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$\mathcal{L}$-INVARIANTS AND LOCAL–GLOBAL COMPATIBILITY FOR THE GROUP GL_2/F

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Abstract

Let F be a totally real number field, $\wp$ a place of F above p. Let ρ be a 2-dimensional p-adic representation of $\mathrm{Gal}(\overline{F}/F)$ which appears in the étale cohomology of quaternion Shimura curves (thus ρ is associated to Hilbert eigenforms). When the restriction $\rho_\wp := \rho|_{D_\wp}$ at the decomposition group of $\wp$ is semistable noncrystalline, one can associate to $\rho_\wp$ the so-called Fontaine–Mazur $\mathcal{L}$-invariants, which are however invisible in the classical local Langlands correspondence. In this paper, we prove one can find these $\mathcal{L}$-invariants in the completed cohomology group of quaternion Shimura curves, which generalizes some of Breuil's results [Breuil, *Astérisque*, **331** (2010), 65–115] in the $\mathrm{GL}_2/\mathbb{Q}$-case.

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1. Introduction

Let F be a totally real number field, B a quaternion algebra of center F such that there exists only one real place of F where B is split. One can associate to B a system of quaternion Shimura curves $\{M_K\}_K$, proper and smooth over F, indexed by open compact subgroups K of $(B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$. We fix a prime number p, and suppose that there exists only one prime $\wp$ of F above p. Suppose B is split at $\wp$, that is, $(B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \cong \mathrm{GL}_2(F_\wp)$ (where $F_\wp$ denotes the completion of F at $\wp$). Let E be a finite extension of $\mathbb{Q}_p$ sufficiently large with $\mathcal{O}_E$ its ring of integers and ϖ_E a uniformizer of $\mathcal{O}_E$.

Let ρ be a 2-dimensional continuous representation of $\mathrm{Gal}(\overline{F}/F)$ over E such that ρ appears in the étale cohomology of M_K for K sufficiently small (so ρ is associated to Hilbert eigenforms). By the theory of completed cohomology of

Emerton [23], one can associate to ρ a unitary admissible Banach representation $\widehat{\Pi}(\rho)$ of $\mathrm{GL}_2(F_\wp)$ as follows: put

$$\widehat{H}^1(K^p, E) := \Big(\varprojlim_n \varinjlim_{K'_p} H^1_{\text{ét}}\big(M_{K^pK'_p} \times_F \overline{F}, \mathcal{O}_E/\varpi_E^n\big)\Big) \otimes_{\mathcal{O}_E} E$$

where K^p denotes the component of K outside p, and K'_p runs over open compact subgroups of $\mathrm{GL}_2(F_\wp)$. This is an E-Banach space equipped with a continuous action of $\mathrm{GL}_2(F_\wp) \times \mathrm{Gal}(\overline{F}/F) \times \mathcal{H}^p$ where $\mathcal{H}^p$ denotes the E-algebra of Hecke operators outside p. Put

$$\widehat{\Pi}(\rho) := \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\rho, \widehat{H}^1(K^p, E)\big).$$

The representation $\widehat{\Pi}(\rho)$ is supposed to be (a finite direct sum of) the right representation of $\mathrm{GL}_2(F_\wp)$ corresponding to $\rho_\wp := \rho|_{\mathrm{Gal}(\overline{F_\wp}/F_\wp)}$ in the p-adic Langlands program (cf. [8]). Nowadays, we know quite little about $\widehat{\Pi}(\rho)$ except when $F_\wp = \mathbb{Q}_p$, for example, we do not know whether it depends only on the local Galois representation $\rho_\wp$. By local–global compatibility of the classical local Langlands correspondence for GL_2/F (for $\ell = p$), one can indeed describe the locally algebraic vectors of $\widehat{\Pi}(\rho)$ in terms of the Weil–Deligne representation $\mathrm{WD}(\rho_\wp)$ associated to $\rho_\wp$ and the Hodge–Tate weights $\mathrm{HT}(\rho_\wp)$ of $\rho_\wp$. However, in general, (unlike the $\ell \neq p$ case), when passing to $(\mathrm{WD}(\rho_\wp), \mathrm{HT}(\rho_\wp))$, a lot of information about $\rho_\wp$ is lost. Finding the lost information in $\widehat{\Pi}(\rho)$ is thus one of the key problems in p-adic Langlands program (this is in fact the starting point of Breuil's initial work on p-adic Langlands program, cf. [6]).

In this paper, we consider the case where $\rho_\wp$ is semistable noncrystalline and noncritical (that is, $\rho_\wp$ satisfies the hypothesis 1). In this case, the missing data, when passing from $\rho_\wp$ to $(\mathrm{WD}(\rho_\wp), \mathrm{HT}(\rho_\wp))$, can be explicitly described by the so-called *Fontaine–Mazur $\mathcal{L}$-invariants* $\underline{\mathcal{L}}_{\Sigma_\wp} = (\mathcal{L}_\sigma)_{\sigma\in\Sigma_\wp} \in E^d$ associated to $\rho_\wp$ (for example, see Section 5.1), where $\Sigma_\wp$ denotes the set of $\mathbb{Q}_p$-embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$. Using these $\mathcal{L}$-invariants, Schraen has associated to $\rho_\wp$ a locally $\mathbb{Q}_p$-analytic representation $\Sigma(\mathrm{WD}(\rho_\wp), \mathrm{HT}(\rho_\wp), \underline{\mathcal{L}}_{\Sigma_\wp})$ of $\mathrm{GL}_2(F_\wp)$ over E (cf. [38, Section 4.2], see also Section 5.2), which generalizes Breuil's theory [5] in $\mathrm{GL}_2(\mathbb{Q}_p)$-case. Note that one can indeed recover $\rho_\wp$ from $\Sigma(\mathrm{WD}(\rho_\wp), \mathrm{HT}(\rho_\wp), \underline{\mathcal{L}}_{\Sigma_\wp})$. The main result of this paper is

THEOREM 1 (cf. Theorem 8). *Keep the above notation and suppose that ρ is absolutely irreducible modulo ϖ_E, there exists a continuous injection of* $\mathrm{GL}_2(F_\wp)$*-representations*

$$\Sigma\big(\mathrm{WD}(\rho_\wp), \mathrm{HT}(\rho_\wp), \underline{\mathcal{L}}_{\Sigma_\wp}\big) \longrightarrow \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}},$$

where $\widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}$ denotes the locally $\mathbb{Q}_p$-analytic vectors of $\widehat{\Pi}(\rho)$.

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Such a result is called local–global compatibility, since the left side of this injection depends only on the local representation $\rho_\wp$ while the right side is globally constructed. Moreover, one can prove the 'uniqueness' (in the sense of Corollary 6) of $\Sigma(\mathrm{WD}(\rho_\wp), \mathrm{HT}(\rho_\wp), \underline{\mathcal{L}}_{\Sigma_\wp})$ as subrepresentation of $\widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}$. As a result, we see the local Galois representation $\rho_\wp$ is determined by $\widehat{\Pi}(\rho)$. Such a result in the $\mathbb{Q}_p$-case, proved by Breuil [7], was the first discovered local–global compatibility in the p-adic local Langlands correspondence. In fact, the $\mathcal{L}$-invariants appearing in the automorphic representation side are often referred to as *Breuil's $\mathcal{L}$-invariants*. Theorem 1 thus shows the equality of Fontaine–Mazur $\mathcal{L}$-invariants and Breuil's $\mathcal{L}$-invariants. Our approach is by using some p-adic family arguments on both GL_2-side and Galois side, thus different from that of Breuil (by using modular symbols).

In the following (of the introduction), we sketch how we manage to 'find' $\{\mathcal{L}_\sigma\}_{\sigma\in\Sigma_\wp}$ in $\widehat{\Pi}(\rho)$. For simplicity, suppose $\rho_\wp$ is of Hodge–Tate weights $(-1, 0)_{\Sigma_\wp}$ (thus $\rho_\wp$ is associated to Hilbert eigenforms of weights $(2, \ldots, 2; 0)$ in the notation of [14]). Let $\tau \in \Sigma_\wp$; it is enough to find $\mathcal{L}_\tau$ in $\widehat{\Pi}(\rho)_{\tau-\mathrm{an}}$ (the maximal locally τ-analytic subrepresentation of $\widehat{\Pi}(\rho)$) in the sense of (2) below:

Denote by $Z_1 := \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \middle| a \in 1 + 2\varpi\mathcal{O}_\wp\right\}$ (where $\mathcal{O}_\wp$ denotes the ring of integers of $F_\wp$ and ϖ is a uniformizer of $\mathcal{O}_\wp$), consider $\widehat{H}^1(K^p, E)^{Z_1}_{\tau-\mathrm{an}}$ (where '$(\cdot)^{Z_1}$' signifies the vectors fixed by Z_1, and 'τ − an' signifies the locally τ-analytic subrepresentation). By applying Jacquet–Emerton functor, one gets an essentially admissible locally τ-analytic representation of $T(F_\wp)$: $J_B(\widehat{H}^1(K^p, E)^{Z_1}_{\tau-\mathrm{an}})$, which is moreover equipped with an action of $\mathcal{H}^p$ commuting with that of $T(F_\wp)$. Following Emerton, one can construct an eigenvariety $\mathcal{V}_\tau$ from $J_B(\widehat{H}^1(K^p, E)^{Z_1}_{\tau-\mathrm{an}})$, which is in particular a rigid space finite over $\widehat{T}_\tau$, the rigid space parameterizing locally τ-analytic characters of $T(F_\wp)$ (cf. Theorem 3). A closed point of $\mathcal{V}_\tau$ can be written as (χ, λ) where χ is a locally τ-analytic character of $T(F_\wp)$ and λ is a system of Hecke eigenvalues (for $\mathcal{H}^p$).

One can associate to ρ an E-point $z_\rho = (\chi_\rho, \lambda_\rho)$ of $\mathcal{V}_\tau$, where $\chi_\rho = \mathrm{unr}(\alpha/q) \otimes \mathrm{unr}(q\alpha)$ ($\mathrm{unr}(a)$ denotes the unramified character of $F_\wp^\times$ sending ϖ to a), λ_ρ denotes the system of eigenvalues of $\mathcal{H}^p$ associated to ρ (via the Eichler–Shimura relations), $\{\alpha, q\alpha\}$ are the eigenvalues of φ^{d_0} on $D_{\mathrm{st}}(\rho_\wp)$ (where d_0 is the degree of the maximal unramified extension of $\mathbb{Q}_p$ in $F_\wp$, $q := p^{d_0}$). Moreover, by multiplicity one result on automorphic representations of $(B \otimes_{\mathbb{Q}} \mathbb{A})^\times$, one can prove as in [17, Section 4.4] that $\mathcal{V}_\tau$ is smooth at z_ρ (cf. Theorem 7, note that by the hypothesis 1, z_ρ is in fact a *noncritical* point).

Let $t : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \to \mathcal{V}_\tau$ be a nonzero element in the tangent space of $\mathcal{V}_\tau$ at z_ρ, via the composition

$$t : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \longrightarrow \mathcal{V}_\tau \longrightarrow \widehat{T}_\tau,$$

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one gets a character $\widetilde{\chi}_\rho = \widetilde{\chi}_{\rho,1} \otimes \widetilde{\chi}_{\rho,2} : T(F_\wp)^\times \rightarrow (E[\epsilon]/\epsilon^2)^\times$, which is in fact an extension of χ_ρ by χ_ρ. One key point is that, by applying an adjunction formula in family for the Jacquet–Emerton functor (see [23, Lemma 4.5.12] for the $\mathrm{GL}_2(\mathbb{Q}_p)$-case) to the tangent space of $\mathcal{V}_\tau$ at z_ρ, one gets a nonzero continuous morphism of $\mathrm{GL}_2(F_\wp)$-representations (where $\overline{B}(F_\wp)$ denotes the group of lower triangular matrixes) (see (44))

$$\left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \widetilde{\chi}_\rho \delta^{-1}\right)^{\tau-\mathrm{an}} \longrightarrow \widehat{H}^1(K^p, E)^{Z_1}_{\tau-\mathrm{an}}[\lambda_\rho] \tag{1}$$

where $\delta := \mathrm{unr}(q^{-1}) \otimes \mathrm{unr}(q)$ and we refer to [38, Section 2] for locally τ-analytic parabolic inductions, and where the right term denotes the generalized λ_ρ-eigenspace of $\widehat{H}^1(K^p, E)^{Z_1}_{\tau-\mathrm{an}}$.

Another key point is that one can describe the character $\widetilde{\chi}_\rho$ in term of $\mathcal{L}_\tau$:

LEMMA 1 (cf. Lemma 10). *There exists an additive character χ of $F_\wp^\times$ in E such that $\widetilde{\chi}_\rho$ (as a 2-dimensional representation of $T(F_\wp)$ over E) is isomorphic to $\chi_\rho \otimes_E \psi(\mathcal{L}_\tau, \chi)$ where*

$$\psi(\mathcal{L}_\tau, \chi)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & \log_{\tau,-\mathcal{L}_\tau}(ad^{-1}) + \chi(ad) \\ 0 & 1 \end{pmatrix},$$

and $\log_{\tau,\mathcal{L}}$ denotes the additive character of $F_\wp^\times$ such that $\log_{\tau,\mathcal{L}}|_{\mathcal{O}_\wp^\times} = \tau \circ \log$ and $\log_{\tau,\mathcal{L}}(p) = \mathcal{L}$.

To prove this lemma, one considers the p-adic family of Galois representations over $\mathcal{V}_\tau$. In fact, there exist an admissible neighborhood U of z_ρ in $\mathcal{V}_\tau$ and a continuous representation $\rho_U : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_2(\mathcal{O}_U)$ such that the evaluation of ρ_U at any classical point of U (which thus corresponds to certain Hilbert eigenforms h) is just the Galois representation associated to h. Via the map t, one gets a continuous representation $\widetilde{\rho} : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_2(E[\epsilon]/\epsilon^2)$ which satisfies $\widetilde{\rho} \equiv \rho \pmod{\epsilon}$. By the theory of global triangulation [30], one can obtain an exact sequence (cf. (39)):

$$0 \rightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\mathrm{unr}(q)\widetilde{\chi}_{\rho,1}) \rightarrow D_{\mathrm{rig}}(\widetilde{\rho}_\wp) \rightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}\left(\widetilde{\chi}_{\rho,2} \prod_{\sigma\in\Sigma_\wp} \sigma^{-1}\right) \rightarrow 0,$$

where $\widetilde{\rho}_\wp := \widetilde{\rho}|_{\mathrm{Gal}(\overline{F_\wp}/F_\wp)}$. The lemma then follows by applying the formula in [42, Theorem 1.1] (which generalizes Colmez's formula [19] in $\mathbb{Q}_p$-case) to $\widetilde{\rho}_\wp$.

Return to the map (1). We know $(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \widetilde{\chi}_\rho \delta^{-1})^{\tau-\mathrm{an}}$ lies in an exact sequence

$$\begin{aligned} 0 \rightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi_\rho \delta^{-1}\right)^{\tau-\mathrm{an}} &\rightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \widetilde{\chi}_\rho \delta^{-1}\right)^{\tau-\mathrm{an}} \\ \xrightarrow{s} \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi_\rho \delta^{-1}\right)^{\tau-\mathrm{an}} &\rightarrow 0, \end{aligned}$$

where s depends on $\mathcal{L}_\tau$ and χ as in the Lemma 1. On the other hand, it is known that $(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi_\rho\delta^{-1})^{\tau-\mathrm{an}}$ admits a unique finite dimensional subrepresentation $V(\alpha) := \mathrm{unr}(\alpha) \circ \det$. Put $\Sigma(\alpha, \mathcal{L}_\tau) := s^{-1}(V(\alpha))/V(\alpha)$ (cf. [38, Section 4.2]), which turns out to be *independent* of the character χ in Lemma 1 and thus depends *only* on $\mathcal{L}_\tau$. At last, one can prove that (1) induces actually a continuous injection of locally τ-analytic representations of $\mathrm{GL}_2(F_\wp)$

$$\Sigma(\alpha, \mathcal{L}_\tau) \longhookrightarrow \widehat{\Pi}(\rho)_{\tau-\mathrm{an}}. \tag{2}$$

It seems this argument might work for some other groups and some other Shimura varieties. For example, in the $\mathrm{GL}_2/\mathbb{Q}$-case (with Coleman–Mazur eigencurve, reconstructed by Emerton [23, Section 4] using completed cohomology of modular curves), by restricting the map [23, (4.5.9)] to the tangent space at a semistable noncrystalline point, one can obtain a map as in (1). On the other hand, one can prove a similar result as in Lemma 1 by Kisin's theory in [31] and Colmez's formula [19]. Combining them together, one can actually reprove Breuil's result in [7] for locally analytic representations and thus obtain directly the equality of Fontaine–Mazur $\mathcal{L}$-invariant and Breuil's $\mathcal{L}$-invariant without using Darmon–Orton's $\mathcal{L}$-invariant (as in Breuil's original proof [7]).

We refer to the body of the text for more detailed and more precise statements.

After the results of this paper was firstly announced, Yuancao Zhang informed us that he had proved the existence of $\mathcal{L}$-invariants in $\widehat{\Pi}(\rho)$ in certain cases by using some arguments as in [12, Section 5]; however, the equality between these $\mathcal{L}$-invariants and Fontaine–Mazur $\mathcal{L}$-invariants was not proved.

2. Notations and preliminaries

Let F be a totally real field of degree d over $\mathbb{Q}$, denote by Σ_∞ the set of real embeddings of F. For a finite place $\mathfrak{l}$ of F, we denote by $F_\mathfrak{l}$ the completion of F at $\mathfrak{l}$, $\mathcal{O}_\mathfrak{l}$ the ring of integers of $F_\mathfrak{l}$ with $\varpi_\mathfrak{l}$ a uniformizer of $\mathcal{O}_\mathfrak{l}$. Denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_F$ the ring of adeles of F. For a set S of places of $\mathbb{Q}$ (respectively of F), we denote by $\mathbb{A}^S$ (respectively by $\mathbb{A}_F^S$) the ring of adeles of $\mathbb{Q}$ (respectively of F) outside S, S_F the set of places of F above that in S, and $\mathbb{A}_F^S := \mathbb{A}_F^{S_F}$.

Let p be a prime number, suppose there exists only one prime $\wp$ of F lying above p. Denote by $\Sigma_\wp$ the set of $\mathbb{Q}_p$-embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$; let ϖ be a uniformizer of $\mathcal{O}_\wp$, $F_{\wp,0}$ the maximal unramified extension of $\mathbb{Q}_p$ in $F_\wp$, $d_0 := [F_{\wp,0} : \mathbb{Q}_p]$, $e := [F_\wp : F_{\wp,0}]$, $q := p^{d_0}$ and $\upsilon_\wp$ a p-adic valuation on $\overline{\mathbb{Q}_p}$ normalized by $\upsilon_\wp(\varpi) = 1$. Let E be a finite extension of $\mathbb{Q}_p$ big enough such that E contains all the $\mathbb{Q}_p$-embeddings of F in $\overline{\mathbb{Q}_p}$, $\mathcal{O}_E$ the ring of integers of E and ϖ_E a uniformizer of $\mathcal{O}_E$.

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Let B be a quaternion algebra of center F with $S(B)$ the set (of even cardinality) of places of F where B is ramified, suppose $|S(B)\cap\Sigma_\infty| = d-1$ and $S(B)\cap\Sigma_\wp = \emptyset$, that is, there exists $\tau_\infty\in\Sigma_\infty$ such that $B\otimes_{F,\tau_\infty}\mathbb{R}\cong M_2(\mathbb{R})$, $B\otimes_{F,\sigma}\mathbb{R}\cong\mathbb{H}$ for any $\sigma\in\Sigma_\infty$, $\sigma\neq\tau_\infty$, where $\mathbb{H}$ denotes the Hamilton algebra, and $B\otimes_{\mathbb{Q}}\mathbb{Q}_p\cong M_2(F_\wp)$. We associate to B a reductive algebraic group G over $\mathbb{Q}$ with $G(R) := (B\otimes_{\mathbb{Q}} R)^\times$ for any $\mathbb{Q}$-algebra R. Set $\mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, and denote by h the morphism

$$h : \mathbb{S}(\mathbb{R})\cong\mathbb{C}^\times \longrightarrow G(\mathbb{R})\cong \mathrm{GL}_2(\mathbb{R})\times(\mathbb{H}^*)^{d-1},$$
$$a+bi\mapsto\left(\begin{pmatrix} a & b\\ -b & a\end{pmatrix}, 1,\ldots,1\right).$$

The space of $G(\mathbb{R})$-conjugacy classes of h has a structure of complex manifold, and is isomorphic to $\mathfrak{h}^\pm := \mathbb{C}\backslash\mathbb{R}$ (that is, 2 copies of the Poincaré's upper half plane). We get a projective system of Riemann surfaces indexed by open compact subgroups of $G(\mathbb{A}^\infty)$:

$$M_K(\mathbb{C}) := G(\mathbb{Q})\backslash\big(\mathfrak{h}^\pm\times(G(\mathbb{A}^\infty)/K)\big)$$

where $G(\mathbb{Q})$ acts on $\mathfrak{h}^\pm$ via $G(\mathbb{Q})\hookrightarrow G(\mathbb{R})$ and the transition map is given by

$$G(\mathbb{Q})\backslash(\mathfrak{h}^\pm\times(G(\mathbb{A}^\infty)/K_1)) \longrightarrow G(\mathbb{Q})\backslash(\mathfrak{h}^\pm\times(G(\mathbb{A}^\infty)/K_2)),\quad (x,g)\mapsto(x,g), \tag{3}$$

for $K_1\subseteq K_2$. It is known that $M_K(\mathbb{C})$ has a canonical proper smooth model over F (via the embedding τ_∞), denoted by M_K, and these $\{M_K\}_K$ form a projective system of proper smooth algebraic curves over F (that is, the transition map (3) admits also an F-model). One has a natural isomorphism $G(\mathbb{Q}_p)\xrightarrow{\sim}\mathrm{GL}_2(F_\wp)$.

