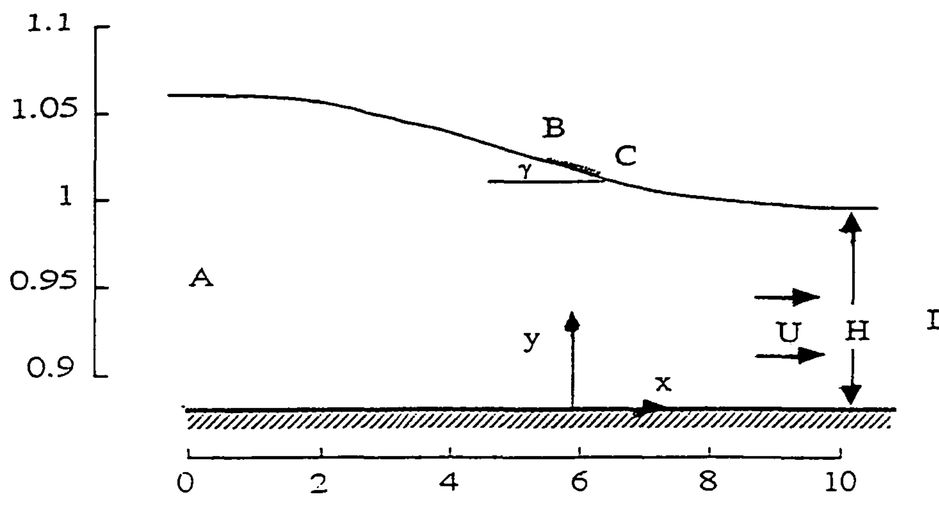

FIGURE 17. Sketch of the flow past a gate inclined at an angle γ. The flow is subcritical upstream and supercritical downstream. The endpoints of the gate are at B and C. The profile shown is a computed solution for $F^2 = 1.1$ and $\beta = \frac{2\pi}{5}$.

In this section we consider the complete sluice gate problem (that is, without the rigid lid approximation) for an an inclined sluice gate (see Figure 17). The gate inclination is denoted by γ. The x-axis is along the bottom AD and the y-axis is directed vertically upwards through the midpoint of the gate. As $|x| \to \infty$, the flows approach uniform streams with velocity $\tilde{U}$ and depth $\tilde{H}$ upstream, and velocity U and depth H downstream. The flow is subcritical upstream and supercritical downstream. The downstream and upstream Froude numbers are defined by (1) and (24) respectively. The flow is assumed to leave tangentially at both ends of the gate.

We introduce dimensionless variables by taking $\left(Q^2/g\right)^{\frac{1}{3}}$ as the unit length and $(Qg)^{\frac{1}{3}}$ as the unit velocity. Here Q denotes the discharge of the flow. As in Section 2, we define the potential function ϕ, the streamfunction ψ, the complex potential function $f = \phi + i\psi$, and the complex velocity function $\zeta = df/dz$. Here $z = x + iy$. Without loss of generality, we choose $\phi = 0$ so that $\phi_b = -\phi_c$. Here ϕ_b and ϕ_c are the values of ϕ at the separation points B and C respectively. The bottom AD is a streamline on which $\psi = 0$. The flow region in the f-plane is the strip $0 \le \psi \le 1$.

In terms of the dimensionless variables, the constant pressure condition on the free

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surface can be written

$$|\zeta|^2 + 2y = 3 \quad \text{on } AB \text{ and } CD. \tag{32}$$

We map the flow region in the f-plane onto the upper half of the unit circle in the complex t-plane by the transformation

$$f = \frac{2}{\pi} \log\left(\frac{1+t}{1-t}\right). \tag{33}$$

The free surface AB, CD and the gate BC are mapped onto the circumference of the half unit circle, whereas the bottom AD of the channel goes onto the real diameter. We use the notation $t = re^{i\sigma}$, so that the free surfaces and the gate are described by $r = 1$ and $0 \le \sigma \le \pi$. The images of points B and C are $t_2 = -e^{-i\beta}$ and $t_1 = e^{i\beta}$ respectively. Here $0 \le \beta \le \pi/2$.

