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# ON UTILITY-BASED SUPERREPLICATION PRICES OF CONTINGENT CLAIMS WITH UNBOUNDED PAYOFFS

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**Abstract**

Consider a financial market in which an agent trades with utility-induced restrictions on wealth. For a utility function which satisfies the condition of reasonable asymptotic elasticity at $-\infty$, we prove that the utility-based superreplication price of an unbounded (but sufficiently integrable) contingent claim is equal to the supremum of its discounted expectations under pricing measures with finite loss-entropy. For an agent whose utility function is unbounded from above, the set of pricing measures with finite loss-entropy can be slightly larger than the set of pricing measures with finite entropy. Indeed, the former set is the closure of the latter under a suitable weak topology. Central to our proof is a proof of the duality between the cone of utility-based superreplicable contingent claims and the cone generated by pricing measures with finite loss-entropy.

*Keywords:* Superreplication; incomplete market; contingent claim; duality theory; weak topology

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## 1. Introduction

Consider a financial market where the discounted prices of $d$ risky assets are modelled over a finite time interval $[0, T]$ by an $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{0 \le t \le T}$, on a filtered probability space $(\Omega, F, (F_t)_{t \in [0,T]}, \mathrm{P})$ satisfying the usual conditions of right continuity and saturatedness. A portfolio on such a market can be represented by a pair $(x, H)$, consisting of an initial wealth $x \in \mathbb{R}$ and a predictable, $S$-integrable process $H$ representing the holdings of the $d$ risky assets. It is assumed that, at any time, all remaining wealth is invested in the numeraire. The discounted wealth process corresponding to the portfolio $(x, H)$ is defined by $X_t^{(x,H)} := x + \int_0^t H_u \, \mathrm{d}S_u$.

Two important theoretical concepts within the above framework for models of financial markets are those of 'no arbitrage' and completeness. An arbitrage opportunity is defined as a trading strategy $H$ such that $X_T^{(0,H)} \ge 0$, P-almost surely (P-a.s.) and such that $\mathrm{P}(X_T^{(0,H)} > 0) > 0$. A model is usually said to satisfy the condition of no arbitrage if there does not exist an *admissible* trading strategy which is an arbitrage opportunity. The condition on $H$ of admissibility is the requirement that the wealth process $X^{(0,H)}$ is uniformly bounded below by a constant; ruling out processes which are unbounded from below is one way to disallow the use of doubling strategies.

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In the celebrated papers [3] and [5] the condition of no arbitrage was weakened to that of no free lunch with vanishing risk (NFLVR), and it was shown that a model satisfies NFLVR if and only if the set, $M_\sigma^{\mathrm{e}}$, of equivalent $\sigma$-martingale measures is nonempty.

A model satisfying NFLVR is then said to be complete if the set $M_\sigma^{\mathrm{e}}$ is a singleton (i.e. if $M_\sigma^{\mathrm{e}} = \{\mathrm{Q}\}$). In a complete market model it is possible, with the use of an admissible trading strategy, to replicate and thereby uniquely price all contingent claims in $L^1(\mathrm{Q})$ with payoffs bounded from below (see [5, Theorem 5.16]). In contrast to this, if a market is incomplete, there exist contingent claims with payoffs bounded from below which cannot be replicated by admissible trading strategies. For such contingent claims there exists an open interval of arbitrage-free prices, rather than a unique price.

Given a general contingent claim with payoff $X$, it is easy to see that an upper bound for the interval of arbitrage-free prices is given by the superreplication price

$$\overline{\pi}(X) := \inf\{x \in \mathbb{R}\colon \text{there exists an admissible } H \text{ such that } X \le X_T^{(x,H)}\}.$$

As a special case of [5, Theorem 5.5], we know that, for a contingent claim with payoff $X$ bounded from below, the superreplication price is in fact the least upper bound for the interval of arbitrage-free prices, in other words

$$\overline{\pi}(X) = \sup_{\mathrm{Q}\in M_\sigma^{\mathrm{e}}} \mathrm{E}_\mathrm{Q}[X]. \tag{1.1}$$