Let $K_{0,\wp} := \mathrm{GL}_2(\mathcal{O}_\wp)$, in the following, we fix an open compact subgroup K^p of $G(\mathbb{A}^{\infty,p})$ small enough such that the open compact subgroup $K^pK_{0,\wp}$ of $G(\mathbb{A}^\infty)$ is neat (cf. [34, Definition 4.11]). Denote by $S(K^p)$ the set of finite places $\mathfrak{l}$ of F such that $\mathfrak{l}\nmid p$, that B is split at $\mathfrak{l}$, that is, $B\otimes_F F_\mathfrak{l}\xrightarrow{\sim}\mathrm{M}_2(F_\mathfrak{l})$, and that $K^p\cap\mathrm{GL}_2(F_\mathfrak{l})\cong\mathrm{GL}_2(\mathcal{O}_\mathfrak{l})$. Denote by $\mathcal{H}^p$ the commutative $\mathcal{O}_E$-algebra generated by the double coset operators $[\mathrm{GL}_2(\mathcal{O}_\mathfrak{l})g_\mathfrak{l}\mathrm{GL}_2(\mathcal{O}_\mathfrak{l})]$ for all $g_\mathfrak{l}\in\mathrm{GL}_2(F_\mathfrak{l})$ with $\det(g_\mathfrak{l})\in\mathcal{O}_\mathfrak{l}$ and for all $\mathfrak{l}\in S(K^p)$. Set

$$T_\mathfrak{l} := \left[\mathrm{GL}_2(\mathcal{O}_\mathfrak{l})\begin{pmatrix}\varpi_\mathfrak{l} & 0\\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_\mathfrak{l})\right],$$
$$S_\mathfrak{l} := \left[\mathrm{GL}_2(\mathcal{O}_\mathfrak{l})\begin{pmatrix}\varpi_\mathfrak{l} & 0\\ 0 & \varpi_\mathfrak{l}\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_\mathfrak{l})\right],$$

then $\mathcal{H}^p$ is the polynomial algebra over $\mathcal{O}_E$ generated by $\{T_\mathfrak{l}, S_\mathfrak{l}\}_{\mathfrak{l}\in S(K^p)}$.

Denote by Z_0 the kernel of the norm map $\mathscr{N} : \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m$ which is a subgroup of $Z = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. We set $G^c := G/Z_0$.

Denote by $\mathrm{Art}_{F_\wp} : F_\wp^\times \xrightarrow{\sim} W_{F_\wp}^{\mathrm{ab}}$ the local Artin map normalized by sending uniformizers to geometric Frobenius elements (where $W_{F_\wp} \subset \mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ denotes the Weil group). Let $\sigma \in \Sigma_\wp$, denote by $\log_\sigma$ the composition $\mathcal{O}_\wp^\times \xrightarrow{\log} \mathcal{O}_\wp \xrightarrow{\sigma} E$. For $\mathcal{L} \in E$, denote by $\log_{\sigma,\mathcal{L},\varpi}$ the (additive) character of $F_\wp^\times$ such that $\log_{\sigma,\mathcal{L},\varpi}|_{\mathcal{O}_\wp^\times} = \log_\sigma$ and $\log_{\sigma,\mathcal{L},\varpi}(\varpi) = \mathcal{L}$. Denote by $\log_{\sigma,\mathcal{L}}$ the (additive) character of $F_\wp^\times$ in E satisfying $\log_{\sigma,\mathcal{L}}|_{\mathcal{O}_\wp^\times} = \log_\sigma$ and $\log_{\sigma,\mathcal{L}}(p) = \mathcal{L}$. Let $\mathcal{L}(\varpi) := e^{-1}(\mathcal{L} - \log_\sigma(p/\varpi^e))$, thus one has

$$\log_{\sigma,\mathcal{L}} = \log_{\sigma,\mathcal{L}(\varpi),\varpi}.$$

Denote by $\mathrm{unr}(a)$ the unramified character of $F_\wp^\times$ sending ϖ to a.

Let V be an E-vector space equipped with an E-linear action of A (with A a set of operators), χ a system of eigenvalues of A, denote by $V^{A=\chi}$ the χ-eigenspace, $V[A=\chi]$ the generalized χ-eigenspace, V^A the vector space of A-fixed vectors.

Let $S \subseteq \Sigma_\wp$, $k_\sigma \in \mathbb{Z}_{\geqslant 2}$ for all $\sigma \in S$, denote by $W(\underline{k}_S) := \otimes_{\sigma\in S}(\mathrm{Sym}^{k_\sigma-2} E^2)^\sigma$ the algebraic representation of $G(\mathbb{Q}_p) \cong \mathrm{GL}_2(F_\wp)$ with $\mathrm{GL}_2(F_\wp)$ acting on $(\mathrm{Sym}^{k_\sigma-2} E^2)^\sigma$ via $\mathrm{GL}_2(F_\wp) \xrightarrow{\sigma} \mathrm{GL}_2(E)$ for $\sigma \in S$. Let $w \in \mathbb{Z}$, $k_\sigma \in \mathbb{Z}_{\geqslant 2}$, $k_\sigma \equiv w \pmod 2$ for all $\sigma \in \Sigma_\wp$, put $W(\underline{k}_{\Sigma_\wp}, w) := \otimes_{\sigma\in\Sigma_\wp}(\mathrm{Sym}^{k_\sigma-2} E^2 \otimes \det^{(w-k_\sigma+2)/2})^\sigma$.

Denote by $B(F_\wp)$ (respectively $\overline{B}(F_\wp)$) the subgroup of $\mathrm{GL}_2(F_\wp)$ of upper (respectively lower) triangular matrixes, $T(F_\wp)$ the group of diagonal matrixes, $N(F_\wp)$ the group of unipotent elements in $B(F_\wp)$, $N_0 := N(F_\wp) \cap \mathrm{GL}_2(\mathcal{O}_\wp)$, $Z' := T(F_\wp) \cap \mathrm{SL}_2(F_\wp)$, $K_{1,\wp} := \{g \in \mathrm{GL}_2(\mathcal{O}_\wp) \mid g \equiv 1 \pmod{2\varpi}\}$, Z_1 the center of $K_{1,\wp}$, $Z_1' := Z' \cap K_{1,\wp}$. Put $\delta := \mathrm{unr}(q^{-1}) \otimes \mathrm{unr}(q)$ being a character of $T(F_\wp)$ (which is in fact the modulus character of $B(F_\wp)$).

Locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F_\wp)$. Recall some notions on locally $\mathbb{Q}_p$-analytic representations. Let V be a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$ over E, that is, a locally analytic representation of $\mathrm{GL}_2(F_\wp)$ with $\mathrm{GL}_2(F_\wp)$ viewed as a p-adic $\mathbb{Q}_p$-analytic group, V is naturally equipped with a $\mathbb{Q}_p$-linear action of the Lie algebra $\mathfrak{g}$ of $\mathrm{GL}_2(F_\wp)$ (thus an E-linear action of $\mathfrak{g}_{\Sigma_\wp} := \mathfrak{g} \otimes_{\mathbb{Q}_p} E$) given by

$$\mathfrak{x} \cdot v := \frac{d}{dt}\exp(t\mathfrak{x})(v)|_{t=0}.$$

Using the isomorphism

$$F_\wp \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma\in\Sigma_\wp} E, \quad a \otimes b \mapsto (\sigma(a)b)_{\sigma: F_\wp \to E}, \tag{4}$$

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one gets a decomposition $\mathfrak{g}_{\Sigma_\wp} \xrightarrow{\sim} \prod_{\sigma\in\Sigma_\wp}\mathfrak{g}_\sigma$ with $\mathfrak{g}_\sigma := \mathfrak{g}\otimes_{F_\wp,\sigma} E$. Let $J\subseteq\Sigma_\wp$, a vector $v\in V$ is called *locally J-analytic* if the action of $\mathfrak{g}_{\Sigma_\wp}$ on v factors through $\mathfrak{g}_J := \prod_{\sigma\in J}\mathfrak{g}_\sigma$ (we put $\mathfrak{g}_\emptyset := \{0\}$), in other words, if v is killed by $\mathfrak{g}_{\Sigma_\wp\setminus J}$ (cf. [38, Definition 2.4]); v is called *quasi-J-classical* if there exist a finite dimensional representation U of $\mathfrak{g}_J$ and a $\mathfrak{g}_J$-invariant map

$$U \hookrightarrow V$$

whose image contains v, if the $\mathfrak{g}_J$-representation U can moreover give rise to an algebraic representation of $\mathrm{GL}_2(F_\wp)$, then we say that v is *J-classical*. In particular, v is $\Sigma_\wp\setminus J$-classical if v is locally J-analytic. Note that v is $\emptyset$-analytic is equivalent to that v is a smooth vector for the action of $\mathrm{GL}_2(F_\wp)$ (that is, v is fixed by certain open compact subgroup of $\mathrm{GL}_2(F_\wp)$) which implies in particular v is $\Sigma_\wp$-classical.

Let V be a Banach representation of $\mathrm{GL}_2(F_\wp)$ over E, denote by $V_{\mathbb{Q}_p-\mathrm{an}}$ the E-vector subspace generated by the locally $\mathbb{Q}_p$-analytic vectors of V, which is stable by $\mathrm{GL}_2(F_\wp)$ and hence is a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$. If V is moreover admissible, by [37, Theorem 7.1], $V_{\mathbb{Q}_p-\mathrm{an}}$ is an admissible locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$ and dense in V. For $J\subseteq\Sigma_\wp$, denote by $V_{J-\mathrm{an}}$ the subrepresentation generated by locally J-analytic vectors of $V_{\mathbb{Q}_p-\mathrm{an}}$, put $V_\infty := V_{\emptyset-\mathrm{an}}$.

Let χ be a continuous (thus locally $\mathbb{Q}_p$-analytic) character of $F_\wp^\times$ (or any open compact subgroup of $F_\wp^\times$) over E, then χ induces a natural $\mathbb{Q}_p$-linear map (where $F_\wp$ is viewed as the Lie algebra of $F_\wp^\times$)

$$F_\wp \longrightarrow E, \quad \mathfrak{x}\mapsto \frac{d}{dt}\chi(\exp(t\mathfrak{x}))|_{t=0},$$

and hence an E-linear map $d_\chi : F_\wp\otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma\in\Sigma_\wp} E \to E$. So there exist $k_{\chi,\sigma}\in E$, called the σ-weight of χ, for all $\sigma\in\Sigma_\wp$ such that $d_\chi((a_\sigma)_{\sigma\in\Sigma_\wp}) = \sum_{\sigma\in\Sigma_\wp} a_\sigma k_{\chi,\sigma}$.

Let $\chi=\chi_1\otimes\chi_2$ be a locally $\mathbb{Q}_p$-analytic character of $T(F_\wp)$ over E. Put

$$C(\chi) := \{\sigma\in\Sigma_\wp \mid k_{\chi_1,\sigma}-k_{\chi_2,\sigma}\in\mathbb{Z}_{\geqslant 0}\}. \tag{5}$$

Denote by $\mathfrak{t}$ the Lie algebra of $T(F_\wp)$, the character χ induces a character $d\chi$ of $\mathfrak{t}_{\Sigma_\wp} := \mathfrak{t}\otimes_{\mathbb{Q}_p} E$ given by $d\chi : \mathfrak{t}_{\Sigma_\wp}\to E$, $d\chi\begin{pmatrix} a_\sigma & 0\\ 0 & d_\sigma\end{pmatrix} = a_\sigma k_{\chi_1,\sigma} + d_\sigma k_{\chi_2,\sigma}$ for $\begin{pmatrix} a_\sigma & 0\\ 0 & d_\sigma\end{pmatrix}\in\mathfrak{t}_\sigma := \mathfrak{t}\otimes_{F_\wp,\sigma} E$, $\sigma\in\Sigma_\wp$.

3. Completed cohomology of quaternion Shimura curves

Recall some facts on completed cohomology of quaternion Shimura curves, following [23] and [34].

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3.1. Generalities. Let W be a finite dimensional algebraic representation of G^c over E, as in [14, Section 2.1], one can associate to W a local system $\mathcal{V}_W$ of E-vector spaces over M_K. Let W_0 be $\mathcal{O}_E$-lattice of W, denote by $\mathcal{S}_{W_0}$ the set (ordered by inclusions) of open compact subgroups of $G(\mathbb{Q}_p) \cong \mathrm{GL}_2(F_\wp)$ which stabilize W_0. For any $K_p \in S_{W_0}$, one can associate to W_0 (respectively to W_0/ϖ_E^s for $s \in \mathbb{Z}_{\geqslant 1}$) a local system $\mathcal{V}_{W_0}$ (respectively $\mathcal{V}_{W_0/\varpi_E^s}$) of $\mathcal{O}_E$-modules (respectively of $\mathcal{O}_E/\varpi_E^s$-modules) over $M_{K^pK_p}$. Following Emerton [23], we put

$$\begin{aligned} H^i_{\text{ét}}(K^p, W_0) &:= \varinjlim_{K_p \in \mathcal{S}_{W_0}} H^i_{\text{ét}}(M_{K_pK^p,\overline{\mathbb{Q}}}, \mathcal{V}_{W_0}) \\ &\cong \varinjlim_{K_p \in \mathcal{S}_{W_0}} \varprojlim_{s} H^i_{\text{ét}}(M_{K_pK^p,\overline{\mathbb{Q}}}, \mathcal{V}_{W_0/\varpi_E^s}); \\ \widetilde{H}^i_{\text{ét}}(K^p, W_0) &:= \varprojlim_{s} \varinjlim_{K_p \in \mathcal{S}_{W_0}} H^i_{\text{ét}}(M_{K_pK^p,\overline{\mathbb{Q}}}, \mathcal{V}_{W_0/\varpi_E^s}); \\ H^i_{\text{ét}}(K^p, W_0)_E &:= H^i_{\text{ét}}(K^p, W_0) \otimes_{\mathcal{O}_E} E; \\ \widetilde{H}^i_{\text{ét}}(K^p, W_0)_E &:= \widetilde{H}^i_{\text{ét}}(K^p, W_0) \otimes_{\mathcal{O}_E} E. \end{aligned}$$

All these groups ($\mathcal{O}_E$-modules or E-vector spaces) are equipped with a natural topology induced from the discrete topology on the finite groups

$$H^i_{\text{ét}}(M_{K_pK^p,\overline{\mathbb{Q}}}, \mathcal{V}_{W_0/\varpi_E^s}),$$

and equipped with a natural continuous action of $\mathcal{H}^p \times \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and of $K_p \in \mathcal{S}_{W_0}$. Moreover, for any $\mathfrak{l} \in S(K^p)$, the action of $\mathrm{Gal}(\overline{F_{\mathfrak{l}}}/F_{\mathfrak{l}})$ (induced by that of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$) is unramified and satisfies the Eichler–Shimura relation:

$$\mathrm{Frob}_{\mathfrak{l}}^{-2} - T_{\mathfrak{l}}\,\mathrm{Frob}_{\mathfrak{l}}^{-1} + \ell^{f_{\mathfrak{l}}} S_{\mathfrak{l}} = 0 \tag{6}$$

where $\mathrm{Frob}_{\mathfrak{l}}$ denotes the arithmetic Frobenius, ℓ the prime number lying below $\mathfrak{l}$, $f_{\mathfrak{l}}$ the degree of the maximal unramified extension (of $\mathbb{Q}_\ell$) in $F_{\mathfrak{l}}$ over $\mathbb{Q}_\ell$ (thus $\ell^{f_{\mathfrak{l}}} = |\mathcal{O}_{\mathfrak{l}}/\varpi_{\mathfrak{l}}|$). Note that $\widetilde{H}^i_{\text{ét}}(K^p, W_0)_E$ is an E-Banach space with norm defined by the $\mathcal{O}_E$-lattice $\widetilde{H}^i_{\text{ét}}(K^p, W_0)$.

Consider the ordered set (by inclusion) $\{W_0\}$ of $\mathcal{O}_E$-lattices of W, following [23, Definition 2.2.9], we put

$$\begin{aligned} H^i_{\text{ét}}(K^p, W) &:= \varinjlim_{W_0} H^i_{\text{ét}}(K^p, W_0)_E, \\ \widetilde{H}^i_{\text{ét}}(K^p, W) &:= \varinjlim_{W_0} \widetilde{H}^i_{\text{ét}}(K^p, W_0)_E, \end{aligned}$$

where all the transition maps are topological isomorphisms (cf. [23, Lemma 2.2.8]). These E-vector spaces are moreover equipped with a natural continuous action of $\mathrm{GL}_2(F_\wp)$.

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THEOREM 2 (cf. [23, Theorem 2.2.11 (i), Theorem 2.2.17]).

(1) *$\widetilde{H}^i_{ét}(K^p, W)$ is an admissible Banach representation of $\mathrm{GL}_2(F_\wp)$. If W is the trivial representation, the representation $\widetilde{H}^i_{ét}(K^p, W)$ is unitary.*

(2) *One has a natural isomorphism of Banach representations of $\mathrm{GL}_2(F_\wp)$ invariant under the action of $\mathcal{H}^p \times \mathrm{Gal}(\overline{F}/F)$:*

$$\widetilde{H}^i_{ét}(K^p, W) \xrightarrow{\sim} \widetilde{H}^i_{ét}(K^p, E) \otimes_E W. \tag{7}$$

(3) *One has a natural $\mathrm{GL}_2(F_\wp) \times \mathcal{H}^p \times \mathrm{Gal}(\overline{F}/F)$-invariant map*

$$H^i_{ét}(K^p, W) \longrightarrow \widetilde{H}^i_{ét}(K^p, W). \tag{8}$$

3.2. Localization at a non-Eisenstein maximal ideal. Let ρ be a 2-dimensional continuous representation of $\mathrm{Gal}(\overline{F}/F)$ over E such that ρ is unramified at all $\mathfrak{l} \in S(K^p)$. Let ρ_0 be a $\mathrm{Gal}(\overline{F}/F)$-invariant lattice of ρ, and $\overline{\rho}^{\mathrm{ss}}$ the semisimplification of ρ_0/ϖ_E, which is in fact independent of the choice of ρ_0. To $\overline{\rho}^{\mathrm{ss}}$, one can associate a maximal ideal of $\mathcal{H}^p$, denoted by $\mathfrak{m}(\overline{\rho}^{\mathrm{ss}})$, as the kernel of the following morphism

$$\mathcal{H}^p \longrightarrow k_E := \mathcal{O}_E/\varpi_E, \quad T_{\mathfrak{l}} \mapsto \mathrm{tr}\left(\overline{\rho}^{\mathrm{ss}}(\mathrm{Frob}_{\mathfrak{l}}^{-1})\right), \quad S_{\mathfrak{l}} \mapsto \det\left(\overline{\rho}^{\mathrm{ss}}(\mathrm{Frob}_{\mathfrak{l}}^{-1})\right)$$

for all $\mathfrak{l} \in S(K^p)$.

NOTATION 1. For an $\mathcal{H}^p$-module M, denote by $M_{\overline{\rho}^{\mathrm{ss}}}$ the localization of M at $\mathfrak{m}(\overline{\rho}^{\mathrm{ss}})$.

Keep the notation in Section 3.1. As in [26, Section 5.2, 5.3], one can show that $\widetilde{H}^1_{ét}(K^p, W)_{\overline{\rho}^{\mathrm{ss}}}$ is a direct summand of $\widetilde{H}^1_{ét}(K^p, W)$. Suppose in the following that ρ is absolutely irreducible modulo ϖ_E and put $\overline{\rho} := \overline{\rho}^{\mathrm{ss}}$.

PROPOSITION 1 [34, Proposition 5.2]. *The map (8) induces an isomorphism*

$$H^1_{ét}(K^p, W)_{\overline{\rho}} \xrightarrow{\sim} \widetilde{H}^1_{ét}(K^p, W)_{\overline{\rho},\infty},$$

where $\widetilde{H}^1_{ét}(K^p, W)_{\overline{\rho},\infty}$ denotes the smooth vectors (for the action of $\mathrm{GL}_2(F_\wp)$) in $\widetilde{H}^1_{ét}(K^p, W)_{\overline{\rho}}$.

PROPOSITION 2 [34, Corollary 5.8]. *Let H be an open compact prop-p subgroup of $K_{0,\wp}$, then there exists $r \in \mathbb{Z}_{\geqslant 1}$ such that*

$$\widetilde{H}^1_{ét}(K^p, E)_{\overline{\rho}} \xrightarrow{\sim} \mathcal{C}\left(H/\overline{(Z(\mathbb{Q}) \cap K^p H)_p}, E\right)^{\oplus r}$$

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as representations of H, where $\mathcal{C}(H/\overline{(Z(\mathbb{Q})\cap K^pH)_p}, E)$ denotes the space of continuous functions from $H/\overline{(Z(\mathbb{Q})\cap K^pH)_p}$ to E, on which H acts by the right regular action, $\overline{(Z(\mathbb{Q})\cap K^pH)_p}$ the closure of $(Z(\mathbb{Q})\cap K^pH)_p$ in $G(\mathbb{Q}_p)$, and $(Z(\mathbb{Q})\cap K^pH)_p$ the image of $Z(\mathbb{Q})\cap K^pH$ in $G(\mathbb{Q}_p)$ via the projection $G(\mathbb{A}^\infty)\twoheadrightarrow G(\mathbb{Q}_p)$.

Let ψ be a continuous character of Z_1 over E. We see $\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\psi}$ is also an admissible Banach representation of $\mathrm{GL}_2(F_\wp)$ stable under the action $\mathrm{Gal}(\overline{F}/F)\times\mathcal{H}^p$. Put

$$U_1 := \{g_\wp \in K_{1,\wp} \mid \det(g_\wp)=1\}. \tag{9}$$

Let $H_\wp := Z_1U_1$ which is an open compact subgroup of $K_{1,\wp}$, ($H_\wp = K_{1,\wp}$ when $p\neq 2$), we see the center of $H_\wp$ is Z_1. By Proposition 2 applied to $H=H_\wp$, one has (note that $\overline{(Z(\mathbb{Q})\cap K^pH_\wp)_p}$ is a subgroup of Z_1)

COROLLARY 1. *Let ψ be a continuous character of Z_1 such that $\psi|_{\overline{(Z(\mathbb{Q})\cap K^pH_\wp)_p}}=1$, then one has an isomorphism of $H_\wp$-representations*

$$\widetilde{H}^1_{\acute{e}t}(K^p, E)^{Z_1=\psi}_{\overline{\rho}} \xrightarrow{\sim} \mathcal{C}(U_1, E)^{\oplus r}$$

where Z_1 acts on $\mathcal{C}(U_1,E)^{\oplus r}$ by the character ψ, and U_1 by the right regular action.