As $\phi \to \infty$, the flow approaches a uniform supercritical stream. The local behavior of ζ as $\phi \to \infty$ is

$$\zeta \sim E + Ke^{-\pi\lambda f} \quad \text{as } \phi \to \infty. \tag{34}$$

Here E and K are constants to be found as part of the solution and λ is the smallest positive root of (8).

It follows from our choice of dimensionless variables that the Froude number F is related to the velocity downstream by

$$F = |\zeta(1)|^{\frac{3}{2}}. \tag{35}$$

Furthermore, the Froude number and the ratio $H/\tilde{H}$ of the depths at infinity are related by (30).

At the separation points B and C, the behavior of the flow is similar to those of free-streamline problems. Therefore, we expect the flow to behave at these separation points like

$$\zeta \sim G + H[f - (\mp b + i)]^{\frac{1}{2}} \quad \text{as } f \to (\mp b + i). \tag{36}$$

Here $b + i$ and $-b + i$ denote the values of f at C and B respectively, and G and H are constants to be found as part of the solution.

We solve the problem by following the series truncation procedure of Section 2. Using (33), (34) and (36), we represent ζ by the expansion

$$\zeta(t) = F^{\frac{2}{3}} \exp\Big[A(1-t)^{2\lambda} + B_1(t^2 + 1 - 2t\cos\beta)^{\frac{1}{2}} + B_2(t^2 + 1 + 2t\cos\beta)^{\frac{1}{2}}$$
$$- B_1(2 - 2\cos\beta)^{\frac{1}{2}} - B_2(2 + 2\cos\beta)^{\frac{1}{2}} + \sum_{n+1}^{\infty} a_n(t^n - 1)\Big]. \tag{37}$$

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The coefficients a_n and the constants $A,\ B_1,\ B_2$ are to be determined so that $\zeta(t)$ satisfies (32) and the kinematic conditions on AD and BC. It can easily be seen that the kinematic condition $v = 0$ on AD is satisfied if we require $a_n,\ A,\ B_1,\ B_2$ to be real. Differentiating (32) with respect to σ, we obtain

$$u(\sigma)u_\sigma(\sigma) + v(\sigma)v_\sigma(\sigma) - \frac{2}{\pi \sin\sigma}\frac{v(\sigma)}{u^2(\sigma)+v^2(\sigma)} = 0 \quad \text{on } AB \text{ and } CD. \tag{38}$$

The kinematic condition on BC can be expressed as

$$v(\sigma) = -u(\sigma)\tan\gamma \quad \text{on } BC. \tag{39}$$

Note that γ is measured clockwise from the negative x-axis. The problem is now to find the coefficients a_n and the constants $A,\ B_1,\ B_2$ in (37) so that (38) and (39) are satisfied. Once ζ is known, we can calculate the profile of the free surface by integrating $\frac{\partial x}{\partial\phi} + i\frac{\partial y}{\partial\phi} = \frac{1}{\zeta}$ numerically along the circumference of the unit circle.

We approximate the problem numerically by truncating the infinite series in (37) after N terms. There are $N+5$ unknowns, $\gamma,\ \lambda,\ A,\ B_1,\ B_2$ and $\{a_i\}_{i=1}^N$, for given values of β and F^2. We define the $N+3$ collocation points

$$\sigma_i = \frac{\pi}{N+3}\left(i - \frac{1}{2}\right), \quad 1,\dots,N+3. \tag{40}$$

For simplicity, we restrict the angle β to the form

$$\beta = \frac{\pi}{N+3}B, \tag{41}$$

where B is an integer smaller than $(N+3)/2$. We obtain $N+3$ equations by satisfying (38) at σ_i, $i = 1,\dots,B-1$ and $i = (N+3)/2 + B,\dots,N+3$, and (39) at σ_i, $i = B,\dots,(N+3)/2 + B - 1$. Relation (30) provides another equation. The last equation is obtained by relating the Froude number to the upstream and the downstream velocities by using (30) and the conservation of mass $UH = \tilde{U}\tilde{H}$. This yields

$$F^2 = \frac{2|\zeta(1)|^2}{|\zeta(-1)|(|\zeta(1)| + |\zeta(-1)|)}. \tag{42}$$

The system of $N+5$ nonlinear equations is solved by Newton's method.

The numerical scheme was used to compute solutions for various values of F^2 and β. For F^2 close to one, the coefficients a_n decrease rapidly (see Table 2). Most of the results presented here were obtained with $N = 400$.