However, for contingent claims with payoffs unbounded from below, admissible trading strategies are unsuitable for superreplication, and this dual representation of the superreplication price does not always hold. Indeed, Biagini and Frittelli [1, Example 8] constructed a market model and a contingent claim with payoff $X$ unbounded from below such that

$$\overline{\pi}(X) > \sup_{\mathrm{Q}\in M_\sigma^{\mathrm{e}}} \mathrm{E}_\mathrm{Q}[X]. \tag{1.2}$$

The reason for the breakdown of (1.1) is that the cone

$$K^{\mathrm{adm}} := \{X_T^{(0,H)}\colon H \text{ is admissible}\}$$

is not closed with respect to a weak topology induced by the set of pricing measures.

It is useful at this point to slightly extend the definition of a superreplication price to allow terminal wealths from an arbitrary convex cone $K \subseteq L^0(\mathrm{P})$. Let

$$\overline{\pi}(X; K) := \inf\{x \in \mathbb{R}\colon X \le x + Y \text{ for some } Y \in K\}.$$

Of course, $\overline{\pi}(X) = \overline{\pi}(X; K^{\mathrm{adm}})$. Note that if $K$ is a solid cone in a subspace $F$ of $L^0(\mathrm{P})$ (i.e. $X \in F$ and $X \le Y \in K$ imply that $X \in K$) then

$$\overline{\pi}(X; K) := \inf\{x \in \mathbb{R}\colon X - x \in K\}.$$

If $K$ is not solid then we may of course replace $K$ by the smallest solid cone containing $K$ without affecting the superreplication price.

Although the cone $K$ is arbitrary, it will be fixed throughout the paper. Several candidates for $K$ may be appropriate including, among many other classes, *admissible* strategies (see above), *acceptable* strategies (see, e.g. [4]), or *permissible* strategies (see, e.g. [9]).

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We are now able to formulate the following natural question with (1.2) in mind. Given an arbitrary convex cone $K$ of contingent claims, is it possible to find a minimal solid, closed convex cone $C \supseteq K$, and a suitable set $M$ of pricing measures, such that

$$\overline{\pi}(X; C) = \sup_{Q \in M} \mathrm{E}_Q[X] \tag{1.3}$$

for *all* (possibly) unbounded $X$ which are integrable with respect to measures in $M$?

A positive answer to this question was given in [1]. In this paper the preferences of an investor were incorporated in the construction of a weakly closed, enlarged cone $C$ by means of the convex conjugate of the investor's utility function. The set $M$ of measures consisted of those absolutely continuous separating measures with finite entropy. A dual representation of the form of (1.3) was obtained for utility functions which are bounded from above.

Inspired by [1], this result has since been extended in [8] to unbounded utility functions with reasonable asymptotic elasticity at both $-\infty$ and $+\infty$, with subsequent alternative proofs given in [2] and [7].

In this paper we show that the condition of reasonable asymptotic elasticity at $+\infty$ is unnecessary, and can be dropped, by formulating the superreplication result in terms of the set of separating measures with finite loss-entropy (see Section 3).

## 2. Assumptions on $U$

The following assumption holds throughout the paper.

**Assumption 2.1.** *We assume that the investor has a critical wealth $a \in [-\infty, \infty)$ and a utility function $U\colon (a, \infty) \to \mathbb{R}$ which is increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions,*

$$\lim_{x \downarrow a} U'(x) = \infty \quad \textit{and} \quad \lim_{x \uparrow \infty} U'(x) = 0.$$

*Furthermore, if the domain of $U$ is the whole real line (i.e. $a = -\infty$) then we assume that $U$ has reasonable asymptotic elasticity at $-\infty$, in the sense that*

$$\mathrm{AE}_{-\infty}(U) := \liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1.$$

*This condition was introduced and discussed in detail in [11].*

The convex conjugate $V$ of the utility function $U$ is defined, for $y > 0$, by

$$V(y) = \sup_{x \in (a, \infty)} \{U(x) - xy\}.$$

It is well known (see [10, Section 26]) that under the conditions of Assumption 2.1, $V$ is strictly convex, continuously differentiable, and satisfies

$$V'(0) := \lim_{y \downarrow 0} V'(y) = -\infty \quad \text{and} \quad V'(\infty) := \lim_{y \uparrow \infty} V'(y) = -a. \tag{2.1}$$

We should point out to the reader that the interesting case is when $a = -\infty$. The following lemma, which is a simple consequence of [11, Proposition 4.1(iii)], provides an equivalent formulation of reasonable asymptotic elasticity at $-\infty$.