4. Eigenvarieties

4.1. Generalities. Consider the admissible locally $\mathbb{Q}_p$-analytic representation $\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}}$, by applying the functor of Jacquet–Emerton (cf. [24]), one obtains an essentially admissible locally $\mathbb{Q}_p$-analytic representation

$$J_B(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}})$$

of $T(F_\wp)$ (cf. [22, Section 6.4]). Denote by $\widehat{T}_{\Sigma_\wp}$ the rigid space over E parameterizing the locally $\mathbb{Q}_p$-analytic characters of $T(F_\wp)$. By definition (of essentially admissible locally $\mathbb{Q}_p$-analytic representations, cf. [22, Definition 6.4.9]), the action of $T(F_\wp)$ on $J_B(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}})^\vee_b$ (where 'b' signifies the strong topology) can extend to a continuous action of $\mathcal{O}(\widehat{T}_{\Sigma_\wp})$ (being a Fréchet–Stein algebra) such that $J_B(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}})^\vee_b$ is a coadmissible $\mathcal{O}(\widehat{T}_{\Sigma_\wp})$-module. Thus there exists a coherent sheaf $\mathcal{M}_0$ on $\widehat{T}_{\Sigma_\wp}$ such that

$$\mathcal{M}_0\big(\widehat{T}_{\Sigma_\wp}\big) \xrightarrow{\sim} J_B\big(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}}\big)^\vee_b.$$

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The action of $\mathcal{H}^p$ on $J_B(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}})$ induces a natural $\mathcal{O}_{\widehat{T}_{\Sigma_\wp}}$-linear action of $\mathcal{H}^p$ on $\mathcal{M}_0$. Following Emerton, one can construct an *eigenvariety* $\mathcal{V}(K^p)$ from the triple $\{\mathcal{M}_0, \widehat{T}_{\Sigma_\wp}, \mathcal{H}^p\}$:

THEOREM 3 (cf. [23, Section 2.3]). *There exists a rigid analytic space $\mathcal{V}(K^p)$ over E together with a finite morphism of rigid spaces*

$$i : \mathcal{V}(K^p) \longrightarrow \widehat{T}_{\Sigma_\wp}$$

and a morphism of E-algebras with dense image (see Remark 1 below)

$$\mathcal{H}^p \otimes_{\mathcal{O}_E} \mathcal{O}\big(\widehat{T}_{\Sigma_\wp}\big) \longrightarrow \mathcal{O}\big(\mathcal{V}(K^p)\big) \tag{10}$$

such that

(1) *a closed point z of $\mathcal{V}(K^p)$ is uniquely determined by its image χ in $\widehat{T}_{\Sigma_\wp}(\overline{E})$ and the induced morphism $\lambda : \mathcal{H}^p \to \overline{E}$, called a system of eigenvalues of $\mathcal{H}^p$, so z would be denoted by (χ, λ);*

(2) *for a finite extension L of E, a closed point $(\chi, \lambda) \in \mathcal{V}(K^p)(L)$ if and only if the corresponding eigenspace*

$$J_B\big(\widetilde{H}^1_{\acute{e}t}(K^p, E)_{\mathbb{Q}_p-\text{an}} \otimes_E L\big)^{T(F_\wp)=\chi, \mathcal{H}^p=\lambda}$$

is nonzero;

(3) *there exists a coherent sheaf over $\mathcal{V}(K^p)$, denoted by $\mathcal{M}$, such that $i_*\mathcal{M} \cong \mathcal{M}_0$ and that for an L-point $z = (\chi, \lambda)$, the special fiber $\mathcal{M}|_z$ is naturally dual to the (finite dimensional) L-vector space*

$$J_B\big(\widetilde{H}^1_{\acute{e}t}(K^p, E)_{\mathbb{Q}_p-\text{an}} \otimes_E L\big)^{T(F_\wp)=\chi, \mathcal{H}^p=\lambda}.$$

REMARK 1. Indeed, by construction as in [23, Section 2.3], for any affinoid admissible open $U = \operatorname{Spm} A$ in $\widehat{T}_{\Sigma_\wp}$, one has $i^{-1}(U) \cong \operatorname{Spm} B$ where B is the affinoid algebra over A generated by the image of $\mathcal{H}^p \to \operatorname{End}_A(\mathcal{M}_0(U))$, from which we see (10) has a dense image.

Denote by $\mathcal{V}(K^p)_{\text{red}}$ the reduced closed rigid subspace of $\mathcal{V}(K^p)$.

LEMMA 2. *The image of $\mathcal{H}^p$ in $\mathcal{O}(\mathcal{V}(K^p)_{\text{red}})$ via (10) lies in*

$$\mathcal{O}\big(\mathcal{V}(K^p)_{\text{red}}\big)^0 := \big\{f \in \mathcal{O}\big(\mathcal{V}(K^p)_{\text{red}}\big) \mid \|f(x)\| \leqslant 1,\ \forall x \in \mathcal{V}(K^p)(\overline{E})\big\}.$$

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Proof. It is sufficient to prove for any closed point (χ, λ) of $\mathcal{V}(K^p)$, the morphism $\lambda : \mathcal{H}^p \to \overline{E}$ factors through $\overline{\mathcal{O}_E}$. But this is clear since $\widetilde{H}^1_{\text{ét}}(K^p, E)$ has an $\mathcal{H}^p$-invariant $\mathcal{O}_E$-lattice (see Section 3.1). □

Since the rigid space $\widehat{T}_{\Sigma_\wp}$ is nested, by [1, Lemma 7.2.11], one has

PROPOSITION 3. *The rigid space $\mathcal{V}(K^p)$ is nested, and $\mathcal{O}(\mathcal{V}(K^p)_{\text{red}})^0$ is a compact subset of $\mathcal{O}(\mathcal{V}(K^p)_{\text{red}})$ (where we refer to the discussion after* [1, Lemma 7.2.11] *for the topology).*

It would be convenient to fix a central character (in the quaternion Shimura curve case), let $w \in \mathbb{Z}$, consider the (essentially admissible) locally $\mathbb{Q}_p$-analytic representation

$$J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\mathbb{Q}_p-\text{an}}\Big) \cong J_B\big(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}}\big)^{Z_1=\mathcal{N}^{-w}}. \tag{11}$$

One can construct an eigenvariety, denoted by $\mathcal{V}(K^p, w)$, in the same way as in Theorem 3 which satisfies all the properties in Theorem 3 with $\widetilde{H}^1_{\text{ét}}(K^p, E)$ replaced by $\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}$. Denote by $\widehat{T}_{\Sigma_\wp}(w)$ the closed rigid subspace of $\widehat{T}_{\Sigma_\wp}$ such that

$$\widehat{T}_{\Sigma_\wp}(w)(\overline{E}) = \big\{\chi \in \widehat{T}_{\Sigma_\wp} \bigm| \chi|_{Z_1} = \mathcal{N}^{-w}\big\}, \tag{12}$$

moreover, if we denote by $\mathcal{M}_0(w)$ the coherent sheaf over $\widehat{T}_{\Sigma_\wp}(w)$ associated to $J_B(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\mathbb{Q}_p-\text{an}})$, by (11), one has $\mathcal{M}_0(w) \cong \mathcal{M}_0 \otimes_{\mathcal{O}(\widehat{T}_{\Sigma_\wp})} \mathcal{O}(\widehat{T}_{\Sigma_\wp}(w))$ and thus $\mathcal{V}(K^p, w)_{\text{red}} \cong (\mathcal{V}(K^p) \times_{\widehat{T}_{\Sigma_\wp}} \widehat{T}_{\Sigma_\wp}(w))_{\text{red}}$.

4.2. Classicality and companion points.

Let

$$T(F_\wp)^+ := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in T(F_\wp) \,\middle|\, v_\wp(a) \geqslant v_\wp(d) \right\},$$

one can equip V^{N_0} with a continuous action of $T(F_\wp)^+$ by

$$\pi_t(v) := \big|N_0/tN_0t^{-1}\big|^{-1} \sum_{n \in N_0/tN_0t^{-1}} (nt)(v).$$

One has a natural $T(F_\wp)^+$-invariant injection

$$J_B(V) \hookrightarrow V^{N_0} \tag{13}$$

which induces a bijection (cf. [24, Proposition 3.4.9])

$$J_B(V)^{T(F_\wp)=\chi} \xrightarrow{\sim} V^{N_0, T(F_\wp)^+=\chi} \tag{14}$$

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for any continuous (locally $\mathbb{Q}_p$-analytic) characters χ of $T(F_\wp)$. In fact, by the same argument in [24, Proposition 3.2.12], one can show (13) induces a bijection between generalized eigenspaces (note that the action of $T(F_\wp)^+$ on $V^{N_0}[T(F_\wp)^+ = \chi]$ extends naturally to an action of $T(F_\wp)$)

$$J_B(V)[T(F_\wp) = \chi] \xrightarrow{\sim} V^{N_0}[T(F_\wp)^+ = \chi]. \tag{15}$$

DEFINITION 1. For an L-point $z = (\chi, \lambda)$ of $\mathcal{V}(K^p)$, $S \subseteq \Sigma_\wp$, z is called S-classical (respectively quasi-S-classical) if there exists a nonzero vector

$$v \in \left(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}} \otimes_E L\right)^{N_0, T(F_\wp)^+=\chi, \mathcal{H}^p=\lambda}$$

such that v is S-classical (respectively quasi-S-classical). We call z classical (quasi-classical) if z is $\Sigma_\wp$-classical (respectively quasi-$\Sigma_\wp$-classical).

DEFINITION 2. Let $z = (\chi_1 \otimes \chi_2, \lambda)$ be a closed point in $\mathcal{V}(K^p)$, for $S \subseteq C(\chi)$ (cf. (5)), put

$$\chi_S^c = \chi_{1,S}^c \otimes \chi_{2,S}^c := \chi_1 \prod_{\sigma\in S} \sigma^{k_{\chi_2,\sigma} - k_{\chi_1,\sigma} - 1} \otimes \chi_2 \prod_{\sigma \in S} \sigma^{k_{\chi_1,\sigma} - k_{\chi_2,\sigma}+1}; \tag{16}$$

we say that z admits an S-companion point if $z_S^c := (\chi_S^c, \lambda)$ is also a closed point in $\mathcal{V}(K^p)$. If so, we say the companion point z_S^c is effective if it is moreover quasi-$C(\chi)\backslash S$-classical (note that $C(\chi_S^c) = C(\chi)\backslash S$).

As in [20, Lemma 6.2.24], one has

PROPOSITION 4. *Let $z = (\chi, \lambda)$ be an L-point in $\mathcal{V}(K^p)$, $\sigma \in C(\chi)$, suppose there exists a nonquasi-σ-classical vector (see Section 2 for $d\chi$)*

$$v \in \left(\widetilde{H}^1_{ét}(K^p, E)_{\mathbb{Q}_p-\text{an}} \otimes_E L\right)^{N_0, \mathfrak{t}_{\Sigma_\wp} = d\chi}[T(F_\wp)^+ = \chi, \mathcal{H}^p = \lambda],$$

then z admits a σ-companion point. Moreover, there exists $S \subseteq C(\chi)$ containing σ such that z admits an effective S-companion point.

Proof. We sketch the proof. Let $k_\sigma := k_{\chi_1,\sigma} - k_{\chi,\sigma} + 2 \in \mathbb{Z}_{\geqslant 2}$ (since $\sigma \in C(\chi)$), if v is not quasi-σ-classical, we deduce that $v_\sigma^c = X_{-,\sigma}^{k_\sigma - 1} \cdot v \neq 0$ (where $X_{-,\sigma} := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{g}_\sigma$), moreover, as in [24, Proposition 4.4.4] (see also [20, Lemma 6.3.15]), one can prove v_σ^c is a generalized (χ_σ^c, λ)-eigenvector for $T(F_\wp) \times \mathcal{H}^p$. From which we deduce z admits a σ-companion point. If v_σ^c is not quasi-σ'-classical for some $\sigma' \in C(\chi)\backslash\{\sigma\} = C(\chi_\sigma^c)$, one can repeat this argument to find companion points of z_σ^c until one gets $S \subseteq C(\chi)$ and an effective S-companion point of z. $\square$

REMARK 2. One can also deduce this proposition from the adjunction formula in [11, Theorem 4.3].

As in [20, Proposition 6.2.27], one has

THEOREM 4 (Classicality). *Let $z = (\chi = \chi_1 \otimes \chi_2, \lambda)$ be an L-point in $\mathcal{V}(K^p)$. For $\sigma \in C(\chi)$, put $k_\sigma := k_{\chi_1,\sigma} - k_{\chi_2,\sigma} + 2 \in \mathbb{Z}_{\geqslant 2}$. Let $S \subseteq C(\chi)$, if*

$$v_\wp(q\chi_1(\varpi)) < \inf_{\sigma \in S}\{k_\sigma - 1\},$$

then any vector in

$$\left(\widetilde{H}^1_{ét}(K^p, E)_{\mathbb{Q}_p-\mathrm{an}} \otimes_E L\right)^{N_0, \mathfrak{t}_{\Sigma_\wp} = d\chi}[T(F_\wp)^+ = \chi, \mathcal{H}^p = \lambda]$$

in quasi-S-classical, in particular, the point z is quasi-S-classical.

Proof. We sketch the proof. For $\sigma \in S$, if there exists a non quasi-σ-classical vector

$$v \in \left(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\mathrm{an}} \otimes_E L\right)^{N_0, \mathfrak{t}_{\Sigma_\wp} = d\chi}[T(F_\wp)^+ = \chi, \mathcal{H}^p = \lambda],$$

by Proposition 4, z admits an effective S'-companion point $z^c_{S'}$ with $S' \subseteq C(\chi)$ containing σ. Using [20, Proposition 6.2.23], this point would induce a continuous injection from a locally $\mathbb{Q}_p$-analytic parabolic induction twisted with certain algebraic representation (as in [20, Proposition 6.2.23] by replacing J, S, $C_{\overline{B}}(\chi)$ by $\Sigma_\wp$, S', $C(\chi)$ respectively) into $\widetilde{H}^1_{\text{ét}}(K^p, E) \otimes_E L$. Since $\widetilde{H}^1_{\text{ét}}(K^p, E)$ is unitary, one can apply [10, Proposition 5.1], and get (as in [20, Corollary 6.2.24]) $v_\wp(q\chi_1(\varpi)) \geqslant \sum_{\sigma \in S'}(k_\sigma - 1)$, a contradiction. □

COROLLARY 2. *Let $w \in \mathbb{Z}$, $z = (\chi = \chi_1 \otimes \chi_2, \lambda)$ be an L-point in $\mathcal{V}(K^p, w)$ with $C(\chi) = \Sigma_\wp$, $k_\sigma := k_{\chi_1,\sigma} - k_{\chi_2,\sigma} + 2 \in 2\mathbb{Z}_{\geqslant 1}$ such that $k_\sigma \equiv w \pmod 2$ for all $\sigma \in \Sigma_\wp$. There exist thus smooth characters ψ_1, ψ_2 such that (note that $k_{\chi_1,\sigma} + k_{\chi_2,\sigma} = -w$)*

$$\chi_1 \otimes \chi_2 = \prod_{\sigma \in \Sigma_\wp} \sigma^{-(w-k_\sigma+2)/2}\psi_1 \otimes \prod_{\sigma \in \Sigma_\wp} \sigma^{-(w+k_\sigma-2)/2}\psi_2.$$

Let $S \subseteq \Sigma_\wp$, if

$$v_\wp(q\psi_1(\varpi)) < \sum_{\sigma \in \Sigma_\wp} \frac{w - k_\sigma + 2}{2} + \inf_{\sigma \in S}\{k_\sigma - 1\},$$

then the point z is S-classical.

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REMARK 3. We invite the reader to compare this corollary with conjectures of Breuil in [9] and results of Tian–Xiao in [40].

4.3. Localization at a non-Eisenstein maximal ideal. Let ρ be a 2-dimensional continuous representation of $\mathrm{Gal}(\overline{F}/F)$ over E, suppose that ρ is absolutely irreducible modulo ϖ_E and there exists an irreducible algebraic representation W of G^c such that $H^1_{\text{ét}}(K^p, W)_{\overline{\rho}} \neq 0$ (ρ is thus called *modular*). It is known that there exist $w \in \mathbb{Z}$, $k_\sigma \in Z_{\geqslant 2}$, $k_\sigma \equiv w \pmod 2$ for all $\sigma \in \Sigma_\wp$ such that $W \cong W(\underline{k}_{\Sigma_\wp}, w)$. We fix this w in the following. Consider the essentially admissible locally $\mathbb{Q}_p$-analytic representation $J_B(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho}}^{Z_1=\mathcal{N}^{-w}})$, whose strong dual gives rise to a coherent sheaf $\mathcal{M}_0(K^p, w)_{\overline{\rho}}$ over $\widehat{T}_{\Sigma_\wp}$. As in Theorem 3, one can obtain an eigenvariety $\mathcal{V}(K^p, w)_{\overline{\rho}}$ together with a coherent sheaf $\mathcal{M}(K^p, w)_{\overline{\rho}}$ over $\mathcal{V}(K^p, w)_{\overline{\rho}}$, which satisfies the properties in Theorem 3 with $\widetilde{H}^1_{\text{ét}}(K^p, E)$ replaced by $\widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho}}^{Z_1=\mathcal{N}^{-w}}$. Since $\widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho}}^{Z_1=\mathcal{N}^{-w}}$ is a direct summand of $\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}$, $\mathcal{V}(K^p, w)_{\overline{\rho}}$ is a closed rigid subspace of $\mathcal{V}(K^p, w)$ (cf. [20, Lemma 6.2.6]). By Proposition 1, one can describe the classical vectors of $\widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho}}^{Z_1=\mathcal{N}^{-w}}$ as follows:

COROLLARY 3. *With the notation in Corollary 2, suppose moreover z in $\mathcal{V}(K^p, w)_{\overline{\rho}}$, let v be a vector in*

$$\left(\widetilde{H}^1_{\acute{e}t}(K^p, E)_{\mathbb{Q}_p-\mathrm{an},\overline{\rho}} \otimes_E L\right)^{N_0, Z_1=\mathcal{N}^{-w}, \mathfrak{t}_{\Sigma_\wp}=d\chi}[T(F_\wp)^+ = \chi, \mathcal{H}^p = \lambda], \tag{17}$$

if v is classical, then v lies in (see Remark 4 below)

$$\begin{aligned}(H^1_{\acute{e}t}(K^p, W(\underline{k}_{\Sigma_\wp}, w))_{\overline{\rho}} \otimes_E L)^{N_0, Z_1=\psi_1\psi_2}[T(F_\wp)^+ \\ = \psi_1 \otimes \psi_2, \mathcal{H}^p = \lambda] \otimes_E \chi(\underline{k}_{\Sigma_\wp}, w),\end{aligned} \tag{18}$$

with $\chi(\underline{k}_{\Sigma_\wp}, w) := \prod_{\sigma\in\Sigma_\wp} \sigma^{-(w-k_\sigma+2)/2} \otimes \prod_{\sigma\in\Sigma_\wp} \sigma^{-(w+k_\sigma-2)/2}$ (being a character of $T(F_\wp)$).

REMARK 4. Note that $T(F_\wp)$ acts on $(W(\underline{k}_{\Sigma_\wp}, w)^\vee)^{N_0}$ via $\chi(\underline{k}_{\Sigma_\wp}, w)$, the embedding of the vector space (18) into (17) is obtained by taking N_0-invariant vectors of the following $\mathrm{GL}_2(F_\wp) \times \mathcal{H}^p \times \mathrm{Gal}(\overline{F}/F)$-invariant injection (cf. Proposition 1)

$$\begin{aligned}&H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big)_{\overline{\rho}} \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee \otimes_E L \\ &\xrightarrow{\sim} \big(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\mathrm{an},\overline{\rho}} \otimes_E W(\underline{k}_{\Sigma_\wp}, w)\big)_\infty \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee \otimes_E L \\ &\hookrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\mathrm{an},\overline{\rho}} \otimes_E L.\end{aligned}$$

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We study in details the structure of $\mathcal{V}(K^p, w)_{\overline{\rho}}$. Let

$$T' := Z_1' \times \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}} \times \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix}^{\mathbb{Z}} \times \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \,\middle|\, z_i \in \mathcal{O}_{\wp}^{\times}, \quad z_i^{q-1} = 1 \right\}. \tag{19}$$

One has thus a finite morphism of rigid spaces (which is moreover an isomorphism when $p \neq 2$)

$$\widehat{T}_{\Sigma_\wp} \xrightarrow{(\mathrm{pr}_1, \mathrm{pr}_2)} \widehat{(T')}_{\Sigma_\wp} \times \widehat{(Z_1)}_{\Sigma_\wp}, \quad \chi \mapsto (\chi|_{T'}, \chi|_{Z_1}), \tag{20}$$

where $\widehat{(T')}_{\Sigma_\wp}$ and $\widehat{(Z_1)}_{\Sigma_\wp}$ denote the rigid spaces parameterizing locally $\mathbb{Q}_p$-analytic characters of T' and Z_1, respectively. Note that $\mathcal{M}_1 := \mathrm{pr}_{1,*}\, \mathcal{M}_0(K^p, w)_{\overline{\rho}}$ is in fact a coherent sheaf over $\widehat{(T')}_{\Sigma_\wp}$: the support of $\mathcal{M}_0(K^p, w)_{\overline{\rho}}$ is contained in $\widehat{T}_{\Sigma_\wp}(w)$ (as a closed subspace of $\widehat{T}_{\Sigma_\wp}$), which is finite over $\widehat{(T')}_{\Sigma_\wp}$.