Typical profiles are shown in Figures 17 and 18. As $\beta \to \pi/2$, the length L of the gate tends to zero and the flow reduces to a uniform stream. As $\beta \to 0$, $L \to \infty$. Numerical values of L versus F^2 for $\beta = \frac{\pi}{3}$, $\frac{2\pi}{5}$ and $\frac{19\pi}{40}$ are shown in Figure 19.

TABLE 2. Some values of the coefficients a_n for $\beta = \frac{2\pi}{5}$ and various values of F^2.

F^2	a_1	a_{100}	a_{200}	a_{300}	a_{400}
1.05	0.71×10^{-1}	0.15×10^{-3}	-0.11×10^{-4}	0.11×10^{-7}	-0.50×10^{-9}
1.10	0.19	0.37×10^{-3}	-0.42×10^{-4}	-0.17×10^{-4}	0.89×10^{-8}
1.15	0.38	0.68×10^{-3}	-0.76×10^{-4}	-0.31×10^{-4}	0.45×10^{-7}
1.20	0.65	0.11×10^{-2}	-0.12×10^{-3}	-0.49×10^{-4}	0.12×10^{-6}

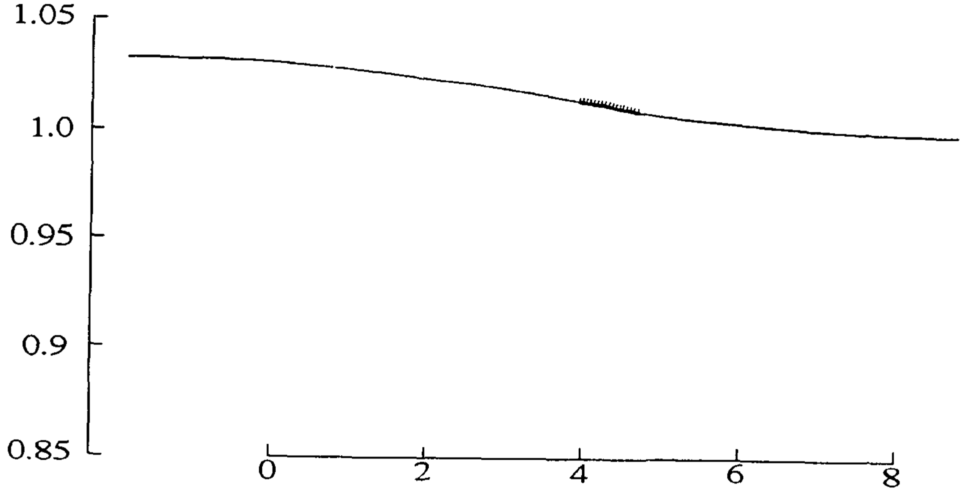

FIGURE 18. Computed profile for $F^2 = 1.05$ and $\beta = \frac{\pi}{3}$.

In Figure 20 we present numerical values of the gate inclination γ versus F^2 for various values of β. In Figure 21 we present numerical values of C_c versus F^2. For $F^2 = 1$, the flow reduces to a uniform stream and $C_c = 1$. For each value of F^2, the ratio $\tilde{H}/H$ can be evaluated by using (30).

As F^2 increases, the coefficients a_n decrease less rapidly and more collocation points are needed to obtain accurate solutions. More than 600 collocation points are required to compute accurate solutions for $F^2 \geq 1.35$. Therefore we have presented solutions only for $1 \leq F^2 < 1.35$.