**Lemma 2.1.** *Let $U$ be a utility function defined on the whole real line. The following conditions are equivalent:*

(i) *$U$ has reasonable asymptotic elasticity at $-\infty$;*

(ii) *there exists $b > 0$ such that $V$ is positive and increasing on $(b, \infty)$ and, for any $\alpha > 1$, there exists $D > 0$ such that $V(\alpha y) \leq DV(y)$ for all $y \in (b, \infty)$.*

*Proof. Condition (i) implies condition (ii).* Since $U$ has reasonable asymptotic elasticity at $-\infty$, a repeated application of [11, Proposition 4.1(iii)] implies that there exist constants $y_0 > 0$, $\lambda > 1$, and $C > 0$ such that $V(\lambda^n y) \leq C^n V(y)$ for $y \geq y_0$ and $n \in \mathbb{N}$.

From (2.1) we see that $V'(\infty) = \infty$, so there exists a $b_0 > 0$ such that $V$ is positive and increasing on $(b_0, \infty)$. Set $b := \max\{y_0, b_0\} > 0$. Given any $\alpha > 1$, there exists $n \in \mathbb{N}$ such that $\alpha \leq \lambda^n$. For $y \geq b$, we have $V(\alpha y) \leq V(\lambda^n y) \leq C^n V(y) = DV(y)$, where $D := C^n$.

*Condition (ii) implies condition (i).* This is an immediate consequence of the fact that [11, Proposition 4.1(iii)] implies [11, Proposition 4.1(i)].

## 3. Finite loss-entropy measures

Recall that $K \subseteq L^0(\mathrm{P})$ is an arbitrary cone which has been fixed throughout the paper. Relative to the cone $K$, we define the convex set of *separating*, or *pricing* measures by

$$M_1 := \{\mathrm{Q} \ll \mathrm{P} \colon X \in L^1(\mathrm{Q}) \text{ and } \mathrm{E}_{\mathrm{Q}}[X] \leq 0 \text{ for all } X \in K\}.$$

In what follows, we refer frequently to the function $V^+ := \max\{V, 0\}$. Note, however, that in most places we can drop the '+', since $V$ is convex, and its graph can be bounded from below by a straight line.

**Definition 3.1.** A measure $\mathrm{Q} \ll \mathrm{P}$ is said to have finite loss-entropy if there exists a constant $b > 0$ such that

$$\mathrm{E}_{\mathrm{P}}\biggl[V^+\biggl(\frac{\mathrm{dQ}}{\mathrm{dP}}\biggr)\mathbf{1}_{\{\mathrm{dQ}/\mathrm{dP} \geq b\}}\biggr] < \infty. \tag{3.1}$$

The set of pricing measures with finite loss-entropy is denoted by

$$\widehat{M}_V := \{\mathrm{Q} \in M_1 \colon \mathrm{Q} \text{ has finite loss-entropy}\}.$$

**Remark 3.1.** (i) We use the notation '$\widehat{M}_V$' to distinguish the set of finite loss-entropy measures from the set $M_V$ of finite-entropy measures used in [1].

(ii) It is easy to see that if (3.1) holds for some constant $b > 0$ then it holds for all $b > 0$. In other words, if $\mathrm{Q} \in \widehat{M}_V$ then $\mathrm{E}_{\mathrm{P}}[V^+(\mathrm{dQ}/\mathrm{dP})\,\mathbf{1}_{\{\mathrm{dQ}/\mathrm{dP} \geq b\}}] < \infty$ for all $b > 0$.