Put $\Pi := \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, let $R_\wp$ be a (finite) set of representatives of $T(F_\wp)/T'Z_1$ in $T(F_\wp)$ (note $T(F_\wp) = T'Z_1$ when $p \neq 2$), let $\mathcal{H}$ be the $\mathcal{H}^p$-algebra generated by $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$, $\left\{\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right) \,\middle|\, z_i \in \mathcal{O}_\wp^\times, z_i^{q-1} = 1\right\}$, and the elements in $R_\wp$. Denote by $\mathcal{W}_{1,\Sigma_\wp}$ the rigid space over E which parameterizes locally $\mathbb{Q}_p$-analytic characters of $1 + 2\varpi\mathcal{O}_\wp \cong Z_1'$, by the decomposition of groups (19), one gets a natural projection $\mathrm{pr} : \widehat{(T')}_{\Sigma_\wp} \to \mathcal{W}_{1,\Sigma_\wp} \times \mathbb{G}_m$, $\chi \mapsto (\chi|_{Z_1'}, \chi(\Pi))$. By Corollary 1 and the argument in the proof of [24, Proposition 4.2.36] (for example, see [24, (4.2.43)]), we see $\mathcal{M}_1(\widehat{(T')}_{\Sigma_\wp})$ is a coadmissible $\mathcal{O}(\mathcal{W}_{1,\Sigma_\wp})\{\{X, X^{-1}\}\}$-module with X acting on $\mathcal{M}_1(\widehat{(T')}_{\Sigma_\wp})$ by the operator Π. Let $\mathcal{M}_2 := \mathrm{pr}_*\, \mathcal{M}_1$ which is thus a coherent sheaf over $\mathcal{W}_{1,\Sigma_\wp} \times \mathbb{G}_m$ equipped with an $\mathcal{O}_{\mathcal{W}_{1,\Sigma_\wp} \times \mathbb{G}_m}$-linear action of $\mathcal{H}$. One can thus construct $\mathcal{V}(K^p, w)_{\overline{\rho}}$ from the triple $\{\mathcal{M}_2, \mathcal{W}_{1,\Sigma_\wp} \times \mathbb{G}_m, \mathcal{H}\}$ as in [23, Section 2.3].

Let $\{\mathrm{Spm}\, A_i\}_{i \in I}$ be an admissible covering of $\mathcal{W}_{1,\Sigma_\wp}$ by increasing affinoid opens, by Corollary 1, [24, (4.2.43)] and the results in [20, Section 5.A], for any $i \in I$, there exists a Fredholm series $F_i(z) \in 1 + zA_i\{\{z\}\}$ (which is hence a global section over $\mathrm{Spm}\, A_i \times \mathbb{G}_m$) such that the coherent sheaf $\mathcal{M}_2|_{\mathrm{Spm}\, A_i \times \mathbb{G}_m}$ is supported at $\mathcal{Z}_i$ where $\mathcal{Z}_i$ is the closed rigid subspace of $\mathrm{Spm}\, A_i \times \mathbb{G}_m$ defined by $F_i(z)$. Moreover, it is known that (cf. [13, Section 4]) $\mathcal{Z}_i$ admits an admissible covering $\{U_{ij} \cong \mathrm{Spm}\, A_i[z]/P_j(z)\}$ such that

- $P_j(z) \in 1 + zA_i[z]$ is a polynomial of degree d_j with leading coefficient being a unit;
- there exists $Q_j(z) \in 1 + zA_i\{\{z\}\}$ such that $F_i(z) = P_j(z)Q_j(z)$ and that $(P_j(z), Q_j(z)) = 1$.

As in the proof of [20, Proposition 5.A.6], one can show $\mathcal{M}_2(U_{ij})$ is a finite locally free A_i-module of rank d_j, equipped with an A_i-linear action of $\mathcal{H}$ such

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that the characteristic polynomial of Π is given by $P_j(z)$, and that $Q_j(\Pi)$ acts on $\mathcal{M}_2(U_{ij})$ via an invertible operator. Denote by $\mathcal{H}_{ij}$ the $A_i[z]/P_j(z)$-algebra generated by the image of the natural map

$$\mathcal{H} \longrightarrow \operatorname{End}_{A_i[z]/P_j(z)}(\mathcal{M}_2(U_{ij})),$$

which is also the A_i-algebra generated by the image of $\mathcal{H} \to \operatorname{End}_{A_i}(\mathcal{M}_2(U_{ij}))$ (since $\Pi \in \mathcal{H}$). One can check the restriction $\mathcal{V}(K^p, w)_{\overline{\rho}}|_{U_{ij}}$ is isomorphic to $\operatorname{Spm} \mathcal{H}_{ij}$ (cf. [23, Section 2.3]). In particular, we see that the construction of $\mathcal{V}(K^p, w)_{\overline{\rho}}$ coincides with the construction of eigenvarieties by Coleman–Mazur (formalized by Buzzard in [13]). Since $\mathcal{W}_{1,\Sigma_\wp}$ is equidimensional of dimension d, by [16, Proposition 6.4.2], we have

PROPOSITION 5. *The rigid analytic space $\mathcal{V}(K^p, w)_{\overline{\rho}}$ is equidimensional of dimension d.*

DEFINITION 3. For a character χ of $T(F_\wp)$, we say that χ is spherically algebraic if χ is the twist of an algebraic character by an unramified character. We call a closed point $z = (\chi, \lambda)$ of $\mathcal{V}(K^p, w)$ semistable classical if z is classical and χ is spherically algebraic.

Denote by $C(w)$ the set of semistable classical points in $\mathcal{V}(K^p, w)_{\overline{\rho}}$. By the same argument as in the proof of [16, Proposition 6.2.7, Proposition 6.4.6], the following proposition follows from Corollary 2.

PROPOSITION 6.

(1) *Let $z = (\chi, \lambda)$ be a closed point of $\mathcal{V}(K^p, w)_{\overline{\rho}}$, suppose moreover χ spherically algebraic, then the set $C(w)$ accumulates over the point z, that is, for any admissible open U containing z, there exists an admissible open $V \subseteq U$, $z \in V(\overline{E})$ such that $C(w) \cap V(\overline{E})$ is Zariski-dense in V.*

(2) *The set $C(w)$ is Zariski-dense in $\mathcal{V}(K^p, w)_{\overline{\rho}}$.*

4.4. Families of Galois representations.

4.4.1. Families of Galois representations on eigenvarieties Keep the above notation. For $\mathfrak{l} \in S(K^p)$, denote by $\mathfrak{a}_\mathfrak{l} \in \mathcal{O}(\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}})$ (respectively $\mathfrak{b}_\mathfrak{l} \in \mathcal{O}(\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}})$) the image of $T_\mathfrak{l} \in \mathcal{H}^p$ (respectively $S_\mathfrak{l} \in \mathcal{H}^p$) via the natural morphism $\mathcal{H}^p \to \mathcal{O}(\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}})$. Denote by $\mathcal{S}$ the complement of $S(K^p)$ in the set of finite places of F, which is hence a finite set. Denote by $F^{\mathcal{S}}$ the maximal algebraic extension of F which is unramified outside $\mathcal{S}$.

For any $z \in C(w)$, by [14] (and Corollary 3), there exists a 2-dimensional continuous representation ρ_z of $\mathrm{Gal}(\overline{F}/F)$ over $k(z)$, the residue field at z, which is unramified outside S and hence a representation of $\mathrm{Gal}(F^{\mathcal{S}}/F)$, such that (see also (6))

$$\mathrm{Frob}_{\mathfrak{l}}^{-2} - \mathfrak{a}_{\mathfrak{l},z} \mathrm{Frob}_{\mathfrak{l}}^{-1} + \ell^{f_{\mathfrak{l}}} \mathfrak{b}_{\mathfrak{l},z} = 0$$

where $\mathfrak{a}_{\mathfrak{l},z}, \mathfrak{b}_{\mathfrak{l},z} \in k(z)$ denote the respective evaluation of $\mathfrak{a}_{\mathfrak{l}}$ and $\mathfrak{b}_{\mathfrak{l}}$ at z. In particular, one has $\mathrm{tr}(\mathrm{Frob}_{\mathfrak{l}}^{-1}) = \mathfrak{a}_{\mathfrak{l},z}$. Denote by $\mathcal{T}_z : \mathrm{Gal}(F^{\mathcal{S}}/F) \to k(z)$, $g \mapsto \mathrm{tr}(\rho_z(g))$, which is thus a 2-dimensional continuous pseudo-character of $\mathrm{Gal}(F^{\mathcal{S}}/F)$ over $k(z)$. By [16, Proposition 7.1.1] and Proposition 3, one has

PROPOSITION 7. *There exists a unique 2-dimensional continuous pseudo-character $\mathcal{T} : \mathrm{Gal}(F^{\mathcal{S}}/F) \to \mathcal{O}(\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}})$ such that the evaluation of $\mathcal{T}$ at $z \in C(w)$ equals to $\mathcal{T}_z$.*

Let z be a closed point of $\mathcal{V}(K^p, w)_{\overline{\rho}}$, denote by $\mathcal{T}_z := \mathcal{T}|_z$, which is thus a 2-dimensional continuous pseudo-character of $\mathrm{Gal}(F^{\mathcal{S}}/F)$ over $k(z)$. By [39, Theorem 1(2)], there exists a unique 2-dimensional continuous semisimple representation ρ_z of $\mathrm{Gal}(F^{\mathcal{S}}/F)$ such that $\mathrm{tr}(\rho_z) = \mathcal{T}_z$. By Eichler–Shimura relations, one has $\overline{\rho_z} \cong \overline{\rho}$, in particular, ρ_z is absolutely irreducible. By [2, Lemma 5.5], one has

PROPOSITION 8. *For any closed point z of $\mathcal{V}(K^p, w)_{\overline{\rho}}$, there exist an admissible open affinoid U containing z in $\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}}$ and a continuous representation*

$$\rho_U : \mathrm{Gal}(F^{\mathcal{S}}/F) \longrightarrow \mathrm{GL}_2(\mathcal{O}_U)$$

such that $\rho_U|_z \cong \rho_{z'}$ for any $z' \in U(\overline{E})$.

In general, by [1, Lemma 7.8.11], one has

PROPOSITION 9. *Let U be an open affinoid of $\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}}$, there exist a rigid space $\widetilde{U}$ over U, and an $\mathcal{O}_{\widetilde{U}}$-module $\mathcal{M}$ locally free of rank 2 equipped with a continuous $\mathcal{O}_{\widetilde{U}}$-linear action of $\mathrm{Gal}(F^{\mathcal{S}}/F)$ such that*

(1) *the morphism $g : \widetilde{U} \to U$ factors through a rigid space U' such that $\widetilde{U}$ is a blow-up over U' of $U' \backslash U''$ with U'' an Zariski-open Zariski-dense subspace of U' and that U' is finite, dominant over U;*

(2) *for any $z \in \widetilde{U}(\overline{E})$, the 2-dimensional representation $\mathcal{M}|_z$ of $\mathrm{Gal}(F^{\mathcal{S}}/F)$ is isomorphic to $\rho_{g(z)}$.*

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REMARK 5. Let Z be a Zariski-dense subset of closed points in U, then $g^{-1}(U)$ is Zariski-dense in $\widetilde{U}$: denote by $g' : U' \to U$ the morphism as in (1), by [16, Lemma 6.2.8], we see $(g')^{-1}(Z)$ is Zariski-dense in U', so $g^{-1}(Z)$ is Zariski-dense in $\widetilde{U}$.

4.4.2. Trianguline representations Consider the restriction $\rho_{z,\wp} := \rho_z|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)}$. Denote by $F_{\wp,\infty} := \cup_n F_\wp(\zeta_{p^n})$ where ζ_{p^n} is a root of unity primitive of order p^n. Set $\Gamma := \mathrm{Gal}(F_{\wp,\infty}/F_\wp)$, $H_\wp := \mathrm{Gal}(\overline{\mathbb{Q}_p}/F_{\wp,\infty})$. One has a ring $B^\dagger_{\mathrm{rig}}$ (cf. [3, Section 3.4]) which is equipped with an action of φ and $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ such that $B^\dagger_{\mathrm{rig},F_\wp} := (B^\dagger_{\mathrm{rig}})^{H_{F_\wp}}$ is naturally isomorphic to the Robba ring with coefficients in $F'_\wp$ where $F'_\wp$ denotes the maximal unramified extension of $\mathbb{Q}_p$ in $F_{\wp,\infty}$ (which is finite over $F_{\wp,0}$). For an n-dimensional continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ over E, $D_{\mathrm{rig}}(V) := (B^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{F}_\wp}}$ is an étale (φ, Γ)-module of rank n over $\mathcal{R}_E := B^\dagger_{\mathrm{rig},F_\wp} \otimes_{\mathbb{Q}_p} E$ (that is, an étale (φ, Γ)-module over $B^\dagger_{\mathrm{rig},F_\wp}$ equipped with an E-action which commutes with that of φ and Γ) (cf. [3, Proposition 3.4]). Let $\delta : F_\wp^\times \to E^\times$ be a continuous character, following [33, Section 1.4], one can associate to δ a (φ, Γ)-module, denoted by $\mathcal{R}_E(\delta)$, free of rank 1 over $\mathcal{R}_E$. The converse is also true, that is, for any (φ, Γ)-module D free of rank 1 over $\mathcal{R}_E$, there exists a continuous character $\delta : F_\wp^\times \to E^\times$ such that $D \cong \mathcal{R}_E(\delta)$.

DEFINITION 4 (Cf. [18, Definition 4.1], [33, Definition 1.15]). Let ρ be a 2-dimensional continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ over E, ρ is called trianguline if there exist continuous characters δ_1, δ_2 of $F_\wp^\times$ over E such that $D_{\mathrm{rig}}(\rho)$ lies in an exact sequence as follows:

$$0 \to \mathcal{R}_E(\delta_1) \to D_{\mathrm{rig}}(V) \to \mathcal{R}_E(\delta_2) \to 0.$$

Such an exact sequence is called a triangulation, denoted by $(\rho, \delta_1, \delta_2)$, of $D_{\mathrm{rig}}(\rho)$ (and of ρ).

We refer to [33] for a classification of 2-dimensional trianguline representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$. Note that if ρ is semistable, then ρ is trianguline, if ρ is moreover noncrystalline, then the triangulation of ρ is unique.

DEFINITION 5 (cf. [32, Definition 4.3.1]). Let ρ be a 2-dimensional trianguline representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ over E with $(\rho, \delta_1, \delta_2)$ a triangulation of ρ. For $\sigma \in \Sigma_\wp$, we say that ρ is non-σ-critical if $k_{\delta_1,\sigma} - k_{\delta_2,\sigma} \in \mathbb{Z}_{\geqslant 1}$. More generally, for $J \subseteq \Sigma_\wp$, we say ρ is non-J-critical if ρ is non-σ-critical for all $\sigma \in J$, we say ρ is noncritical if ρ is non-$\Sigma_\wp$-critical.

For a closed point $z = (\chi_z = \chi_{z,1} \otimes \chi_{z,2}, \lambda_z)$ of $\mathcal{V}(K^p, w)_{\overline{\rho}}$, put $k_{z,\sigma} := k_{\chi_{z,1},\sigma} - k_{\chi_{z,2},\sigma} + 2 \in \mathbb{Z}_{\geqslant 2}$, for $\sigma \in C(\chi_z)$. If $z \in C(w)$, we see $k_{z,\sigma} \equiv w \pmod 2$ for all $\sigma \in \Sigma_\wp$ (note that $C(\chi_z) = \Sigma_\wp$ in this case). The following theorem can be easily deduced from the results in [35] (together with Corollary 3 and the results of [33] on triangulations of semistable representations, see [20, Proposition 6.2.44] for the unitary Shimura curves case).

THEOREM 5. *Let $z = (\chi_{z,1} \otimes \chi_{z,2}, \lambda_z) \in C(w)$, then $\rho_{z,\wp}$ is semistable (hence trianguline) with a triangulation given by*

$$0 \to \mathcal{R}_{k(z)}(\delta_{z,1}) \to D_{\mathrm{rig}}(\rho_{z,\wp}) \to \mathcal{R}_{k(z)}(\delta_{z,2}) \to 0 \tag{21}$$

where

$$\begin{cases} \delta_{z,1} = \mathrm{unr}(q)\chi_{z,1} \prod_{\sigma\in\Sigma_z} \sigma^{1-k_{z,\sigma}} \\ \delta_{z,2} = \chi_{z,2} \prod_{\sigma\in\Sigma_\wp} \sigma^{-1} \prod_{\sigma\in\Sigma_z} \sigma^{k_{z,\sigma}-1} \end{cases}$$

with $\Sigma_z \subseteq \Sigma_\wp$ (maybe empty). Put $\psi_{z,1} := \chi_{z,1} \prod_{\sigma\in\Sigma_\wp} \sigma^{(w-k_{z,\sigma}+2)/2}$ (being an unramified character of $F_\wp^\times$), for $S \subseteq \Sigma_\wp$, if one has

$$v_\wp(q\psi_{z,1}(\varpi)) < \sum_{\sigma\in\Sigma_\wp} \frac{w-k_{z,\sigma}+2}{2} + \inf_{\sigma\in S}\{k_{z,\sigma}-1\},$$

then $\Sigma_z \cap S = \emptyset$, in particular, in this case the triangulation (21) is non-S-critical.

Denote by $C(w)_0$ the subset of $C(w)$ of points z such that

$$v_\wp(q\psi_{z,1}(\varpi)) < \sum_{\sigma\in\Sigma_\wp} \frac{w-k_{z,\sigma}+2}{2} + \inf_{\sigma\in\Sigma_\wp}\{k_{z,\sigma}-1\}. \tag{22}$$

As in [16, Proposition 6.2.7, Proposition 6.4.6], one can prove $C(w)_0$ is Zariski-dense in $\mathcal{V}(K^p, w)_{\bar{\rho}}$, and is an accumulation subset (cf. [1, Section 3.3.1]). By the theory of global triangulation, one has

THEOREM 6. *Let $z = (\chi_z = \chi_{z,1} \otimes \chi_{z,2}, \lambda)$ be a closed point of $\mathcal{V}(K^p, w)_{\bar{\rho}}$, then the representation $\rho_{z,\wp}$ is trianguline with a triangulation given by*

$$0 \to \mathcal{R}_{k(z)}(\delta_{z,1}) \to D_{\mathrm{rig}}(\rho_{z,\wp}) \to \mathcal{R}_{k(z)}(\delta_{z,2}) \to 0$$

where

$$\begin{cases} \delta_{z,1} = \mathrm{unr}(q)\chi_{z,1} \prod_{\sigma\in\Sigma_z} \sigma^{1-k_{z,\sigma}} \\ \delta_{z,2} = \chi_{z,2} \prod_{\sigma\in\Sigma_\wp} \sigma^{-1} \prod_{\sigma\in\Sigma_z} \sigma^{k_{z,\sigma}-1} \end{cases}$$

with Σ_z a subset (maybe empty) of $C(\chi_z) \cap \Sigma_\wp$.

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Proof. Since $\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}}$ is nested, and $C(w)_0$ is Zariski-dense, there exists an irreducible affinoid neighborhood of z such that $C(w)_0 \cap U(\overline{E})$ is Zariski-dense in U (for example, see [1, Lemma 7.2.9]). Denote by $g : \widetilde{U} \to U$ the rigid space as in Proposition 9, thus $g^{-1}(C(w)_0 \cap U(\overline{E}))$ is Zariski-dense in $\widetilde{U}$. The theorem then follows from [30, Theorem 6.3.13] and [30, Ex.6.3.14] (see also [32, Theorem 4.4.2]). □

COROLLARY 4. *Keep the notation in Theorem 6, suppose moreover*

$$\mathrm{unr}(q^{-1})\chi_{z,1}^{-1}\chi_{z,2} \neq \prod_{\sigma\in\Sigma_\wp} \sigma^{n_\sigma} \quad \textit{for all } \underline{n}_{\Sigma_\wp} \in \mathbb{Z}^d, \tag{23}$$

let $S \subseteq C(\chi_z)$, if $\Sigma_z \cap S = \emptyset$, then z does not have S'-companion point for any $S' \subseteq S$, $S' \neq \emptyset$. As a result, the point z is quasi-S-classical.

Proof. The second part follows from the first part and Proposition 4. We prove the first part. Let $S' \subseteq S$, $S' \neq \emptyset$, suppose z admits an S'-companion point $z_{S'}^c$, by applying Theorem 6 to the point $z_{S'}^c$, one can get a triangulation $(\rho_{z_{S'}^c,\wp}, \delta_{z_{S'}^c,1}, \delta_{z_{S'}^c,2})$ for $\rho_{z_{S'}^c,\wp} \cong \rho_{z,\wp}$. Note that $S' \cap C((\chi_z)_{S'}^c) = \emptyset$, so $S' \cap \Sigma_{z_{S'}^c,\wp} = \emptyset$. By the hypothesis (23) and [33, Theorem 3.7], one can check the triangulations $(\rho_{z_{S'}^c,\wp}, \delta_{z_{S'}^c,1}, \delta_{z_{S'}^c,2})$ and $(\rho_{z,\wp}, \delta_{z,1}, \delta_{z,2})$ are the same. As a result, one sees $S' \subseteq \Sigma_z$, a contradiction. □

COROLLARY 5. *Keep the notation in Theorem 6, suppose χ_z is spherically algebraic (thus $\chi_{z,i} = \psi_{z,i} \prod_{\sigma\in\Sigma_\wp} \sigma^{k_{\chi_i,\sigma}}$ with $\psi_{z,i}$ an unramified character of $F_\wp^\times$) and satisfies*

$$\psi_{z,1}(p)^{-1}\psi_{z,2}(p)q^{-e} \neq 1$$

(note this condition is slightly stronger than the hypothesis (23)), then there exists an open affinoid neighborhood U of z in $\mathcal{V}(K^p, w)_{\overline{\rho},\mathrm{red}}$ containing z such that for any closed point $z' = (\chi_{z'}, \lambda') \in U(\overline{E})$, $\Sigma_{z'} = \emptyset$ and z' does not have companion point.