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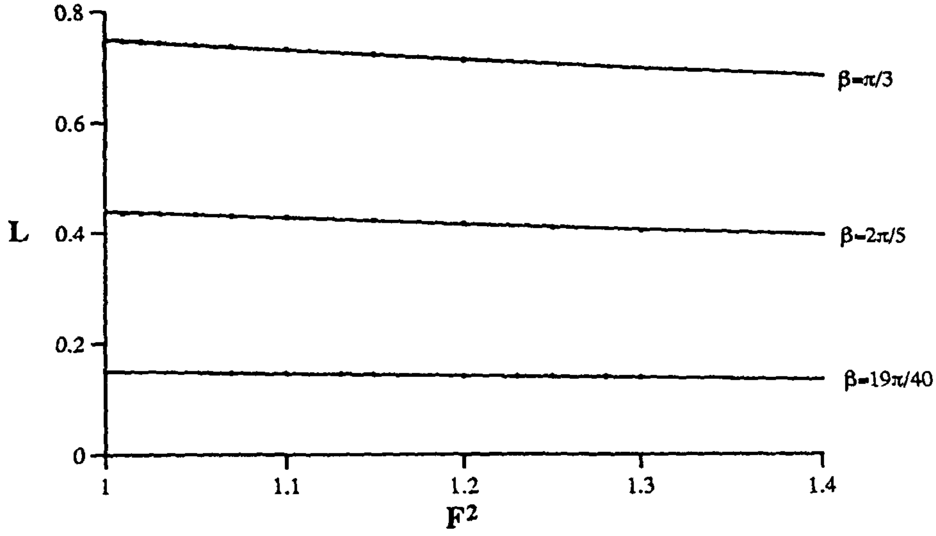

FIGURE 19. The length L of the gate is shown as a function of F^2 for three values of β.

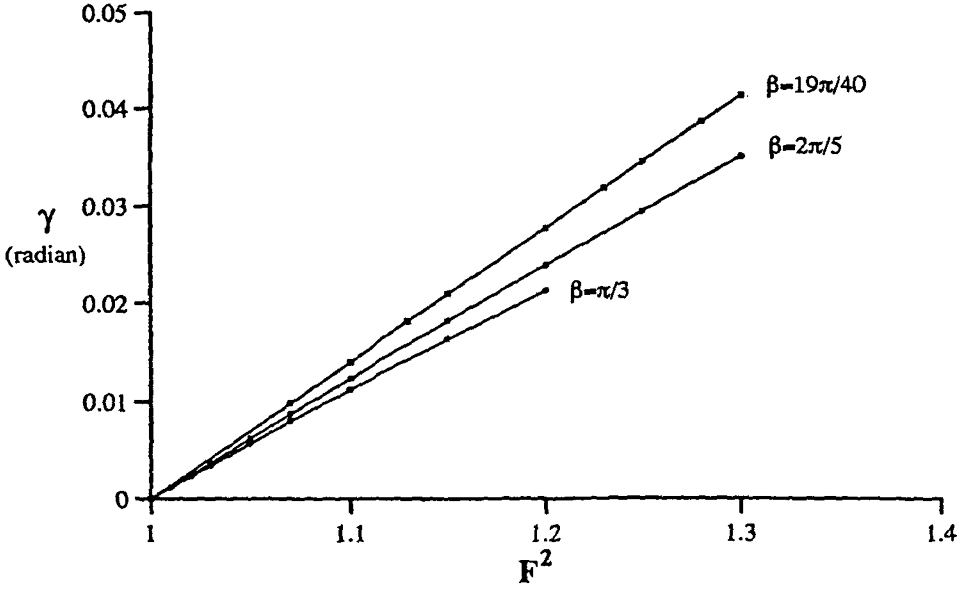

FIGURE 20. The gate inclination γ is shown as a function of F^2 for three values of β.

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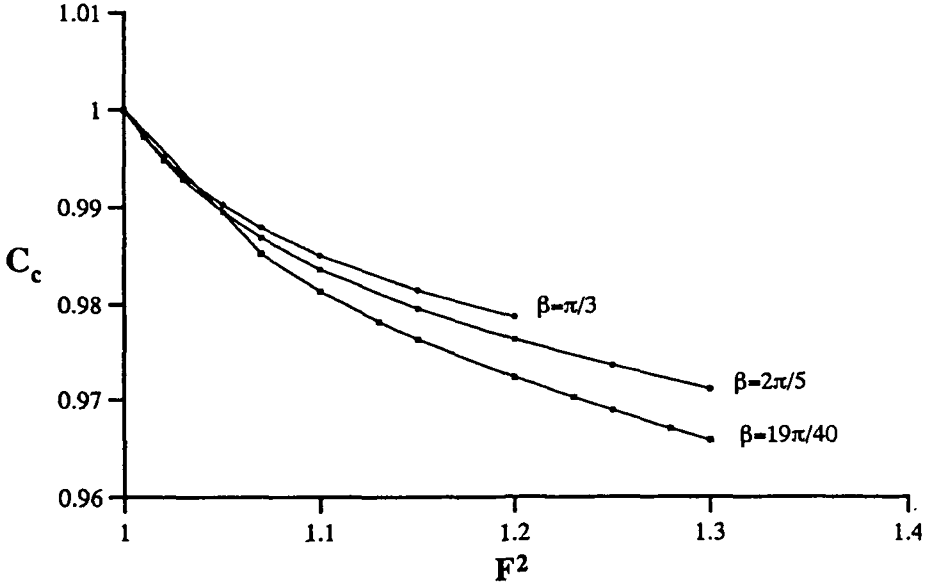