(iii) If the domain of $U$ is a half-line then the 'truncated' convex conjugate function $V(y)\,\mathbf{1}_{\{y \geq b\}}$ can be bounded above and below by linear functions. In such cases it immediately follows that $\widehat{M}_V = M_1$, i.e. all measures have finite entropy.

**Lemma 3.1.** *The set $\widehat{M}_V$ is convex.*

*Proof.* For the case where $U$ is defined on a half real line (i.e. $a \in (-\infty, \infty)$) convexity is trivial, since $\widehat{M}_V = M_1$. Nevertheless, we give a universal proof. Since the function $V^+$ is

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convex and nonnegative, given arbitrary constants $0 < y \le z$ and $x, b > 0$, we have

$$V^+(y)\,\mathbf{1}_{[b,\infty)}(y) \le V^+(b)\,\mathbf{1}_{[b,\infty)}(y) + V^+(z)\,\mathbf{1}_{[b,\infty)}(y) \\ \le V^+(x)\,\mathbf{1}_{[b,\infty)}(x) + V^+(z)\,\mathbf{1}_{[b,\infty)}(z) + V^+(b). \tag{3.2}$$

Take $\alpha \in [0, 1]$, $\mathrm{Q}_1, \mathrm{Q}_2 \in \widehat{M}_V$, and define $\mathrm{Q}_\alpha := \alpha\mathrm{Q}_1 + (1-\alpha)\mathrm{Q}_2$. Let $b > 0$ be an arbitrary constant. Applying (3.2), we have

$$\begin{aligned} &\mathrm{E}_{\mathrm{P}}\biggl[V^+\biggl(\frac{\mathrm{dQ}_\alpha}{\mathrm{dP}}\biggr)\,\mathbf{1}_{\{\mathrm{dQ}_\alpha/\mathrm{dP}\ge b\}}\biggr] \\ &\quad\le \mathrm{E}_{\mathrm{P}}\biggl[V^+\biggl(\frac{\mathrm{dQ}_1}{\mathrm{dP}}\biggr)\,\mathbf{1}_{\{\mathrm{dQ}_1/\mathrm{dP}\ge b\}}\biggr] + \mathrm{E}_{\mathrm{P}}\biggl[V^+\biggl(\frac{\mathrm{dQ}_2}{\mathrm{dP}}\biggr)\,\mathbf{1}_{\{\mathrm{dQ}_2/\mathrm{dP}\ge b\}}\biggr] + V^+(b) \\ &\quad< \infty. \end{aligned}$$

## 4. The superreplicable contingent claims

Let $L_U := \bigcap_{\mathrm{Q}\in\widehat{M}_V} L^1(\mathrm{Q})$ denote the vector space of all $\widehat{M}_V$-integrable contingent claims. Note that owing to the definition of $M_1$ it follows that the fixed cone $K$ is a subset of $L_U$. Consider the solid convex cone

$$K_U := \{X \in L_U : X \le \widetilde{X} \text{ for some } \widetilde{X} \in K\},$$

of all $\widehat{M}_V$-integrable contingent claims that can be dominated by a terminal wealth in $K$. Throughout we shall adopt the common practise of identifying probability measures $\mathrm{Q} \ll \mathrm{P}$ with their Radon–Nikodym derivative $\mathrm{dQ}/\mathrm{dP} \in L^1_+(\mathrm{P})$. Following this convention, we let $L_V^+$ denote the subspace of $L^1(\mathrm{P})$, formed by taking the linear span of the Radon–Nikodym derivatives of all finite loss-entropy pricing measures. When endowed with the bilinear form

$$\langle X, X^+\rangle = \mathrm{E}_{\mathrm{P}}[XX^+],$$

it is easy to show that the pair $(L_U, L_V^+)$ becomes a left dual system (see [7] and [8]). We now define

$$C_U := \overline{K_U}^{\sigma(L_U, L_V^+)}.$$

The set $C_U$ is the smallest solid, closed, convex cone in $L_U$ containing $K$, and is useful for a duality theory. We use the subscript $U$ to stress the weak dependence of this set upon $U$. In Section 6 we give a characterisation of $C_U$ in terms of intersections of all $L^1(\mathrm{Q})$ closures of $K_U$. We call $\overline{\pi}(X; C_U)$ the *utility-based superreplication price* of $X$.