Proof. As in the proof of [16, Proposition 6.2.7], one can prove $C(w)_0$ accumulates over z. Thus one can choose an open affinoid neighborhood U_0 of z such that

(1) $C(w)_0 \cap U_0(\overline{E})$ is Zariski-dense in U_0;

(2) $\mathrm{unr}(q^{-1})\chi_{z',1}^{-1}\chi_{z',2} \neq \prod_{\sigma\in\Sigma_\wp} \sigma^{n_\sigma}$ for any $\underline{n}_{\Sigma_\wp} \in \mathbb{Z}^d$, $z' \in U_0(\overline{E})$ (see Lemma 3(1) below).

By [30, Theorem 6.3.9], $Z_{U_0} := \{z' \in U_0(\overline{E}) \mid \Sigma_{z'} \neq \emptyset\}$ is a Zariski-closed subset of U_0 and $z \notin Z_{U_0}$. So there exists an open affinoid U of U_0 containing z such that Z_U (defined in the same way as Z_{U_0} by replacing U_0 by U) is empty. The corollary follows. □

LEMMA 3. *Let $\chi_1 \otimes \chi_2$ be a spherically algebraic character of $T(F_\wp)$ (which can be seen as a closed point of $\widehat{T}_{\Sigma_\wp}$), and let $\psi_i := \chi_i \prod_{\sigma\in\Sigma_\wp} \sigma^{-k_{\chi_i,\sigma}}$.*

(1) *Suppose $\psi_1(p)^{-1}\psi_2(p)q^{-e} \neq 1$, then there exists an admissible neighborhood U of $\chi_1 \otimes \chi_2$ in $\widehat{T}_{\Sigma_\wp}$ such that* $\mathrm{unr}(q^{-1})(\chi_1')^{-1}\chi_2' \neq \prod_{\sigma\in\Sigma_\wp}\sigma^{n_\sigma}$ *for any $\underline{n}_{\Sigma_\wp} \in \mathbb{Z}^d$ and $\chi_1' \otimes \chi_2' \in U(\overline{E})$.*

(2) *Suppose $\psi_1(\varpi)^{-1}\psi_2(\varpi)^{-1}q^{-1} \neq 1$, then there exists an admissible neighborhood U of $\chi_1 \otimes \chi_2$ in $\widehat{T}_{\Sigma_\wp}$ such that* $\mathrm{unr}(q^{-1})(\chi_1')^{-1}\chi_2' \neq \prod_{\sigma\in\Sigma_\wp}\sigma^{n_\sigma}$ *for any $\underline{n}_{\Sigma_\wp} \in \mathbb{Z}^d_{\geqslant k_{\chi_2,\sigma}-k_{\chi_1,\sigma}}$ and $\chi_1' \otimes \chi_2' \in U(\overline{E})$.*

Proof. Denote by $\widehat{Z}_{\Sigma_\wp}$ the rigid space parameterizing locally $\mathbb{Q}_p$-analytic characters of $F_\wp^\times$. One has a morphism of rigid spaces:

$$\widehat{T}_{\Sigma_\wp} \longrightarrow \widehat{Z}_{\Sigma_\wp}, \quad (\chi_1')^{-1}\otimes\chi_2' \mapsto \chi_1'\chi_2'. \tag{24}$$

Let $\psi_0 := \mathrm{unr}(q^{-1})\psi_1^{-1}\psi_2$, we claim that

- if $\psi_0(p) \neq 1$ (respectively $\psi_0(\varpi) \neq 1$), then there exists an admissible open U_0 of $\widehat{Z}_{\Sigma_\wp}$ containing ψ_0 (where ψ_0 is seen as a closed point of $\widehat{Z}_{\Sigma_\wp}$) such that $\prod_{\sigma\in\Sigma_\wp}\sigma^{n_\sigma} \notin U(\overline{E})$ for any $\underline{n}_{\Sigma_\wp} \in \mathbb{Z}^d$ (respectively $\prod_{\sigma\in\Sigma_\wp}\sigma^{n_\sigma} \notin U(\overline{E})$ for any $\underline{n}_{\Sigma_\wp} \in \mathbb{Z}^d_{\geqslant 0}$).

Assuming this claim, and let U be the preimage of the admissible open

$$\Big(\prod_{\sigma\in\Sigma_\wp}\sigma^{-k_{\chi_1,\sigma}+k_{\chi_2,\sigma}}\Big)U_0$$

(of $\widehat{Z}_{\Sigma_\wp}$) in $\widehat{T}_{\Sigma_\wp}$ via (24). Thus U satisfies the property in the lemma (1) (respectively (2)).

We prove the claim. Consider the projection $\widehat{Z}_{\Sigma_\wp} \to \mathcal{W}_{\Sigma_\wp} \times \mathbb{G}_m$, $\chi \mapsto (\chi|_{\mathcal{O}_\wp^\times}, \chi(p))$ (respectively the isomorphism $\widehat{Z}_{\Sigma_\wp} \xrightarrow{\sim} \mathcal{W}_{\Sigma_\wp} \times \mathbb{G}_m$, $\chi \mapsto (\chi|_{\mathcal{O}_\wp^\times}, \chi(\varpi))$), set $a := \psi_0(p)$ (respectively $a := \psi_0(\varpi)$), which is the image of ψ_0 in $\mathbb{G}_m$. We discuss in the following two cases:

If $v_\wp(a) \neq 0$, then choose $n \in \mathbb{Z}_{\geqslant 1}$ such that $v_\wp(a) \notin p^n\mathbb{Z}$; let U_1 be an admissible open in $\mathcal{W}_{\Sigma_\wp}$ containing the trivial character such that if $\prod_{\sigma\in\Sigma_\wp}\sigma^{n_\sigma}|_{\mathcal{O}_\wp^\times} \in U_1(\overline{E})$ then $p^n | n_\sigma$ for all σ, U_2 be an admissible open in $\mathbb{G}_m$

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containing a such that $\upsilon_{\wp}(a') = \upsilon_{\wp}(a)$ for all $a' \in U(\overline{E})$, one easily check the admissible open $U_0 := U_1 \times U_2$ satisfies the property in the claim.

If $\upsilon_{\wp}(a) = 0$, since $a \neq 1$ by hypothesis, let U_2 be an admissible open in $\mathbb{G}_m$ such that $a \in U_2(E)$, $1 \notin U_2(E)$ and for all $a' \in U_2(\overline{E})$, $\upsilon_{\wp}(a') = 0$; put $U_0 := \mathcal{W}_{\Sigma_{\wp}} \times U_2$, we see if $\chi' = \prod_{\sigma \in \Sigma_L} \sigma^{n_\sigma} \in U_0(\overline{E})$ for $\underline{n}_{\Sigma_{\wp}} \in \mathbb{Z}^d$ (respectively for $\underline{n}_{\Sigma_{\wp}} \in \mathbb{Z}^d_{\geqslant 0}$), thus $\upsilon_{\wp}(\chi'(p)) = 0$ (respectively $\upsilon_{\wp}(\chi'(\varpi)) = 0$), thus $\sum_{\sigma \in \Sigma_L} n_\sigma = 0$, so $\chi'(p) = 1 \notin U_2(\overline{E})$ (respectively $n_\sigma = 0$ for all $\sigma \in \Sigma_{\wp}$ hence $\chi'(\varpi) = 1 \notin U_2(\overline{E})$), a contradiction, so U_0 satisfies the property in the claim. □

4.4.3. Étaleness of eigenvarieties at noncritical classical points Let $z = (\chi_z = \chi_{z,1} \otimes \chi_{z,2}, \lambda_z)$ be a semistable classical point of $\mathcal{V}(K^p, w)_{\overline{\rho}}$, for $\sigma \in \Sigma_{\wp}$, we say that z is *noncritical* if

(1) the triangulation $(\rho_{z,\wp}, \delta_{z,1}, \delta_{z,2})$ (cf. Theorem 6) is noncritical (that is, $\Sigma_z = \emptyset$);

(2) $\mathrm{unr}(q^{-1})\chi_{z,1}^{-1}\chi_{z,2} \neq \prod_{\sigma \in \Sigma_{\wp}} \sigma^{n_\sigma}$ for any $\underline{n}_{\Sigma_{\wp}} \in \mathbb{Z}^d$.

Let $\psi_{z,1}$, $\psi_{z,2}$ be unramified characters of $F_{\wp}^{\times}$ such that $\chi_{z,i} = \psi_{z,i} \prod_{\sigma \in \Sigma_{\wp}} \sigma^{k_{\chi_{z,i},\sigma}}$, then the condition (2) is equivalent to $\psi_{z,1}(q\varpi) \neq \psi_{z,2}(\varpi)$. If one considers the Galois representation $\rho_{z,\wp}$ (which is semistable), this condition means the eigenvalues of φ^{d_0} on $D_{\mathrm{st}}(\rho_{z,\wp})$ are different.

Consider the natural morphism

$$\kappa : \mathcal{V}(K^p, w)_{\overline{\rho}} \longrightarrow \widehat{T}_{\Sigma_{\wp}} \longrightarrow \mathcal{W}_{1,\Sigma_{\wp}},$$

where the last map is induced by the inclusion $Z_1' \to T(F_{\wp})$ (see also (20)). This section is devoted to prove the following result.

THEOREM 7. *Let z be a noncritical semistable classical point of the rigid space $\mathcal{V}(K^p, w)_{\overline{\rho}}$, then $\mathcal{V}(K^p, w)_{\overline{\rho}}$ is étale over $\mathcal{W}_{1,\Sigma_{\wp}}$ at z.*

The theorem follows by the same argument as in the proof of [17, Theorem 4.8]. Let $z = (\chi_z = \chi_{z,1} \otimes \chi_{z,2}, \lambda_z) : \operatorname{Spec} \overline{E} \to \mathcal{V}(K^p, w)_{\overline{\rho}}$ be a noncritical semistable classical point of $C(w)$, by the construction of $\mathcal{V}(K^p, w)_{\overline{\rho}}$ as in Section 4.3, one can find a connected affinoid neighborhood U of $\kappa(z)$ in $\mathcal{W}_{1,\Sigma_{\wp}}$ and a finite locally free $\mathcal{O}(U)$-module M equipped with an $\mathcal{O}(U)$-linear action of $\mathcal{H}$ such that (see also the proof of [17, Theorem 4.8])

(1) the affinoid spectrum V of $\mathrm{Im}(\mathcal{O}(U) \otimes_{\mathcal{O}_E} \mathcal{H} \to \mathrm{End}_{\mathcal{O}(U)}(M))$ is an affinoid neighborhood of z in $\mathcal{V}(K^p, w)_{\overline{\rho}}$ (thus one has $\mathcal{M}(K^p, w)_{\overline{\rho}}(V) \cong M$ as $\mathcal{O}(V)$-module);

(2) for each continuous character $\chi \in \mathcal{W}_{1,\Sigma_\wp}(\overline{E})$, there is a $T(F_\wp)\times\mathcal{H}^p$-invariant isomorphism

$$M \otimes_{\mathcal{O}(U),\chi} \overline{E} \cong \bigoplus_{(\chi_{z'},\lambda_{z'})\in\kappa^{-1}(\chi)} \big(J_B(\widetilde{H}^1_{\text{ét}}(K^p,E)_{\mathbb{Q}_p-\text{an}}) \otimes_E \overline{E}\big)^{Z_1=\mathcal{N}^{-w}, Z_1'=\chi} [T(F_\wp)=\chi_{z'},\mathcal{H}^p=\lambda_{z'}]^\vee;$$

(3) $\kappa^{-1}(\kappa(z))^{\text{red}} = \{z\}$ and the natural surjection $\mathcal{O}(V) \to k(z)$ has a section.

Let $Z_0 \subseteq U$ be the set of closed points χ such that any point in $\kappa^{-1}(\chi)\cap V(\overline{E})$ is classical, thus Z_0 is Zariski-dense in U (by Theorem 4, note that $v_\wp(\chi_{z',1}(\varpi))$ is bounded for $z' \in V(\overline{E})$), and $Z := \kappa^{-1}(Z_0)\cup\{z\}$ is Zariski-dense in V. Up to shrinking Z_0, one can assume that for any $z' \in Z$,

(a) $\Sigma_{z'} = \emptyset$ (since z is supposed to be noncritical, for $z' \neq z$, this would follow from Theorem 5);

(b) $\mathrm{unr}(q^{-1})\chi_{z',1}^{-1}\chi_{z',2} \neq \prod_{\sigma\in\Sigma_\wp}\sigma^{n_\sigma}$ for any $\underline{n}_{\Sigma_\wp} \in \mathbb{Z}^d$ (for example, by Lemma 3 (2), since by shrinking Z_0, one can assume $k_{\chi_{z',2},\sigma} - k_{\chi_{z',1},\sigma} \geqslant k_{\chi_{z,2},\sigma} - k_{\chi_{z,1},\sigma}$ for all $\sigma \in \Sigma_\wp$, and $z' \in Z_0$).

Let $z' = (\chi_{z'},\lambda_{z'}) \in Z$, denote by $k(z')$ the residue field at z'. One has an isomorphism (cf. (15))

$$J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p,E)_{\mathbb{Q}_p-\text{an}} \otimes_E k(z')\Big)^{Z_1=\mathcal{N}^{-w},Z_1'=\kappa(z')}[T(F_\wp)=\chi_{z'},\mathcal{H}^p=\lambda_{z'}]$$
$$\xrightarrow{\sim} \Big(\widetilde{H}^1_{\text{ét}}(K^p,E)_{\mathbb{Q}_p-\text{an}} \otimes_E k(z')\Big)^{N_0,Z_1=\mathcal{N}^{-w},Z_1'=\kappa(z')}[T(F_\wp)=\chi_{z'},\mathcal{H}^p=\lambda_{z'}].$$

LEMMA 4. *Keep the above notation, any vector in*

$$\Big(\widetilde{H}^1_{\acute{e}t}(K^p,E)_{\mathbb{Q}_p-\text{an}} \otimes_E k(z')\Big)^{N_0,Z_1=\mathcal{N}^{-w},Z_1'=\kappa(z')}[T(F_\wp)=\chi_{z'},\mathcal{H}^p=\lambda_{z'}]$$

is classical.

Proof. Suppose there exists a nonclassical vector v, thus there exists $\sigma \in \Sigma_\wp$, such that v is non-σ-classical. By Proposition 4, one can prove z' admits a σ-companion point, which would lead to a contradiction by the same argument as in the proof of Corollary 4. □

Keep the above notation (so $z' \in Z$), and put

$$\psi_{z'} := \chi_{z'}\Bigg(\prod_{\sigma\in\Sigma_\wp}\sigma^{-k_{\chi_{1,z'},\sigma}} \otimes \prod_{\sigma\in\Sigma_\wp}\sigma^{-k_{\chi_{2,z'},\sigma}}\Bigg)$$

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(being a smooth character of $T(F_\wp)$), $T_0 := Z_1 Z_1'$, by Lemma 4, Corollary 3, one has an isomorphism of $k(z')$-vector spaces

$$\Big(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}} \otimes_E k(z')\Big)^{N_0, Z_1=\mathcal{N}^{-w}, Z_1'=\kappa(z')}[T(F_\wp) = \chi_{z'}, \mathcal{H}^p = \lambda_{z'}]$$
$$\xrightarrow{\sim} \Big(H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big) \otimes_E k(z')\Big)^{N_0, T_0=\psi_{z'}}[T(F_\wp) = \psi_{z'}, \mathcal{H}^p = \lambda_{z'}],$$

where $k_\sigma := k_{\chi_{1,z'},\sigma} - k_{\chi_{2,z'},\sigma} + 2$ for all $\sigma \in \Sigma_\wp$. So

$$J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}} \otimes_E k(z')\Big)^{Z_1=\mathcal{N}^{-w}, Z_1'=\kappa(z')}[T(F_\wp) = \chi_{z'}, \mathcal{H}^p = \lambda_{z'}]$$
$$\xrightarrow{\sim} J_B\Big(H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big) \otimes_E k(z')\Big)^{T_0=\psi_{z'}}[T(F_\wp) = \psi_{z'}, \mathcal{H}^p = \lambda_{z'}].$$

Denote by $\delta(z')$ the dimension of the above vector space over $k(z')$. Set

$$H^1_{\text{ét}}\big(W(\underline{k}_{\Sigma_\wp}, w)\big) := \varinjlim_{(K^p)'} H^1_{\text{ét}}\big((K^p)', W(\underline{k}_{\Sigma_\wp}, w)\big) \otimes_E \overline{E}$$

where $(K^p)'$ runs over open compact subgroups of K^p, this is a smooth admissible representation of $G(\mathbb{A}^\infty)$ equipped with a continuous action of $\mathrm{Gal}(\overline{F}/F)$. One has a decomposition of $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{F}/F)$-representations

$$H^1_{\text{ét}}\big(W(\underline{k}_{\Sigma_\wp}, w)\big) \cong \oplus_\pi \big(\rho(\pi) \otimes \pi\big)$$

where π runs over irreducible smooth admissible representations of $G(\mathbb{A}^\infty)$. It is known that if $\rho(\pi) \neq 0$, then $\dim_{\overline{E}} \rho(\pi) = 2$ (for example, see [**14**, Section 2.2.4]). A necessary condition for $\rho(\pi)$ to be nonzero is that there exists an admissible representation π_∞ of $G(\mathbb{R})$ such that $\pi_\infty \otimes \pi$ is an automorphic representation of $G(\mathbb{A})$ (we fix an isomorphism $\overline{E} \xrightarrow{\sim} \mathbb{C}$). Note that one has

$$H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big) \otimes_E \overline{E} \xrightarrow{\sim} H^1_{\text{ét}}\big(W(\underline{k}_{\Sigma_\wp}, w)\big)^{K^p} \cong \oplus_\pi \big(\rho(\pi) \otimes \pi^{K^p}\big).$$

For an irreducible smooth admissible representation π of $G(\mathbb{A}^\infty)$, π admits thus a decomposition $\pi \cong \otimes'_{\mathfrak{l}} \pi_{\mathfrak{l}}$ with $\pi_{\mathfrak{l}}$ an irreducible smooth admissible representation of $(B \otimes_F F_{\mathfrak{l}})^\times$, where $\mathfrak{l}$ runs over the finite places of F. Recall (for example, see [**29**, Theorem VI.1.1(4)])

Proposition 10. *Let π_1, π_2 be two automorphic representations of $G(\mathbb{A})$, if $\pi_{1,\mathfrak{l}} \cong \pi_{2,\mathfrak{l}}$ for all but finitely many places $\mathfrak{l}$ of F, then $\pi_1 \cong \pi_2$.*

By this proposition (and the above discussions), there exists a unique irreducible smooth admissible representation $\pi_{z'}$ of $G(\mathbb{A}^\infty)$ such that the action

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of $\mathcal{H}^p$ on $(\pi_{z'})^{K^p}$ is given by $\lambda_{z'}$ and that $\rho(\pi_{z'}) \neq 0$ (one has in fact $\rho_{z'} \cong \rho(\pi_{z'})$). Thus

$$H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big)[\mathcal{H}^p = \lambda_{z'}] \cong \rho_{z'} \otimes \pi_{z',\wp} \otimes \bigg(\Big(\otimes_{\mathfrak{l}\neq\wp} \pi_{z',\mathfrak{l}}^{(K^p)_{\mathfrak{l}}}\Big)[\mathcal{H}^p = \lambda_{z'}]\bigg).$$

We have the following facts:

- $\dim_{\overline{E}} J_B(\pi_{z',\wp})[T(F_\wp) = \psi_{z'}] = 1$ (by classical Jacquet module theory and the condition (b));
- $\dim_{\overline{E}} \pi_{z',\mathfrak{l}}^{(K^p)_{\mathfrak{l}}} = 1$, for all $\mathfrak{l} \in S(K^p)$;

from which we deduce

$$\begin{aligned} &J_B\big(H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big) \otimes_E k(z')\big)^{T(F_\wp)=\psi_{z'},\mathcal{H}^p=\lambda_{z'}} \\ &\quad \xrightarrow{\sim} J_B(H^1_{\text{ét}}(K^p, W(\underline{k}_{\Sigma_\wp}, w)) \otimes_E k(z'))^{T_0=\psi_{z'}}[T(F_\wp) = \psi_{z'}, \mathcal{H}^p = \lambda_{z'}]. \end{aligned}$$

Denote by S' the complement of $S(K^p) \cup \{\wp\}$ in the set of finite places of F (thus S' is a finite set), we also deduce (compare with [17, (4.21)])

$$\delta(z') = 2 \sum_{\mathfrak{l}\in S'} \dim_{\overline{E}} \big(\pi_{z',\mathfrak{l}}^{(K^p)_{\mathfrak{l}}}\big).$$

By the same argument as in the proof of [17, Theorem 4.8], one can prove $\delta(z') \geqslant \delta(z)$ for all $z' \in Z$, and then deduce that $\mathcal{O}(V) \cong \mathcal{O}(U) \otimes_E k(z)$. The theorem follows.

REMARK 6. Keep the above notation, if z is moreover an E-point of $\mathcal{V}(K^p, w)_{\overline{\rho}}$ (in practice, one can always enlarge E if necessary), thus one has $\mathcal{O}(V) \cong \mathcal{O}(U)$. So the action of $T(F_\wp)$ on M is given by the character $T(F_\wp) \to \mathcal{O}(V)^\times \cong \mathcal{O}(U)^\times$ induced by the natural morphism $V \to \widehat{T}_{\Sigma_\wp}$.

5. $\mathcal{L}$-invariants and local–global compatibility

5.1. Fontaine–Mazur $\mathcal{L}$-invariants. Recall Fontaine–Mazur $\mathcal{L}$-invariants for 2-dimensional semistable noncrystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$.