FIGURE 21. Relationship between the contraction coefficient C_c and F^2.

7. Conclusion

We have presented accurate numerical solutions for nonlinear free surface flows emerging from a vessel. The results show that there are two families of solutions. Each family depends continuously on two parameters, b and F. Some of the solutions have a free surface profile with a continuous slope at the separation point. Others have a stagnation point at the separation point with an angle of 90° or 120° between the free surface and the wall. It can be shown that these are the only three possible local behaviors at the intersection of a free surface and a rigid wall in the abscence of surface tension (see Dagan and Tulin [5], Tayler [10] and Lee and Vanden-Broeck [7]). Recently Vanden-Broeck and Tuck [14] constructed similarity solutions to describe these local behaviors.

We have used the same flow configuration to model the flow under a sluice gate when the upstream free surface is replaced by a rigid lid. We have shown that there is a solution for each value of $F \geq 1$. These results extend previous calculations, for F large.

Finally we have considered the complete sluice gate problem with two free surfaces and constructed solutions for which the free surfaces leave tangentially at both separation points. For a fixed inclination γ of the gate, there is a one-parameter family of such solutions. Accurate solutions could be obtained only for small values of γ. Numerical calculations based on a boundary integral equation method are now in progress to extend the results to larger values of γ.

Acknowledgement

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References

[1] T. B. Benjamin, "On the flow in channels when rigid obstacles are placed in the stream", *J. Fluid Mech.* **1** (1956) 227–248.

[2] T. B. Benjamin, "Gravity currents and related phenomena", *J. Fluid Mech.* **31** (1968) 209–248.

[3] A. M. Binnie, "The flow of water under a sluice gate", *Quart. J. Mech. and Applied Math.* **5** (1952) 395–414.

[4] P. Budden and J. Norbury, "Uniqueness of free-boundary flows under gravity", *Arch. Rat. Math. Anal.* **78** (1982) 361–380.

[5] G. Dagan and M. P. Tulin, "Two-dimensional free-surface gravity flow past blunt bodies", *J. Fluid Mech.* **51** (1972) 529–543.

[6] D. D. Fangmeier and T. S. Strelkoff, "Solution for gravity flow under a sluice gate", *J. Eng. Mech.* **94** (1968) 153–176.

[7] M. I. Gurevich, *The theory of jets in an ideal fluid*, International Series of Monographs in Pure and Applied Mathematics, Vol. 93 (Pergamon Press, 1961).

[8] J. Lee and J.-M. Vanden-Broeck, "Two-dimensional jets falling from funnels and nozzles", *Phys. Fluids A* **5** (1993) 2454–2460.

[9] J. H. Masliyah, K. Nandakumar, F. Hemphill and L. Fung, "Body-fitted coordinates for flow under sluice gates", *J. Hydr. Eng.* **111** (1985) 922–933.

[10] A. B. Tayler, "Singularities at flow separation points", *Quart. Journ. Mech. and Applied Math.* **26** (1973) 153–172.

[11] J.-M. Vanden-Broeck, "Flow under a gate", *Phys. Fluids* **29** (1986) 3148–3151.

[12] J.-M. Vanden-Broeck and J. B. Keller, "Weir flows", *J. Fluid Mech.* **176** (1987) 283–293.

[13] J.-M. Vanden-Broeck and J. B. Keller, "Surfing on solitary waves", *J. Fluid Mech.* **198** (1989) 115–125.

[14] J.-M. Vanden-Broeck and E. O. Tuck, "Flow near the intersection of a free surface with a vertical wall", *SIAM J. Appl. Math.* **54** (1994) 1–13.