## 5. Main results

Our main results are Theorems 5.1 and 5.2. In both theorems we assume that $U$ satisfies Assumption 2.1, and that $\widehat{M}_V \ne \varnothing$. Note that since only the asymptotic growth of $V$ at $+\infty$ is required to decide if a measure has finite loss-entropy, it turns out that the latter condition is a delicate joint condition on the asymptotic behaviour of the utility function at $-\infty$ and the cone $K$. We also remark that the existence of an equivalent separating measure for $K$ is intimately connected to the absence of arbitrage within $K$; however, we require only the slightly weaker existence of an absolutely continuous separating measure.

Given a nonempty set $A \subseteq L_U$, we let $\operatorname{cone}(A)$ denote the smallest convex cone containing $A$, and we define its *polar cone* $A^{\lhd} \subseteq L_V^+$ by

$$A^{\lhd} := \{X^+ \in L_V^+ \colon \langle X, X^+ \rangle \leq 0 \text{ for all } X \in A\}.$$

For a nonempty set $B \subseteq L_V^+$, we define in a similar way $\operatorname{cone}(B)$, and we define its polar cone $B^{\lhd} \subseteq L_U$ by

$$B^{\lhd} := \{X \in L_U \colon \langle X, X^+ \rangle \leq 0 \text{ for all } X^+ \in B\}.$$

The following simple result about polar cones follows immediately from [12, Theorem 0.8].

**Lemma 5.1.** *Let $A \subseteq L_U$ be nonempty. Then $A^{\lhd} \subseteq L_V^+$ is a $\sigma(L_V^+, L_U)$-closed convex cone. Moreover,*

$$(\operatorname{cone}(A))^{\lhd} = \overline{A}^{\lhd} = A^{\lhd} \quad \textit{and} \quad A^{\lhd\lhd} = \overline{\operatorname{cone}(A)},$$

*where closures are taken in the $\sigma(L_U, L_V^+)$ topology.*

**Theorem 5.1.** *Suppose that $U$ satisfies Assumption 2.1 and that $\widehat{M}_V \neq \varnothing$. Then*

$$L_V^+ \cap \operatorname{cone}(M_1) = \operatorname{cone}(\widehat{M}_V) = K_U^{\lhd} = C_U^{\lhd} \tag{5.1}$$

*and*

$$C_U = (\widehat{M}_V)^{\lhd}.$$

*Proof.* To obtain (5.1), we first show that $L_V^+ \cap M_1 \subseteq \widehat{M}_V$. To this end, take any $\mathrm{Q} \in L_V^+ \cap M_1$. Since $\mathrm{Q}$ is a probability measure, it follows, from the definition of $L_V^+$, that $\mathrm{Q} = \alpha \mathrm{Q}_0 - (\alpha - 1)\mathrm{Q}_1$ for some $\mathrm{Q}_0, \mathrm{Q}_1 \in \widehat{M}_V$ and some $\alpha \geq 1$. We now show that $\mathrm{Q} \in \widehat{M}_V$.

In the case where $U$ is defined on the whole real line, let $b > 0$ be the constant from the statement of Lemma 2.1. Owing to Lemma 2.1, we have

$$\begin{aligned} \mathrm{E}_\mathrm{P}\left[V^+\left(\frac{\mathrm{dQ}}{\mathrm{dP}}\right)\mathbf{1}_{\{\mathrm{dQ}/\mathrm{dP} \geq \alpha b\}}\right] &\leq \mathrm{E}_\mathrm{P}\left[V^+\left(\alpha\frac{\mathrm{dQ}_0}{\mathrm{dP}}\right)\mathbf{1}_{\{\mathrm{dQ}_0/\mathrm{dP} \geq b\}}\right] \\ &\leq D\,\mathrm{E}_\mathrm{P}\left[V^+\left(\frac{\mathrm{dQ}_0}{\mathrm{dP}}\right)\mathbf{1}_{\{\mathrm{dQ}_0/\mathrm{dP} \geq b\}}\right] \\ &< \infty, \end{aligned}$$