Let $k_{1,\sigma}, k_{2,\sigma} \in \mathbb{Z}$, $k_{1,\sigma} < k_{2,\sigma}$ for all $\sigma \in \Sigma_\wp$; let ρ be a 2-dimensional semistable noncrystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ over E of Hodge–Tate weights $(-k_{2,\sigma}, -k_{1,\sigma})_{\sigma\in\Sigma_\wp}$. By Fontaine's theory (cf. [27], [28]), one can associate to ρ a filtered (φ, N)-module (D_0, D) where

$$D_0 := D_{\mathrm{st}}(\rho) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \rho)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)}$$

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is a free $F_{\wp,0}\otimes_{\mathbb{Q}_p} E$-module of rank 2 equipped with a bijective ($F_{\wp,0}$-semilinear and E-linear) endomorphism φ and a nilpotent $F_{\wp,0}\otimes_{\mathbb{Q}_p} E$-linear operator N such that $N\varphi = p\varphi N$, and that $D := D_0\otimes_{F_{\wp,0}} F_\wp \cong D_{\mathrm{dR}}(\rho) := (B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\rho)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)}$ is a free $F_\wp\otimes_{\mathbb{Q}_p} E$-module of rank 2 equipped with a decreasing exhaustive separated filtration by $F_\wp\otimes_{\mathbb{Q}_p} E$-submodules.

Using the isomorphism

$$F_{\wp,0}\otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma_0:F_{\wp,0}\to E} E, \quad a\otimes b\mapsto \big(\sigma_0(a)b\big)_{\sigma_0:F_{\wp,0}\to E},$$

one can decompose D_0 as $D_0 \xrightarrow{\sim} \prod_{\sigma_0:F_{\wp,0}\to E} D_{\sigma_0}$. Each D_{σ_0} is an E-vector space of rank 2 equipped with an E-linear action of φ^{d_0} and N, moreover, the operator φ (on D_0) induces a bijection: $D_{\sigma_0}\xrightarrow{\sim} D_{\sigma_0\circ\varphi^{-1}}$. It is known that $\mathrm{Ker}(N)$ is a free $F_{\wp,0}\otimes_{\mathbb{Q}_p} E$-module of rank 1, and thus admits a decomposition

$$\mathrm{Ker}(N)\xrightarrow{\sim}\prod_{\sigma_0:F_{\wp,0}\to E}\mathrm{Ker}(N)_{\sigma_0}.$$

Let $e_{0,\sigma_0}\in D_{\sigma_0}$ such that $Ee_{0,\sigma_0} = \mathrm{Ker}(N)_{\sigma_0}$. In fact, one can choose e_{0,σ_0} such that

$$\varphi(e_{0,\sigma_0}) = e_{0,\sigma_0\circ\varphi^{-1}}. \tag{25}$$

Since $\mathrm{Ker}(N)_{\sigma_0}$ is stable by φ^{d_0}, there exists $\alpha\in E^\times$ such that $\varphi^{d_0}(e_{0,\sigma_0}) = \alpha e_{0,\sigma_0}$ (by (25), we see α is independent of σ_0). Since $N\varphi = p\varphi N$, there exists a unique $e_{1,\sigma_0}\in D_{\sigma_0}$ such that $Ne_{1,\sigma_0} = e_{0,\sigma_0}$ and $\varphi^{d_0}(e_{1,\sigma_0}) = q\alpha e_{1,\sigma_0}$ (thus $D_{\sigma_0} = Ee_{0,\sigma_0}\oplus Ee_{1,\sigma_0}$).

Using the isomorphism

$$F_\wp\otimes_{\mathbb{Q}_p} E\xrightarrow{\sim}\prod_{\sigma\in\Sigma_\wp} E,\quad a\otimes b\mapsto (\sigma(a)b)_{\sigma\in\Sigma_\wp},$$

one can decompose D as $D\xrightarrow{\sim}\prod_{\sigma\in\Sigma_\wp} D_\sigma$. One has $D_{\sigma_0}\otimes_{F_{\wp,0}} F_\wp\cong\prod_{\substack{\sigma\in\Sigma_\wp\\ \sigma|_{F_{\wp,0}}=\sigma_0}} D_\sigma$ for any $\sigma_0: F_{\wp,0}\to E$. For $\sigma\in\Sigma_\wp$, $i=0,1$, let $e_{i,\sigma}\in D_\sigma$, such that

$$e_{i,\sigma_0}\otimes 1 = \big(e_{i,\sigma}\big)_{\substack{\sigma\in\Sigma_\wp\\ \sigma|_{F_{\wp,0}}=\sigma_0}}.$$

Since ρ is of Hodge–Tate weights $(-k_{2,\sigma}, -k_{1,\sigma})_{\sigma\in\Sigma_\wp}$, for all $\sigma\in\Sigma_\wp$, there exists $(a_\sigma, b_\sigma)\in E\times E\backslash\{(0,0)\}$ such that

$$\mathrm{Fil}^i D_\sigma = \begin{cases} D_\sigma & i\leqslant k_{1,\sigma},\\ E\big(a_\sigma e_{1,\sigma}+b_\sigma e_{0,\sigma}\big) & k_{1,\sigma}<i\leqslant k_{2,\sigma},\\ 0 & i>k_{2,\sigma}.\end{cases}$$

We suppose ρ satisfies the following hypothesis.

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HYPOTHESIS 1. For all $\sigma \in \Sigma_\wp$, $a_\sigma \neq 0$.

Pose $\mathcal{L}_\sigma := b_\sigma / a_\sigma$, for $\sigma \in \Sigma_\wp$. One sees easily that $\mathcal{L}_\sigma$ is independent of the choice of $e_{0,\sigma}$. An important fact is that one can recover $(D_{\mathrm{st}}(\rho), D_{\mathrm{dR}}(\rho))$ (and hence ρ) by the data:

$$\big\{(-k_{2,\sigma}, -k_{1,\sigma})_{\sigma\in\Sigma_\wp}; \alpha, q\alpha; \{\mathcal{L}_\sigma\}_{\sigma\in\Sigma_\wp}\big\}.$$

Note that by the Hypothesis 1, ρ admits a unique triangulation given by

$$0 \to \mathcal{R}_E\Big(\mathrm{unr}(\alpha) \prod_{\sigma\in\Sigma_\wp} \sigma^{-k_{1,\sigma}}\Big) \to D_{\mathrm{rig}}(\rho) \to \mathcal{R}_E\Big(\mathrm{unr}(q\alpha) \prod_{\sigma\in\Sigma_\wp} \sigma^{-k_{2,\sigma}}\Big) \to 0,$$

in particular, ρ is noncritical.

REMARK 7. This hypothesis is automatically satisfied when $F_\wp = \mathbb{Q}_p$ by the weak admissibility of (D_0, D). Note also that in critical case, one can still define Fontaine–Mazur $\mathcal{L}$-invariants $\mathcal{L}_\sigma$ for the embeddings σ with $a_\sigma \neq 0$; we refer to [21] for results in this case (where a key ingredient is the Colmez–Greenberg–Stevens formula in critical case).

5.2. Breuil's $\mathcal{L}$-invariants. Keep the above notation, following [38] (which generalizes results in [5]), one can associate to ρ a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$. We recall the construction (note that the log maps that we use are slightly different from those in [38]) and introduce some notations. Let $w \in \mathbb{Z}$, $k_\sigma \in \mathbb{Z}_{\geqslant 2}$ for all $\sigma \in \Sigma_\wp$ such that $k_\sigma \equiv w \pmod 2$, suppose ρ is of Hodge–Tate weights $(-(w+k_\sigma)/2, -(w-k_\sigma+2)/2)_{\sigma\in\Sigma_\wp}$. Put

$$\chi(\underline{k}_{\Sigma_\wp}, w; \alpha) := \mathrm{unr}(\alpha) \prod_{\sigma\in\Sigma_\wp} \sigma^{-(w-k_\sigma+2)/2} \otimes \mathrm{unr}(\alpha) \prod_{\sigma\in\Sigma_\wp} \sigma^{-(w+k_\sigma-2)/2},$$

which is a continuous character of $T(F_\wp)$ over E. Consider the parabolic induction

$$\Big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\Big)^{\mathbb{Q}_p-\mathrm{an}}, \tag{26}$$

we have the following facts

- the unique finite dimensional subrepresentation of (26) is $V(\underline{k}_{\Sigma_\wp}, w; \alpha) := (\mathrm{unr}(\alpha) \circ \det) \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee$;
- the maximal locally algebraic subrepresentation of the quotient

$$\Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha) := \Big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\Big)^{\mathbb{Q}_p-\mathrm{an}} / V(\underline{k}_{\Sigma_\wp}, w; \alpha)$$

is $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha) := \mathrm{St} \otimes_E V(\underline{k}_{\Sigma_\wp}, w; \alpha)$, which is also the socle of $\Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha)$ (where St denotes the Steinberg representation).

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Let $\psi(\underline{\mathcal{L}}_{\Sigma_\wp})$ be the following $(d+1)$-dimensional representation of $T(F_\wp)$ over E

$$\psi(\underline{\mathcal{L}}_{\Sigma_\wp})\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} = \begin{pmatrix} 1 & \log_{\sigma_1,-\mathcal{L}_{\sigma_1}}(ad^{-1}) & \log_{\sigma_2,-\mathcal{L}_{\sigma_2}}(ad^{-1}) & \cdots & \log_{\sigma_d,-\mathcal{L}_{\sigma_d}}(ad^{-1})\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}.$$

One gets thus an exact sequence of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F_\wp)$:

$$\begin{aligned} 0 &\longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\right)^{\mathbb{Q}_p-\mathrm{an}}\\ &\longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\otimes_E \psi(\underline{\mathcal{L}}_{\Sigma_\wp})\right)^{\mathbb{Q}_p-\mathrm{an}}\\ &\xrightarrow{s} \left(\left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\right)^{\mathbb{Q}_p-\mathrm{an}}\right)^{\oplus d} \longrightarrow 0.\end{aligned}$$

Following Schraen [38, Section 4.2], put

$$\Sigma\left(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp}\right) := s^{-1}\left(V(\underline{k}_{\Sigma_\wp}, w; \alpha)^{\oplus d}\right)/V(\underline{k}_{\Sigma_\wp}, w; \alpha). \tag{27}$$

REMARK 8.

(1) By [38, Proposition 4.13], $\Sigma(\underline{k}'_{\Sigma_\wp}, w'; \alpha'; \underline{\mathcal{L}}'_{\Sigma_\wp}) \cong \Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp})$ if and only if $\underline{k}'_{\Sigma_\wp} = \underline{k}_{\Sigma_\wp}$, $w' = w$, $\alpha' = \alpha$ and $\underline{\mathcal{L}}'_{\Sigma_\wp} = \underline{\mathcal{L}}_{\Sigma_\wp}$.

(2) For $\sigma \in \Sigma_\wp$, denote by $\psi(\mathcal{L}_\sigma)$ the following 2-dimensional representation of $T(F_\wp)$:

$$\psi(\mathcal{L}_\sigma)\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} = \begin{pmatrix} 1 & \log_{\sigma,-\mathcal{L}_\sigma}(ad^{-1})\\ 0 & 1\end{pmatrix}.$$

One has thus an exact sequence

$$\begin{aligned} 0 &\longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\right)^{\mathbb{Q}_p-\mathrm{an}}\\ &\longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\otimes_E \psi(\mathcal{L}_\sigma)\right)^{\mathbb{Q}_p-\mathrm{an}}\\ &\xrightarrow{s_\sigma} \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\right)^{\mathbb{Q}_p-\mathrm{an}} \longrightarrow 0.\end{aligned}$$

Put

$$\Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\sigma) := s_\sigma^{-1}(V(\underline{k}_{\Sigma_\wp}, w; \alpha))/V(\underline{k}_{\Sigma_\wp}, w; \alpha).$$

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One has an isomorphism of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F_\wp)$:

$$\Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_{\sigma_1}\big) \oplus_{\Sigma(\underline{k}_{\Sigma_\wp}, w;\alpha)} \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_{\sigma_2}\big) \oplus_{\Sigma(\underline{k}_{\Sigma_\wp}, w;\alpha)} \cdots \oplus_{\Sigma(\underline{k}_{\Sigma_\wp}, w;\alpha)} \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_{\sigma_d}\big) \xrightarrow{\sim} \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp}\big). \tag{28}$$

Let χ_i be a locally σ_i-analytic (additive) character of $F_\wp^\times$ in E, replacing the term $\log_{\sigma_i, -\mathcal{L}_{\sigma_i}}(ad^{-1})$ by $\log_{\sigma_i, -\mathcal{L}_{\sigma_i}}(ad^{-1}) + \chi_i \circ \det$, one can construct a representation $\Sigma'(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp})$ exactly the same way as $\Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp})$. By cohomology arguments as in [38, Section 4.3], one can actually prove

LEMMA 5. *One has an isomorphism of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(F_\wp)$:*

$$\Sigma'\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp}\big) \xrightarrow{\sim} \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp}\big). \tag{29}$$

Proof. We use Ext^1 to denote the extensions in the category of admissible locally $\mathbb{Q}_p$-analytic representations. By the same argument as in [38, Section 4.3], replacing $\overline{G}$, $\overline{T}$ by $\mathrm{GL}_2(F_\wp)$, $T(F_\wp)$ respectively, one has (see in particular [38, Lemma 4.8] and the discussion which follows)

$$\mathrm{Ext}^1_{\mathrm{GL}_2(F_\wp)}\Big(V(\underline{k}_{\Sigma_\wp}, w; \alpha), \big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\big)^{\mathbb{Q}_p-\mathrm{an}}\Big) \cong \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{an}}(T(F_\wp), E),$$

which is hence of dimension $2(d+1)$ over E.

On the other hand, one can prove

$$\mathrm{Ext}^1_{\mathrm{GL}_2(F_\wp)}\Big(V(\underline{k}_{\Sigma_\wp}, w; \alpha), V(\underline{k}_{\Sigma_\wp}, w; \alpha)\Big) \cong \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{an}}(F_\wp^\times, E). \tag{30}$$

Indeed, put $V := V(\underline{k}_{\Sigma_\wp}, w; \alpha)$ for simplicity, then by [38, Proposition 3.5], one has

$$\mathrm{Ext}^1_{\mathrm{GL}_2(F_\wp)}(V, V) \cong H^1_{\mathrm{an}}\big(\mathrm{GL}_2(F_\wp), V \otimes_E V^\vee\big);$$

for any finite dimensional algebraic representation W of $\mathrm{Res}_{L/\mathbb{Q}_p} \mathrm{GL}_2$ over E, by [15, Theorem 3], one has

$$H^1_{\mathrm{an}}(\mathrm{GL}_2(F_\wp), W) \cong H^1(\mathfrak{g} \otimes_{\mathbb{Q}_p} E, W).$$

Using Künneth formula (with respect to the decomposition $\mathfrak{g} \cong \mathfrak{s} \times \mathfrak{z}$, where $\mathfrak{s}$ denotes the Lie algebra of $\mathrm{SL}_2(F_\wp)$ and $\mathfrak{z}$ the Lie algebra of the center $Z(F_\wp)$ of $\mathrm{GL}_2(F_\wp)$) and the first Whitehead lemma (cf. [41, Corollary 7.8.10]), one can show $H^1(\mathfrak{g} \otimes_{\mathbb{Q}_p} E, W) = 0$, if W is irreducible nontrivial; and

$$H^1(\mathfrak{g} \otimes_{\mathbb{Q}_p} E, E) \cong H^1(\mathfrak{z} \otimes_{\mathbb{Q}_p} E, E) \cong H^1_{\mathrm{an}}(F_\wp^\times, E) \cong \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{an}}(F_\wp^\times, E).$$

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Since the trivial representation has multiplicity one in $V \otimes_E V^\vee$, one gets the isomorphism in (30).

From the exact sequence

$$0 \to V(\underline{k}_{\Sigma_\wp}, w; \alpha) \to \big(\mathrm{Ind}_{B(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\big)^{\mathbb{Q}_p-\mathrm{an}} \to \Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha) \to 0$$

one gets

$$\begin{aligned} 0 \longrightarrow\ & \mathrm{Ext}^1_{\mathrm{GL}_2(F_\wp)}\big(V(\underline{k}_{\Sigma_\wp}, w; \alpha), V(\underline{k}_{\Sigma_\wp}, w; \alpha)\big) \\ \longrightarrow\ & \mathrm{Ext}^1_{\mathrm{GL}_2(F_\wp)}\Big(V(\underline{k}_{\Sigma_\wp}, w; \alpha), \big(\mathrm{Ind}_{B(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\big)^{\mathbb{Q}_p-\mathrm{an}}\Big) \\ \xrightarrow{j}\ & \mathrm{Ext}^1_{\mathrm{GL}_2(F_\wp)}\big(V(\underline{k}_{\Sigma_\wp}, w; \alpha), \Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha)\big). \end{aligned}$$

So $\dim_E \mathrm{Im}(j) = d + 1$. This, combined with the discussion above [38, Proposition 4.10], shows that the natural injection (cf. [38, Proposition 3.5], where $\mathrm{PGL}_2 := \mathrm{GL}_2 / Z$)

$$\begin{aligned} & \mathrm{Ext}^1_{\mathrm{PGL}_2(F_\wp)}\big(V(\underline{k}_{\Sigma_\wp}, w; \alpha), \Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha)\big) \\ & \quad \hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(F_\wp)}\big(V(\underline{k}_{\Sigma_\wp}, w; \alpha), \Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha)\big) \end{aligned}$$

induces a bijection between $\mathrm{Ext}^1_{\mathrm{PGL}_2(F_\wp)}(V(\underline{k}_{\Sigma_\wp}, w; \alpha), \Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha))$ and $\mathrm{Im}(j)$, from which the isomorphism (29) follows. □

5.3. Local–global compatibility. Let $w \in \mathbb{Z}$, $k_\sigma \in \mathbb{Z}_{\geqslant 2}$, $k_\sigma \equiv w \pmod 2$ for all $\sigma \in \Sigma_\wp$. Let ρ be a 2-dimensional continuous representation of $\mathrm{Gal}(\overline{F}/F)$ over E such that

(1) ρ is absolutely irreducible modulo ϖ_E;

(2) $\rho_\wp := \rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)}$ is semistable noncrystalline of Hodge–Tate weights $(-(w + k_\sigma)/2, -(w - k_\sigma + 2)/2)_{\sigma \in \Sigma_\wp}$ satisfying the Hypothesis 1 with $\{\mathcal{L}_\sigma\}_{\sigma \in \Sigma_\wp}$ the associated Fontaine–Mazur $\mathcal{L}$-invariants and $\{\alpha, q\alpha\}$ the eigenvalues of φ^{d_0} over $D_{\mathrm{st}}(\rho_\wp)$;

(3) $\mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\rho, H^1_{\text{ét}}(K^p, W(\underline{k}_{\Sigma_\wp}, w))\big) \neq 0$.

Denote by λ_ρ the system of eigenvalues of $\mathcal{H}^p$ associated to ρ (via the Eichler–Shimura relations), put

$$\widehat{\Pi}(\rho) := \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\Big(\rho, \widetilde{H}^1_{\text{ét}}(K^p, E)^{\mathcal{H}^p = \lambda_\rho}\Big).$$

Note that one has

$$\widehat{\Pi}(\rho) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\Big(\rho, \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho}}^{\mathcal{H}^p = \lambda_\rho}\Big).$$

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One can deduce from the isomorphism (cf. Theorem 2(2))

$$\widetilde{H}^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big)_{\mathbb{Q}_p-\mathrm{an}} \xrightarrow{\sim} \widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\mathrm{an}} \otimes_E W(\underline{k}_{\Sigma_\wp}, w)$$

a natural injection (cf. Proposition 1 and [22, Proposition 4.2.4])

$$H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}} \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee \hookrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}, \tag{31}$$

thus $\widehat{\Pi}(\rho)$ is nonzero (by the condition (3)). Moreover, by Theorem 2(1), $\widehat{\Pi}(\rho)$ is a unitary admissible Banach representation of $\mathrm{GL}_2(F_\wp)$ over E. In fact, $\widehat{\Pi}(\rho)$ is supposed to be (a finite sum of) the right representation of $\mathrm{GL}_2(F_\wp)$ corresponding to $\rho_\wp$ in the p-adic Langlands program (cf. [8]). By the local–global compatibility in the classical local Langlands correspondence for $\ell = p$, and Proposition 1 (see also [34, Theorem 5.3]), one can show that there exists $r \in \mathbb{Z}_{\geqslant 1}$, such that (cf. Section 5.2)

$$\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha)^{\oplus r} \xrightarrow{\sim} \widehat{\Pi}(\rho)_{\mathrm{lalg}}, \tag{32}$$

where $\widehat{\Pi}(\rho)_{\mathrm{lalg}}$ denotes the locally algebraic vectors of $\widehat{\Pi}(\rho)$.We can now announce the main result of this article.

THEOREM 8. *Keep the above notation and hypothesis, the natural restriction map*

$$\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big(\Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp}\big), \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}\Big) \longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big(\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha), \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}\Big)$$

is bijective. In particular, one has a continuous injection of locally analytic $\mathrm{GL}_2(F_\wp)$*-representations*

$$\Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \underline{\mathcal{L}}_{\Sigma_\wp}\big)^{\oplus r} \hookrightarrow \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}},$$

which induces an isomorphism between the locally algebraic subrepresentations.

Such a result is called local–global compatibility, since the $\Pi(\rho_\wp)$ are constructed by the local parameters (that is, parameters of $\rho_\wp$) while $\widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}$ is a global object. By the isomorphism (28), the Theorem 8 would follow from the following proposition.

PROPOSITION 11. *For any $\tau \in \Sigma_\wp$, the restriction map*

$$\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big(\Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau\big), \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}\Big) \longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big(\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha), \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}\Big)$$

is bijective.

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Before proving this proposition, we give a corollary on the uniqueness of $\mathcal{L}$-invariants (suggested by Breuil):

COROLLARY 6. *Keep the notation in Proposition 11, let $\mathcal{L}'_\tau \in E$, if there exists a continuous injection of* $\mathrm{GL}_2(F_\wp)$*-representations*

$$i : \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}'_\tau\big) \hookrightarrow \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}},$$

then $\mathcal{L}'_\tau = \mathcal{L}_\tau$.

Proof. By Proposition 11, the restriction on i to $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha)$ gives rise to a continuous injection

$$j : \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau\big) \hookrightarrow \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}.$$

Suppose $\mathcal{L}'_\tau \neq \mathcal{L}_\tau$, one can thus deduce from i and j an injection

$$\Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}'_\tau\big) \oplus_{\Sigma(\underline{k}_{\Sigma_\wp}, w;\alpha)} \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau\big) \hookrightarrow \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}. \tag{33}$$

Put for simplicity

$$V := \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}'_\tau\big) \oplus_{\Sigma(\underline{k}_{\Sigma_\wp}, w;\alpha)} \Sigma\big(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau\big),$$

the key point is the locally algebraic subrepresentation V_{lalg} contains an extension of $V(\underline{k}_{\Sigma_\wp}, w; \alpha)$ by $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha)$, which would contradict to (32):

Denote by $\psi(\mathcal{L}'_\tau, \mathcal{L}_\tau)$ the following 3-dimensional representation of $T(F_\wp)$:

$$\psi(\mathcal{L}'_\tau, \mathcal{L}_\tau)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & \log_{\tau,-\mathcal{L}'_\tau}(ad^{-1}) & \log_{\tau,-\mathcal{L}_\tau}(ad^{-1}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

thus one has an exact sequence

$$\begin{aligned} 0 \longrightarrow & \Big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\Big)^{\mathbb{Q}_p-\mathrm{an}} \\ \longrightarrow & \Big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha) \otimes_E \psi(\mathcal{L}'_\tau, \mathcal{L}_\tau)\Big)^{\mathbb{Q}_p-\mathrm{an}} \\ \xrightarrow{s'} & \Big(\Big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\Big)^{\mathbb{Q}_p-\mathrm{an}}\Big)^{\oplus 2} \longrightarrow 0. \end{aligned}$$

It is straightforward to see

$$V \xrightarrow{\sim} (s')^{-1}(V(\underline{k}_{\Sigma_\wp}, w; \alpha)^{\oplus 2})/V(\underline{k}_{\Sigma_\wp}, w; \alpha).$$

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On the other hand, $\psi(\mathcal{L}'_\tau, \mathcal{L}_\tau)$ admits a smooth subrepresentation

$$\psi_0(\mathcal{L}'_\tau, \mathcal{L}_\tau)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & (\mathcal{L}'_\tau - \mathcal{L}_\tau)\upsilon_\wp(ad^{-1}) \\ 0 & 1 \end{pmatrix},$$

one has thus an exact sequence of smooth representations of $\mathrm{GL}_2(F_\wp)$

$$0 \longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi_\alpha\right)^\infty \longrightarrow \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi_\alpha \otimes_E \psi_0(\mathcal{L}'_\tau, \mathcal{L}_\tau)\right)^\infty \xrightarrow{s''} \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi_\alpha\right)^\infty \longrightarrow 0,$$

where $\chi_\alpha := \mathrm{unr}(\alpha)\otimes\mathrm{unr}(\alpha)$. Note that $\mathrm{unr}(\alpha)\circ\det$ is the socle of $(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi_\alpha)^\infty$, and one can check

$$V' := ((s'')^{-1}(\mathrm{unr}(\alpha)\circ\det)/\,\mathrm{unr}(\alpha)\circ\det) \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee$$

is a locally algebraic subrepresentation of V, which is an extension (nonsplit) of $V(\underline{k}_{\Sigma_\wp}, w; \alpha)$ by $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha)$. We deduce from (33) an injection $V' \hookrightarrow \widehat{\Pi}(\rho)_{\mathrm{lalg}}$, a contradiction with (32). □

REMARK 9. By Theorem 8 and Corollary 6, we see that the local Galois representation $\rho_\wp$ can be determined by $\widehat{\Pi}(\rho)$.

The following lemma has a straightforward proof that is omitted.

LEMMA 6. *Let V be an admissible locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(F_\wp)$ over E, there exists a natural bijection*

$$\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(V, \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\left(\rho, \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}(V, \widetilde{H}^1_{\acute{e}t}(K^p, E)^{\mathcal{H}^p=\lambda_\rho}_{\mathbb{Q}_p-\mathrm{an}})\right).$$

The Proposition 11 thus follows from

PROPOSITION 12. *With the notation in Proposition 11, the restriction map*

$$\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(\Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau), \widetilde{H}^1_{\acute{e}t}(K^p, E)^{\mathcal{H}^p=\lambda_\rho}_{\mathbb{Q}_p-\mathrm{an}}\right) \longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(\mathrm{St}\left(\underline{k}_{\Sigma_\wp}, w; \alpha\right), \widetilde{H}^1_{\acute{e}t}(K^p, E)^{\mathcal{H}^p=\lambda_\rho}_{\mathbb{Q}_p-\mathrm{an}}\right) \quad (34)$$

is bijective.

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The rest of this paper is devoted to the proof of Proposition 12. Given an injection (whose existence follows from (32)) $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha) \hookrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)^{\mathcal{H}^p=\lambda_\rho}_{\mathbb{Q}_p-\text{an}}$, by applying the Jacquet–Emerton functor and Theorem 3, one gets a closed E-point (associate to ρ) in $\mathcal{V}(K^p, w)_{\overline{\rho}}$ given by

$$z := \big(\chi := \chi(\underline{k}_{\Sigma_\wp}, w; \alpha)\delta, \lambda_\rho\big).$$

LEMMA 7. *The restriction map (34) is injective.*

Proof. The proof is the same as in [20, Proposition 6.3.9]. Let f be in the kernel of (34), suppose $f \neq 0$, thus f would induce an injection

$$V_\wp \longleftrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)^{\mathcal{H}^p=\lambda_\rho}_{\mathbb{Q}_p-\text{an}}$$

with $V_\wp$ an irreducible constituent of $\Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau)$ different from $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha)$ (since f lies in the kernel of (34)) and from $V(\underline{k}_{\Sigma_\wp}, w; \alpha)$ (by (32)), from which, by applying the Jacquet–Emerton functor, one would get a companion point of z, which contradicts to the fact that z is noncritical (thus does not admit companion points, cf. Corollary 4). □

In the following, we prove the surjectivity of (34), which is the key of this paper. By assumption, we know the point z is noncritical, thus one may find an open neighborhood $\mathcal{U}$ of z in $\mathcal{V}(K^p, w)_{\overline{\rho}}$ such that (cf. Corollary 5 and [20, Lemma 6.3.12])

(1) $\mathcal{U}$ is strictly quasi-Stein [22, Definition 2.1.17(iv)];

(2) for any $z' \in \mathcal{U}(\overline{E})$, z' does not have companion points.

Denote by $\mathcal{M} := \mathcal{M}(K^p, w)_{\overline{\rho}}$ for simplicity, the natural restriction (with dense image since $\mathcal{U}$ is strictly quasi-Stein)

$$J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho},\mathbb{Q}_p-\text{an}}\Big)^\vee_b \cong \mathcal{M}(\mathcal{V}(K^p, w)_{\overline{\rho}}) \longrightarrow \mathcal{M}(\mathcal{U})$$

induces a continuous injection of locally $\mathbb{Q}_p$-analytic representations of T (invariant under $\mathcal{H}^p$)

$$\mathcal{M}(\mathcal{U})^\vee_b \longleftrightarrow J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho},\mathbb{Q}_p-\text{an}}\Big) \tag{35}$$

where $\mathcal{M}(\mathcal{U})^\vee_b$ denotes the strict dual of $\mathcal{M}(\mathcal{U})$. As in Appendix A, one can show (see [23, Lemma 4.5.12] and the proof of [23, Theorem 4.5.7] for $\mathrm{GL}_2(\mathbb{Q}_p)$-case)

- $\mathcal{M}(\mathcal{U})^\vee_b$ is an allowable subrepresentation of $J_B(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\mathbb{Q}_p-\text{an}})$ (cf. Lemma 14 below);

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- the map (35) is balanced (cf. Lemma 16 below);
- $\mathcal{M}(\mathcal{U})$ is a torsion free $\mathcal{O}(\mathcal{W}_{1,\Sigma_\wp})$-module (cf. Section 4.3).

From which, one can deduce that the map (35) induces a continuous $\mathrm{GL}_2(F_\wp)\times\mathcal{H}^p$-invariant map (cf. Corollary 7 below, see [23, (4.5.9)] for $\mathrm{GL}_2(\mathbb{Q}_p)$-case)

$$\left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}\mathcal{M}(\mathcal{U})_b^\vee[\delta^{-1}]\right)^{\mathbb{Q}_p-\mathrm{an}}\longrightarrow \widetilde{H}^1_{\text{ét}}(K^p,E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w}}, \tag{36}$$

where $\mathcal{M}(\mathcal{U})_b^\vee[\delta^{-1}]$ denotes the twist of $\mathcal{M}(\mathcal{U})_b^\vee$ by δ^{-1}. We would deduce Proposition 12 from this map.

For $\sigma\in\Sigma_\wp$, denote by $\widehat{T}_\sigma$ (respectively $\mathcal{W}_{1,\sigma}$) the rigid space over E parameterizing locally σ-analytic characters of $T(F_\wp)$ (respectively $1+2\varpi\mathcal{O}_\wp$), which is hence a closed subspace of $\widehat{T}_{\Sigma_\wp}$ (respectively $\mathcal{W}_{1,\Sigma_\wp}$) (for example, see [20, Section 5.1.4]).

The character χ induces a closed embedding

$$\chi:\mathcal{W}_{1,\tau}\hookrightarrow\mathcal{W}_{1,\Sigma_\wp},\quad \chi'\mapsto\chi|_{Z_1'}\chi'.$$

Recall that we have a natural morphism $\kappa:\mathcal{V}(K^p,w)_{\overline{\rho}}\to\mathcal{W}_{1,\Sigma_\wp}$ which is étale at z (cf. Theorem 7). Put $\mathcal{V}(K^p,w)_{\overline{\rho},\tau}:=\mathcal{V}(K^p,w)_{\overline{\rho}}\times_{\mathcal{W}_{1,\Sigma_\wp},\chi}\mathcal{W}_{1,\tau}$, one has thus a Cartesian diagram

$$\begin{array}{ccc}\mathcal{V}(K^p,w)_{\overline{\rho},\tau} & \longrightarrow & \mathcal{W}_{1,\tau}\\ \downarrow & & \downarrow{\scriptstyle\chi}\\ \mathcal{V}(K^p,w)_{\overline{\rho}} & \longrightarrow & \mathcal{W}_{1,\Sigma_\wp}\end{array}$$

Denote still by z the preimage of z in $\mathcal{V}(K^p,w)_{\overline{\rho},\tau}$, κ the natural morphism $\mathcal{V}(K^p,w)_{\overline{\rho},\tau}\to\mathcal{W}_{1,\tau}$, thus κ is étale at z. By results in Section 4.4.3, one can choose an open affinoid V of $\mathcal{V}(K^p,w)$ containing z such that V is étale over $\mathcal{W}_{1,\Sigma_\wp}$, and that any point in V does not have companion points. Denote by V_τ the preimage of V in $\mathcal{V}(K^p,w)_{\overline{\rho},\tau}$. We see V_τ is in fact a smooth curve. By Proposition 8 and shrinking V (and hence V_τ) if necessary, one gets a continuous representation

$$\rho_{V_\tau}:\mathrm{Gal}(\overline{F}/F)\to\mathrm{GL}_2(\mathcal{O}(V))\to\mathrm{GL}_2(\mathcal{O}(V_\tau)).$$

Denote by $\chi_{V_\tau}=\chi_{V_\tau,1}\otimes\chi_{V_\tau,2}:T(F_\wp)\to\mathcal{O}(V_\tau)^\times$ the character induced by the natural morphism $V_\tau\to\mathcal{V}(K^p,w)_{\overline{\rho},\tau}\to\mathcal{V}(K^p,w)_{\overline{\rho}}\to\widehat{T}_{\Sigma_\wp}$. By [30, Theorem 6.3.9] applied to the smooth affinoid curve V_τ, together with the fact that any point in V_τ does not have companion points, one has

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LEMMA 8. *There exists an exact sequence of (φ, Γ)-modules over $\mathcal{R}_{\mathcal{O}(V_\tau)} := B^{\dagger}_{\mathrm{rig}, F_\wp} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{O}(V_\tau)$ (see for example* [30, Theorem 2.2.17] *for $D_{\mathrm{rig}}(\rho_{V_\tau,\wp})$, and* [30, Const.6.2.4] *for (φ, Γ)-modules of rank 1 associated to continuous characters of $F_\wp^\times$ with values in $\mathcal{O}(V_\tau)^\times$):*

$$0 \to \mathcal{R}_{\mathcal{O}(V_\tau)}(\mathrm{unr}(q)\chi_{V_\tau,1}) \to D_{\mathrm{rig}}(\rho_{V_\tau,\wp}) \to \mathcal{R}_{\mathcal{O}(V_\tau)}\Big(\chi_{V_\tau,2} \prod_{\sigma \in \Sigma_\wp} \sigma^{-1}\Big) \to 0. \quad (37)$$

Let $t_\tau : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \to \mathcal{W}_{1,\tau}$ be a *nonzero* element in the tangent space of $\mathcal{W}_{1,\tau}$ at the identity point (corresponding to the trivial character), since V_τ is étale over $\mathcal{W}_{1,\tau}$, t_τ gives rise to a nonzero element, still denoted by t_τ, in the tangent space of $\mathcal{V}(K^p, w)_{\overline{\rho},\tau}$ at the point z. The following composition

$$t_\tau : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \longrightarrow \mathcal{V}(K^p, w)_{\overline{\rho},\tau} \longrightarrow \mathcal{V}(K^p, w)_{\overline{\rho}} \longrightarrow \widehat{T}_{\Sigma_\wp} \quad (38)$$

gives rise to a character $\widetilde{\chi}_\tau : T(F_\wp) \to (E[\epsilon]/\epsilon^2)^\times$. We have in fact $\widetilde{\chi}_\tau = t_\tau \circ \chi_{V_\tau}$ ($t_\tau : \mathcal{O}(V_\tau) \to E[\epsilon]/\epsilon^2$). We know $\widetilde{\chi}_\tau \equiv \chi \pmod{\epsilon}$. Since the image of (38) lies in $\widehat{T}_{\Sigma_\wp}(w)$ (cf. (12)), we see $\widetilde{\chi}_\tau|_{Z_1} = \mathcal{N}^{-w}$ and thus $(\widetilde{\chi}_\tau\chi^{-1})|_{Z_1} = 1$.

LEMMA 9. *There exist $\gamma, \eta \in E$, $\mu \in E^\times$ such that*

$$\psi_\tau := \widetilde{\chi}_\tau \chi^{-1} = \mathrm{unr}(1+\gamma\epsilon)(1-\mu\epsilon \log_{\tau,0,\varpi}) \otimes \mathrm{unr}(1+\eta\epsilon)(1+\mu\epsilon\log_{\tau,0,\varpi}).$$

Proof. The lemma is straightforward. Note that $\mu \neq 0$ since t_τ (as an element in the tangent space) is nonzero. □

By multiplying ϵ by constants, we assume $\mu = 1$ and thus

$$\psi_\tau = \mathrm{unr}(1+\gamma\epsilon)(1-\epsilon\log_{\tau,0,\varpi}) \otimes \mathrm{unr}(1+\eta\epsilon)(1+\epsilon\log_{\tau,0,\varpi}).$$

The following lemma, which describes the character $\widetilde{\chi}_\tau$ in terms of the $\mathcal{L}$-invariants, is one of the key points in the proof of Proposition 12.

LEMMA 10. $(\eta-\gamma)/2 = e^{-1}(-\mathcal{L}_\tau - \log_\tau(p/\varpi^e)) = (-\mathcal{L}_\tau)(\varpi)$ *(cf. Section 2).*

Proof. Denote by

$$\widetilde{\rho}_{z,\wp} := t_\tau \circ \rho_{V_\tau,\wp} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp) \to \mathrm{GL}_2(\mathcal{O}(V_\tau)) \to \mathrm{GL}_2(E[\epsilon]/\epsilon),$$

from (37), one gets an exact sequence of (φ, Γ)-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$:

$$0 \to \mathcal{R}_{E[\epsilon]/\epsilon^2}\big(\mathrm{unr}(q)\widetilde{\chi}_{\tau,1}\big) \to D_{\mathrm{rig}}(\widetilde{\rho}_{z,\wp}) \to \mathcal{R}_{E[\epsilon]/\epsilon^2}\Big(\widetilde{\chi}_{\tau,2} \prod_{\sigma\in\Sigma_\wp} \sigma^{-1}\Big) \to 0. \quad (39)$$

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For $\sigma \in \Sigma_\wp$, denote by $\varepsilon_{\sigma,\varpi}$ the character of $F_\wp^\times$ with $\varepsilon_{\sigma,\varpi}|_{\mathcal{O}_\wp^\times} = \sigma|_{\mathcal{O}_\wp^\times}$ and $\varepsilon_{\sigma,\varpi}(\varpi) = 1$. Recall

$$\chi = \mathrm{unr}(q^{-1}\alpha) \prod_{\sigma\in\Sigma_\wp} \sigma^{-(w-k_\sigma+2)/2} \otimes \mathrm{unr}(q\alpha) \prod_{\sigma\in\Sigma_\wp} \sigma^{-(w+k_\sigma-2)/2}.$$

We have thus

$$\begin{aligned}
\delta_1 &:= \mathrm{unr}(q)\widetilde{\chi}_{\tau,1} = \mathrm{unr}\big(\alpha(1+\gamma\epsilon)\big)(1-\epsilon\log_{\tau,0,\varpi}) \prod_{\sigma\in\Sigma_\wp} \sigma^{-(w-k_\sigma+2)/2} \\
&= \mathrm{unr}\left(\alpha(1+\gamma\epsilon)\prod_{\sigma\in\Sigma_\wp}\sigma(\varpi)^{-w-k_\sigma+2/2}\right)(1-\epsilon\log_{\tau,0,\varpi}) \prod_{\sigma\in\Sigma_\wp} \varepsilon_{\sigma,\varpi}^{-(w-k_\sigma+2)/2}, \\
\delta_2 &:= \widetilde{\chi}_{\tau,2}\prod_{\sigma\in\Sigma_\wp}\sigma^{-1} = \mathrm{unr}\left(q\alpha(1+\eta\epsilon)\prod_{\sigma\in\Sigma_\wp}\sigma(\varpi)^{-(w+k_\sigma-1)/2}\right) \\
&\quad \times (1+\epsilon\log_{\tau,0,\varpi})\prod_{\sigma\in\Sigma_\wp}\varepsilon_{\sigma,\varpi}^{-(w+k_\sigma)/2}.
\end{aligned}$$

Let $\chi_0 := (1-\epsilon\log_{\tau,0,\varpi})\prod_{\sigma\in\Sigma_\wp}\varepsilon_{\sigma,\varpi}^{-(w-k_\sigma+2)/2}$, one can view χ_0 as a character of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ over $E[\epsilon]/[\epsilon^2]$ via the local Artin map $\mathrm{Art}_{F_\wp}$. Denote by $\widetilde{\rho} := \widetilde{\rho}_{z,\wp} \otimes_{E[\epsilon]/\epsilon^2} \chi_0^{-1}$,

$$\begin{aligned}
\delta_1' &:= \delta_1\chi_0^{-1} = \mathrm{unr}\left(\alpha(1+\gamma\epsilon)\prod_{\sigma\in\Sigma_\wp}\sigma(\varpi)^{-(w-k_\sigma+2)/2}\right) \\
\delta_2' &:= \delta_2\chi_0^{-1} = \mathrm{unr}\left(q\alpha(1+\eta\epsilon)\prod_{\sigma\in\Sigma_\wp}\sigma(\varpi)^{-(w+k_\sigma-1)/2}\right) \\
&\quad \times (1+2\epsilon\log_{\tau,0,\varpi})\prod_{\sigma\in\Sigma_\wp}\varepsilon_{\sigma,\varpi}^{1-k_\sigma}.
\end{aligned}$$

Thus one has

$$0 \to \mathcal{R}_{E[\epsilon]/\epsilon^2}(\delta_1') \to D_{\mathrm{rig}}(\widetilde{\rho}) \to \mathcal{R}_{E[\epsilon]/\epsilon^2}(\delta_2') \to 0. \tag{40}$$

Denote by $\widetilde{\alpha} := \alpha(1+\gamma\epsilon)\prod_{\sigma\in\Sigma_\wp}\sigma(\varpi)^{-(w-k_\sigma+2)/2}$; by [3, Theorem 0.2], we deduce from (40) that $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \widetilde{\rho})^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp),\varphi^{d_0}=\widetilde{\alpha}}$ is free of rank 1 over $F_{\wp,0} \otimes_{\mathbb{Q}_p} E[\epsilon]/\epsilon^2$; note also that the E-representation $\overline{\widetilde{\rho}}$ (mod ϵ) of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ is semistable noncrystalline of Hodge–Tate weights $(1-k_\sigma, 0)_{\sigma\in\Sigma_\wp}$, and has the same $\mathcal{L}$-invariants as $\rho_\wp$. By applying the formula in [42, Theorem 1.1], one gets

$$\frac{\gamma}{d_0} + \left(-\frac{\gamma+\eta}{2d_0}\right) - \frac{1}{d}\log_\tau\left(\frac{p}{\varpi^e}\right) - \frac{1}{d}\mathcal{L}_\tau = 0.$$

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In fact, with the notation of [42], one has

$$\log_{\tau,0,\varpi}=-\frac{1}{d}\log_{\tau}\left(\frac{p}{\varpi^{e}}\right)\psi_1+1_{\tau}\psi_2$$

where $1_\tau \in F_\wp \otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma\in\Sigma_\wp} E$ such that $(1_\tau)_\sigma=0$ if $\sigma\neq\tau$ and $(1_\tau)_\tau=1$, and where we view $\log_{\tau,0,\varpi}$ as an additive character of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\wp)$ via the local Artin map $\mathrm{Art}_{F_\wp}$. Thus one can apply the formula in [42, Theorem 1.1] to

$$\{V,\alpha,\delta,\kappa\}=\left\{\widetilde{\rho},\widetilde{\alpha},\left(-\frac{\gamma+\eta}{d_0}-\frac{2}{d}\log_{\tau}\left(\frac{p}{\varpi^{e}}\right)\right)\epsilon,2_{\tau}\epsilon\right\}.$$

The lemma follows. □

The following lemma follows directly from Lemma 10.