so $\mathrm{Q} \in \widehat{M}_V$. The case where $U$ is defined on a half real line (i.e. $a \in (-\infty, \infty)$) is trivial owing to Remark 3.1(iii). Consequently,

$$L_V^+ \cap \operatorname{cone}(M_1) \subseteq \operatorname{cone}(L_V^+ \cap M_1) \subseteq \operatorname{cone}(\widehat{M}_V).$$

Now take any $X \in K_U$ and any $\mathrm{Q} \in \widehat{M}_V$. There exists $\widetilde{X} \in K \subseteq L^1(\mathrm{Q})$ such that $X \leq \widetilde{X}$. Hence, $\mathrm{E}_\mathrm{Q}[X] \leq \mathrm{E}_\mathrm{Q}[\widetilde{X}] \leq 0$ and, therefore, $\operatorname{cone}(\widehat{M}_V) \subseteq K_U^{\lhd}$.

Conversely, since $-L_+^\infty(\mathrm{P}) \cup K \subseteq K_U$,

$$\begin{aligned} K_U^{\lhd} &\subseteq (-L_+^\infty(\mathrm{P}))^{\lhd} \cap K^{\lhd} \\ &= \{X^+ \in L^1(\mathrm{P}) \colon\ \mathrm{E}_\mathrm{P}[XX^+] \geq 0 \text{ for all } X \in L_+^\infty(\mathrm{P})\} \cap K^{\lhd} \\ &= L_+^1(\mathrm{P}) \cap K^{\lhd} \\ &= \{X^+ \in L_+^1(\mathrm{P}) \colon\ X^+ \in L_V^+ \text{ and } \mathrm{E}_\mathrm{P}[XX^+] \leq 0 \text{ for all } X \in K\} \\ &= L_V^+ \cap \operatorname{cone}(M_1). \end{aligned}$$

Moreover, applying Lemma 5.1, we obtain

$$C_U^{\lhd} = (\overline{K_U}^{\sigma(L_U, L_V^+)})^{\lhd} = K_U^{\lhd},$$

and (5.1) follows. A final application of Lemma 5.1 shows that

$$C_U = \overline{K_U}^{\sigma(L_U, L_V^+)} = K_U^{\lhd\lhd} = (\operatorname{cone}(\widehat{M}_V))^{\lhd} = (\widehat{M}_V)^{\lhd}.$$

**Theorem 5.2.** *Suppose that $U$ satisfies Assumption 2.1 and that $\widehat{M}_V \neq \varnothing$. Then, for any $X \in L_U$,*

$$\overline{\pi}(X; C_U) = \sup_{\mathrm{Q} \in \widehat{M}_V} \mathrm{E}_{\mathrm{Q}}[X].$$

*Proof.* Owing to Theorem 5.1, $C_U = (\widehat{M}_V)^{\lhd}$. Since $C_U$ is solid in $L_U$, we have

$$\begin{aligned}
\overline{\pi}(X; C_U) &= \inf\{x \in \mathbb{R} \colon X - x \in C_U\} \\
&= \inf\{x \in \mathbb{R} \colon \mathrm{E}_{\mathrm{Q}}[X - x] \leq 0 \text{ for all } \mathrm{Q} \in \widehat{M}_V\} \\
&= \inf\{x \in \mathbb{R} \colon \mathrm{E}_{\mathrm{Q}}[X] \leq x \text{ for all } \mathrm{Q} \in \widehat{M}_V\} \\
&= \sup_{\mathrm{Q} \in \widehat{M}_V} \mathrm{E}_{\mathrm{Q}}[X].
\end{aligned}$$

**Remark 5.1.** Given that $C_U$ is a $\sigma(L_U, L_V^+)$-closed cone, it is specified by its polar set, $\operatorname{cone}(\widehat{M}_V)$. This set depends only on the shape of $V(y)$ for arbitrarily large $y$, which in turn depends only on the values of $U(x)$ for arbitrarily large negative $x$. Consequently, the cone $C_U$ of allowable terminal wealths depends only on the preferences of the investor to asymptotically large losses. This interesting observation also suggests the following open problem. Can the set $C_U$ be parametrised by a real number which is defined in terms of the asymptotic behaviour of $U$ at $-\infty$?

## 6. A representation of $C_U$

Note that the set $K_U$ of Section 4 can be rewritten as

$$K_U = \bigcap_{\mathrm{Q} \in \widehat{M}_V} (K - L_+^1(\mathrm{Q})). \tag{6.1}$$

The next theorem gives two useful alternative representations of the weak closed cone $C_U$, which provides further links with [1]. See Remark 6.1, below.

**Theorem 6.1.**

$$C_U \overset{\text{(i)}}{=} \bigcap_{\mathrm{Q} \in \widehat{M}_V} \overline{K_U}^{L^1(\mathrm{Q})} \overset{\text{(ii)}}{=} \bigcap_{\mathrm{Q} \in \widehat{M}_V} \overline{K - L_+^1(\mathrm{Q})}^{L^1(\mathrm{Q})}.$$

*Proof.* (i) To show one inclusion, let $X \in \bigcap_{\mathrm{Q} \in \widehat{M}_V} \overline{K_U}^{L^1(\mathrm{Q})}$. Then, for each $\mathrm{Q} \in \widehat{M}_V$, there exists a sequence $\{X_n^{\mathrm{Q}}\} \subseteq K_U$ such that

$$X_n^{\mathrm{Q}} \xrightarrow{L^1(\mathrm{Q})} X \quad \text{as } n \to \infty.$$

Since $K_U \subseteq (\widehat{M}_V)^{\lhd}$, it follows that $\mathrm{E}_{\mathrm{Q}}[X] = \lim_{n\to\infty} \mathrm{E}_{\mathrm{Q}}[X_n^{\mathrm{Q}}] \leq 0$ for each $\mathrm{Q} \in \widehat{M}_V$. Consequently, $X \in (\widehat{M}_V)^{\lhd} = C_U$.

For the other inclusion, we proceed along the lines of the proof of the Kreps–Yan theorem (see [6, Theorem 3.5.8]) and consider an arbitrary $Z \in L_U$ such that $Z \notin \overline{K_U}^{L^1(\mathrm{Q}^*)}$ for some $\mathrm{Q}^* \in \widehat{M}_V$. By the Hahn–Banach hyperplane separation theorem there exists a continuous linear functional on $L^1(\mathrm{Q}^*)$ that separates $Z$ from the closed cone $\overline{K_U}^{L^1(\mathrm{Q}^*)}$. In other words, there exists a $\Lambda \in L^\infty(\mathrm{Q}^*)$ such that

$$\mathrm{E}_{\mathrm{Q}^*}[\Lambda X] \leq 0 < \mathrm{E}_{\mathrm{Q}^*}[\Lambda Z] \quad \text{for all } X \in K_U. \tag{6.2}$$

By considering $X = -\mathbf{1}_{\{\Lambda<0\}} \in -L^\infty_+(\mathrm{P}) \subseteq K_U$, we see that $\Lambda \geq 0$ $\mathrm{Q}^*$-a.s. and $\mathrm{E}_{\mathrm{Q}^*}[\Lambda] > 0$. Thus, if we set $\Lambda^* = \Lambda / \mathrm{E}_{\mathrm{Q}^*}[\Lambda]$ then $\mathrm{Q}_0(A) := \mathrm{E}_{\mathrm{Q}^*}[\Lambda^* \mathbf{1}_A]$ defines a probability measure on $(\Omega, F)$, and (6.2) implies that $\mathrm{Q}_0 \in M_1$ and $\mathrm{E}_{\mathrm{Q}_0}[Z] > 0$. To finish the proof of the first equality it suffices to prove that $\mathrm{Q}_0$ has finite loss-entropy, as then it follows, from (6.2), that $\mathrm{Q}_0 \in \widehat{M}_V$ and $Z \notin (\widehat{M}_V)^{\lhd} = C_U$.

In the case where $U$ is defined on the whole real line, let $b > 0$ be the constant from the statement of Lemma 2.1. Owing to Lemma 2.1 it follows that

$$\begin{aligned}
\mathrm{E}_{\mathrm{P}}\bigg[V^+\bigg(\frac{\mathrm{dQ}_0}{\mathrm{dP}}\bigg)\mathbf{1}_{\{\mathrm{dQ}_0/\mathrm{dP} \geq b\|\Lambda^*\|_{L^\infty(\mathrm{Q}^*)}\}}\bigg] &= \mathrm{E}_{\mathrm{P}}\bigg[V^+\bigg(\Lambda^*\frac{\mathrm{dQ}^*}{\mathrm{dP}}\bigg)\mathbf{1}_{\{\Lambda^*(\mathrm{dQ}^*/\mathrm{dP}) \geq b\|\Lambda^*\|_{L^\infty(\mathrm{Q}^*)}\}}\bigg] \\
&\leq \mathrm{E}_{\mathrm{P}}\bigg[V^+\bigg(\|\Lambda^*\|_{L^\infty(\mathrm{Q}^*)}\frac{\mathrm{dQ}^*}{\mathrm{dP}}\bigg)\mathbf{1}_{\{\mathrm{dQ}^*/\mathrm{dP} \geq b\}}\bigg] \\
&\leq D\,\mathrm{E}_{\mathrm{P}}\bigg[V^+\bigg(\frac{\mathrm{dQ}^*}{\mathrm{dP}}\bigg)\mathbf{1}_{\{\mathrm{dQ}^*/\mathrm{dP} \geq b\}}\bigg] \\
&< \infty.
\end{aligned}$$

The case where $U$ is defined on a half real line (i.e. $a \in (-\infty, \infty)$) is trivial owing to Remark 3.1(iii).

(ii) To prove the second equality it suffices to show that

$$\overline{K_U}^{L^1(\mathrm{Q})} = \overline{K - L^1_+(\mathrm{Q})}^{L^1(\mathrm{Q})}$$

for an arbitrary $\mathrm{Q} \in \widehat{M}_V$. Indeed, from (6.1) we have $K_U \subseteq K - L^1_+(\mathrm{Q}) \subseteq L^1(\mathrm{Q})$, so

$$\overline{K_U}^{L^1(\mathrm{Q})} \subseteq \overline{K - L^1_+(\mathrm{Q})}^{L^1(\mathrm{Q})}.$$

Moreover, since $K \cup (-L^\infty_+(\mathrm{Q})) \subseteq K_U$, we have $K - L^\infty_+(\mathrm{Q}) \subseteq K_U$. Since $L^\infty(\mathrm{Q})$ is dense in $L^1(\mathrm{Q})$, it follows that

$$\overline{K - L^1_+(\mathrm{Q})}^{L^1(\mathrm{Q})} = \overline{K - \overline{L^\infty_+(\mathrm{Q})}^{L^1(\mathrm{Q})}}^{L^1(\mathrm{Q})} \subseteq \overline{\overline{K - L^\infty_+(\mathrm{Q})}^{L^1(\mathrm{Q})}}^{L^1(\mathrm{Q})} \subseteq \overline{K_U}^{L^1(\mathrm{Q})}.$$

**Remark 6.1.** In [1] the set $C_U$ is defined by

$$C_U := \bigcap_{\mathrm{Q} \in M_V} \overline{(K - L^1_+(\mathrm{Q}))}^{L^1(\mathrm{Q})}.$$

Under our approach we do not need to explicitly construct $C_U$; we define it as the weak closure of $K_U$. The above result demonstrates, however, that this is essentially the same set—the only difference being that the intersection is taken over all finite loss-entropy measures, as opposed to all finite-entropy measures.

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