LEMMA 11. *As representations of $T(F_\wp)$ (of dimension 2) over E, one has*

$$\widetilde{\chi}_\tau\delta^{-1}\cong\chi(\underline{k}_{\Sigma_\wp},w;\alpha)\otimes_E\psi(\mathcal{L}_\tau)',$$

where $\psi(\mathcal{L}_\tau)'\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}=\begin{pmatrix} 1 & \log_{\tau,-\mathcal{L}_\tau}(ad^{-1})+(\gamma+\eta)v_\wp(ad)/2\\ 0 & 1\end{pmatrix}$.

The parabolic induction $(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}\widetilde{\chi}_\tau\delta^{-1})^{\mathbb{Q}_p-\mathrm{an}}$ lies thus in an exact sequence as follows

$$\begin{aligned}0\longrightarrow\left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}\chi(\underline{k}_{\Sigma_\wp},w;\alpha)\right)^{\mathbb{Q}_p-\mathrm{an}}&\longrightarrow\left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}\widetilde{\chi}_\tau\delta^{-1}\right)^{\mathbb{Q}_p-\mathrm{an}}\\ \xrightarrow{s'_\tau}\left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}\chi(\underline{k}_{\Sigma_\wp},w;\alpha)\right)^{\mathbb{Q}_p-\mathrm{an}}&\longrightarrow 0.\end{aligned}\tag{41}$$

By Lemma 5, one has

LEMMA 12. *One has an isomorphism of locally $\mathbb{Q}_p$-analytic representations of* $\mathrm{GL}_2(F_\wp)$*:*

$$(s'_\tau)^{-1}\left(V\left(\underline{k}_{\Sigma_\wp},w;\alpha\right)\right)/V\left(\underline{k}_{\Sigma_\wp},w;\alpha\right)\cong\Sigma\left(\underline{k}_{\Sigma_\wp},w;\alpha;\mathcal{L}_\tau\right).$$

Consider the composition $t_\tau:\mathrm{Spec}\,E[\epsilon]/\epsilon^2\xrightarrow{t_\tau}\mathcal{V}(K^p,w)_{\overline{\rho},\tau}\hookrightarrow\mathcal{V}(K^p,w)_{\overline{\rho}}$, and $(t_\tau^*\mathcal{M})^\vee$ being a finite dimensional E-vector space equipped with a natural action of $T(F_\wp)\times\mathcal{H}^p$. We claim there exists $n\in\mathbb{Z}_{\geqslant 1}$ such that (as $T(F_\wp)$)-representations)

$$(t_\tau^*\mathcal{M})^\vee\cong\widetilde{\chi}_\tau^{\oplus n}.\tag{42}$$

In fact, as in Section 4.4.3, there exist open affinoids V' of $\mathcal{V}(K^p, w)_{\overline{\rho}}$ and U of $\mathcal{W}_{1,\Sigma_\wp}$ such that V' lies over U, $\mathcal{O}(V') \cong \mathcal{O}(U)$, and that $\mathcal{M}(V')$ is a locally free $\mathcal{O}(U)$-module. The group $T(F_\wp)$ acts on $\mathcal{M}(V')$ via the character $T(F_\wp) \to \mathcal{O}(V')^\times \cong \mathcal{O}(U)^\times$ (with the first map induced by the natural morphism $V' \to \widehat{T}_{\Sigma_\wp}$), the claim follows. We also see that $\mathcal{H}^p$ acts on $\mathcal{M}(V')$ via the natural morphism $\mathcal{H}^p \to \mathcal{O}(V') \cong \mathcal{O}(U)$. Thus $\mathcal{H}^p$ acts on $(t_\tau^*\mathcal{M})^\vee$ via $\mathcal{H}^p \to \mathcal{O}(V') \to E[\epsilon]/\epsilon^2$, in particular $(t_\tau^*\mathcal{M})^\vee$ is a generalized λ_ρ-eigenspace for $\mathcal{H}^p$ (one can view t_τ as a thickening of the point z).

Since $\mathcal{U}$ is strictly quasi-Stein, the restriction map $\mathcal{M}(\mathcal{U}) \to t_\tau^*\mathcal{M}$ is surjective, so we have injections

$$(z^*\mathcal{M})^\vee \hookrightarrow (t_\tau^*\mathcal{M})^\vee \hookrightarrow \mathcal{M}(\mathcal{U})_b^\vee.$$

Firstly note that a nonzero map f in the right term of (34) corresponds to a nonzero vector $v \in (z^*\mathcal{M})^\vee$ in a natural way:

$$\begin{aligned}(z^*\mathcal{M})^\vee &\xrightarrow{\sim} J_B\big(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\mathbb{Q}_p-\text{an}}\big)^{T(F_\wp)=\chi,\mathcal{H}^p=\lambda_\rho} \\ &\xrightarrow{\sim} J_B\big(H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big) \otimes_E W(\underline{k}_{\Sigma_\wp}, w)^\vee\big)^{T(F_\wp)=\chi,\mathcal{H}^p=\lambda_\rho} \\ &\xrightarrow{\sim} J_B\big(H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big)\big)^{T(F_\wp)=\psi,\mathcal{H}^p=\lambda_\rho} \otimes_E \chi(\underline{k}_{\Sigma_\wp}, w) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big(\big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \psi\delta^{-1}\big)^\infty, H^1_{\text{ét}}\big(K^p, W(\underline{k}_{\Sigma_\wp}, w)\big)^{\mathcal{H}^p=\lambda_\rho}\Big) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\Big(\mathrm{St}\big(\underline{k}_{\Sigma_\wp}, w; \alpha\big), \widetilde{H}^1_{\text{ét}}(K^p, E)^{\mathcal{H}^p=\lambda_\rho}_{\mathbb{Q}_p-\text{an}}\Big),\end{aligned} \tag{43}$$

where the first isomorphism follows from Theorem 3, the second from the fact that any vector in the second term is classical (see also Corollary 3),

$$\psi := \chi\chi(\underline{k}_{\Sigma_\wp}, w)^{-1}$$

(cf. Corollary 3), the fourth from the adjunction formula for the classical Jacquet functor, and the last isomorphism follows from (31) and (32) (and [20, Corollary 5.1.6]).

By the isomorphism (42), there exists $\widetilde{v} \in (t_\tau^*\mathcal{M})^\vee$ such that $(E[\epsilon]/\epsilon^2) \cdot \widetilde{v} \cong \widetilde{\chi}$ and that $v \in (E[\epsilon]/\epsilon^2) \cdot \widetilde{v}$. By multiplying $\widetilde{t}$ by scalars in E, one can assume $v = \epsilon\widetilde{v}$. The $T(F_\wp)$-invariant map, by mapping a basis to $\widetilde{v}$,

$$\widetilde{\chi} \hookrightarrow \mathcal{M}(\mathcal{U})_b^\vee[\mathcal{H}^p = \lambda_\rho]$$

induces a $\mathrm{GL}_2(F_\wp)$-invariant map denoted by $\widetilde{v}$

$$\begin{aligned}\widetilde{v} : \big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \widetilde{\chi}_\tau\delta^{-1}\big)^{\mathbb{Q}_p-\text{an}} &\hookrightarrow \big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \mathcal{M}(\mathcal{U})_b^\vee[\delta^{-1}]\big)^{\mathbb{Q}_p-\text{an}}[\mathcal{H}^p = \lambda_\rho] \\ &\xrightarrow{(36)} \widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho},\mathbb{Q}_p-\text{an}}[\mathcal{H}^p = \lambda_\rho].\end{aligned} \tag{44}$$

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Similarly, the $T(F_\wp)$-invariant map $\chi \hookrightarrow (\mathcal{M}(\mathcal{U})_b^\vee)^{\mathcal{H}^p=\lambda_\rho}$, by mapping a basis to v, induces a $\mathrm{GL}_2(F_\wp)$-invariant map, denoted by v

$$v : \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi\delta^{-1}\right)^{\mathbb{Q}_p-\mathrm{an}} \longrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w},\mathcal{H}^p=\lambda_\rho}.$$

It is straightforward to see that the following diagram commutes

$$\begin{array}{ccc} \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \chi\delta^{-1}\right)^{\mathbb{Q}_p-\mathrm{an}} & \xrightarrow{v} & \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w},\mathcal{H}^p=\lambda_\rho} \\ \downarrow & & \downarrow \\ \left(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \widetilde{\chi}_\tau\delta^{-1}\right)^{\mathbb{Q}_p-\mathrm{an}} & \xrightarrow{\widetilde{v}} & \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w}}[\mathcal{H}^p = \lambda_\rho] \end{array} \tag{45}$$

where the left arrow is induced by $\chi \hookrightarrow \widetilde{\chi}$, and the right arrow is the natural injection.

By the same argument as in the proof of Lemma 7, one can prove the map v factors through an injection

$$v : \Sigma\left(\underline{k}_{\Sigma_\wp}, w; \alpha\right) \longrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w},\mathcal{H}^p=\lambda_\rho}.$$

Moreover, we claim the restriction $f_v := v|_{\mathrm{St}(\underline{k}_{\Sigma_\wp}, w;\alpha)}$ is equal to f. In fact, by taking Jacquet–Emerton functor, one sees both the maps f and f_v give rise to the same eigenvector

$$v \in J_B\left(\widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w}}\right)^{T(F_\wp)=\chi,\mathcal{H}^p=\lambda_\rho} \cong (z^*\mathcal{M})^\vee,$$

from which the claim follows.

By the commutative diagram (45) and Lemma 12, we see $\widetilde{v}$ induces a continuous $\mathrm{GL}_2(F_\wp)$-invariant injection

$$\Sigma\left(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau\right) \longrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w}}[\mathcal{H}^p = \lambda_\rho], \tag{46}$$

whose restriction to $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha)$ equals to f (by the above discussion and commutative diagram (45)). It is sufficient to prove the map (46) factors through $\widetilde{H}^1_{\text{ét}}(K^p, E)_{\overline{\rho},\mathbb{Q}_p-\mathrm{an}}^{Z_1=\mathcal{N}^{-w},\mathcal{H}^p=\lambda_\rho}$.

By the same argument as in the proof of Lemma 7, one can prove the following restriction map is injective:

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(\Sigma\left(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau\right), \widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\mathrm{an}}[\mathcal{H}^p = \lambda_\rho]\right) \\ &\quad\longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left(\mathrm{St}\left(\underline{k}_{\Sigma_\wp}, w; \alpha\right), \widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\mathrm{an}}[\mathcal{H}^p = \lambda_\rho]\right). \end{aligned}$$

For any $X \in \mathcal{H}^p$, we know the restriction of the map $(X - \lambda_\rho(X))\widetilde{v}$ to $\mathrm{St}(\underline{k}_{\Sigma_\wp}, w; \alpha)$ is zero (since the image of f lies in the λ_ρ-eigenspace), hence $(X -$

$\lambda_\rho(X))\widetilde{v} = 0$ (here $X\widetilde{v}$ signifies the composition $\Sigma(\underline{k}_{\Sigma_\wp}, w; \alpha; \mathcal{L}_\tau) \xrightarrow{(46)} \widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}}[\mathcal{H}^p = \lambda_\rho] \xrightarrow{X} \widetilde{H}^1_{\text{ét}}(K^p, E)_{\mathbb{Q}_p-\text{an}}[\mathcal{H}^p = \lambda_\rho]$), in other words, $\mathrm{Im}(\widetilde{v}) \in \widetilde{H}^1_{\text{ét}}(K^p, E)^{\mathcal{H}^p=\lambda_\rho}_{\mathbb{Q}_p-\text{an}}$. This concludes the proof of Proposition 12.

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Appendix A. Some locally analytic representation theory

As suggested by one of the referees, in this appendix, we recall some locally analytic representation theory and explain how to deduce the map (36) from Emerton's adjunction formula [**25**] (the results in this section were contained in [**20**]).

Let $\mathcal{U}$ be a strictly quasi-Stein rigid analytic space over E (cf. [**22**, Definition 2.1.17(iv)]), $A := \mathcal{O}(\mathcal{U})$ is thus a (commutative) nuclear Fréchet–Stein E-algebra (cf. [**37**, Section 3]); let $\mathcal{M}$ be a coherent sheaf over $\mathcal{U}$, $M := \mathcal{M}(\mathcal{U})$ is thus a coadmissible A-module (cf. [**37**, Section 3], [**22**, Definition 1.2.8]). The strong dual $U := M_b^\vee$ is then an E-vector space of compact type (cf. [**36**, Section 1]), equipped with a continuous A-action: $(a \cdot U)(m) := u(am)$ for $a \in A$, $u \in U$ and $m \in M$ (that is, $A \times U \to U$ is separately continuous, for example, see [**22**, 1.2.14]). Note $U_b^\vee \cong M$. Let $(U \otimes_E \overline{E})^g$ be the $\overline{E}$-vector subspace of $U \otimes_E \overline{E}$ generated by the generalized eigenvectors for A, and $U^g := (U \otimes_E \overline{E})^g \cap U$.

LEMMA 13. *U^g is dense inside U.*

Proof. For a closed maximal ideal $\mathfrak{m}$ of A, $n \in \mathbb{Z}_{\geqslant 1}$, the natural projection $M \to M/\mathfrak{m}^n$ induces an injection $(M/\mathfrak{m}^n)^\vee \hookrightarrow M_b^\vee$ (note $M/\mathfrak{m}^n$ is finite dimensional over E). Then $U_0^g := \sum_{\mathfrak{m}} \varinjlim_n (M/\mathfrak{m}^n)^\vee \subseteq U$ is contained in U^g, and it is sufficient to prove the closure $\overline{U_0^g}$ equals U. First note by Nakayama's lemma, if $U_0^g = 0$ then $U = 0$. Let $W := U/\overline{U_0^g}$, and the strong dual of the exact sequence

$$0 \to \overline{U_0^g} \to U \to W \to 0$$

gives an exact sequence of coadmissible A-modules (corresponding to coherent sheaves over $\mathcal{U}$):

$$0 \to W_b^\vee \to M \to (\overline{U_0^g})_b^\vee \to 0.$$

We define W_0^g in the same way as U_0^g replacing M by $M' := W_b^\vee$ (and let $\mathcal{M}'$ be the corresponding coherent sheaf over $\mathcal{U}$), and as mentioned above, $W_0^g = 0$ would imply $W = 0$. To prove $W_0^g = 0$, it is sufficient to prove the induced map $U_0^g \to W_0^g$ is surjective. Thus it is sufficient to prove the

$$\varinjlim_n (M/\mathfrak{m}^n)^\vee \longrightarrow \varinjlim_n (M'/\mathfrak{m}^n)^\vee \tag{A.1}$$

is surjective for any closed maximal ideal $\mathfrak{m}$ of A (which corresponds to a closed point z in $\mathcal{U}$). Let $\mathcal{Y} = \operatorname{Spm} B$ be an admissible affinoid neighborhood of the point z in $\mathcal{U}$, $N := \mathcal{M}(\mathcal{Y}) \cong M\widehat{\otimes}_A B$, $N' := \mathcal{M}'(\mathcal{Y}) \cong M'\widehat{\otimes}_A B$ (for example, see [37, Corollary 3.1]), and denote by $\mathfrak{m}_B$ the corresponding maximal ideal of B. Then N and N' are finitely generated B-modules, and we have $M/\mathfrak{m}^n \cong N/\mathfrak{m}_B^n$, $M'/\mathfrak{m}^n \cong N'/\mathfrak{m}_B^n$ for $n \in \mathbb{Z}_{\geqslant 1}$ (see [37, Section 3] for properties of coadmissible modules over Fréchet–Stein algebras, note also that it can be deduced from [4, Proposition 7.2.2/1 (ii)] that $A/\mathfrak{m}^n \cong B/\mathfrak{m}_B^n$). By Artin–Rees lemma, there exists $r(n)$ such that $(\mathfrak{m}_B^{r(n)}N) \cap N' \subseteq \mathfrak{m}_B^n N'$. The natural map $(N/\mathfrak{m}_B^{r(n)}N)^\vee \to (N'/((\mathfrak{m}_B^{r(n)}N) \cap N'))^\vee$ is surjective (since its dual is injective), while, $(N'/\mathfrak{m}_B^n N')^\vee$ is naturally contained in the latter space, from which we see (A.1) is surjective. The lemma follows. □

Now suppose $\mathcal{U} \subset X$ with X a rigid space finite over $\widehat{T}_{\Sigma_\wp}$, then U is naturally equipped with a locally analytic $T(F_\wp)$-action induced by the evaluation map $T(F_\wp) \to \mathcal{O}(\widehat{T}_{\Sigma_\wp})$ (for example, see [36, Corollary 3.3], note by [22, Proposition 6.4.6], the distribution algebra $\mathcal{D}(T(F_\wp), E)$ is contained in $\mathcal{O}(\widehat{T}_{\Sigma_\wp})$).

LEMMA 14. *U is an allowable locally analytic representation of $T(F_\wp)$ (cf. [25, Definition 0.11]).*

Proof. Let $\mathcal{H} := \operatorname{End}_{T(F_\wp)}(U)$, since A acts naturally on U, we have a natural map $A \to \mathcal{H}$. By definition, we need to prove for any two algebraic characters χ_1, χ_2 of $T(F_\wp)$, each element in $\operatorname{Hom}_{\mathcal{H}[T(F_\wp)]}(U \otimes_E \chi_1, U \otimes_E \chi_2)$ is strict where $\mathcal{H}$ acts $U \otimes_E \chi_i$ on the first factor and $T(F_\wp)$ acts on $U \otimes_E \chi_i$ via the diagonal action. However, we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}[T(F_\wp)]}(U \otimes_E \chi_1, U \otimes_E \chi_2) &\subset \operatorname{Hom}_{\mathcal{H}}(U \otimes_E \chi_1, U \otimes_E \chi_2)\\ &= \operatorname{Hom}_{\mathcal{H}}(U, U) \subset \operatorname{Hom}_A(U, U),\end{aligned}$$

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and any element in the right set is strict (since $U^\vee$ is a coadmissible A-module), the lemma follows. □

Denote by $\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N,E)$ the affine E-algebra of the algebraic group

$$\mathrm{Res}^{F_\wp}_{\mathbb{Q}_p} N\times_{\mathbb{Q}_p} E\cong \mathrm{Res}^{F_\wp}_{\mathbb{Q}_p}\mathbb{G}_a\times_{\mathbb{Q}_p} E,$$

thus

$$\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N,E)\cong\otimes_{\sigma\in\Sigma_\wp}E[\sigma(z)]=:E[\underline{z}_{\Sigma_\wp}].$$

Let $\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N,U):=\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N,E)\otimes_E U\cong U[\underline{z}_{\Sigma_\wp}]$. This space can be equipped with a natural $\mathfrak{g}_{\Sigma_\wp}$-action such that for $z^{\underline{m}_{\Sigma_\wp}}:=\prod_{\sigma\in\Sigma_\wp}\sigma(z)^{m_\sigma}\in\mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N,E)$, $\underline{m}_{\Sigma_\wp}\in\mathbb{Z}_{\geqslant 0}^{|\Sigma_\wp|}$, and $u\in U$,

- $Z_\sigma\cdot(uz^{\underline{m}_{\Sigma_\wp}})=m_\sigma(d_\sigma-a_\sigma)uz^{\underline{m}_{\Sigma_\wp}}+(Z_\sigma\cdot u)z^{\underline{m}_{\Sigma_\wp}}$, for $Z_\sigma=\begin{pmatrix}a_\sigma&0\\0&d_\sigma\end{pmatrix}\in\mathfrak{t}_\sigma\subset\mathfrak{g}_\sigma$;
- $X_{+,\sigma}\cdot(uz^{\underline{m}_{\Sigma_\wp}})=\begin{cases}0&\text{if } m_\sigma=0\\ m_\sigma uz^{\underline{m}_{\Sigma_\wp}-1_\sigma}&\text{otherwise}\end{cases}$ where $1_\sigma\in\mathbb{Z}_{\geqslant 0}^{|\Sigma_\wp|}$ with $(1_\sigma)_{\sigma'}=\begin{cases}1&\sigma'=\sigma\\0&\sigma'\neq\sigma\end{cases}$, and $X_{+,\sigma}:=\begin{pmatrix}0&1\\0&0\end{pmatrix}\in\mathfrak{g}_\sigma$;
- $X_{-,\sigma}\cdot(uz^{\underline{m}_{\Sigma_\wp}})=(h_\sigma\cdot u)z^{\underline{m}_{\Sigma_\wp}+1_\sigma}-m_\sigma uz^{\underline{m}_{\Sigma_\wp}+1_\sigma}$ with $X_{-,\sigma}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\in\mathfrak{g}_\sigma$, $h_\sigma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\in\mathfrak{g}_\sigma$.

The embedding $U\hookrightarrow U[\underline{z}_{\Sigma_{F_\wp}}]$ (which can be easily checked to be $\mathfrak{b}_{\Sigma_\wp}$-invariant, where $\mathfrak{b}$ denotes the Lie algebra of $B(F_\wp)$ and $\mathfrak{b}_{\Sigma_\wp}$ acts on U via $\mathfrak{b}_{\Sigma_\wp}\twoheadrightarrow\mathfrak{t}_{\Sigma_\wp}$) thus induces

$$\mathrm{U}(\mathfrak{g}_{\Sigma_\wp})\otimes_{\mathrm{U}(\mathfrak{b}_{\Sigma_\wp})}U\longrightarrow U[\underline{z}_{\Sigma_{F_\wp}}] \tag{A.2}$$

since $\mathrm{U}(\mathfrak{g}_{\Sigma_\wp})\cong\mathrm{U}(\overline{\mathfrak{n}}_{\Sigma_\wp})\otimes_E\mathrm{U}(\mathfrak{b}_{\Sigma_\wp})$, $\mathrm{U}(\mathfrak{g}_{\Sigma_\wp})\otimes_{\mathrm{U}(\mathfrak{b}_{\Sigma_\wp})}U\cong\mathrm{U}(\overline{\mathfrak{n}}_{\Sigma_\wp})\otimes_E U$. One gets thus a map $\mathrm{U}(\overline{\mathfrak{n}}_{\Sigma_\wp})\otimes_E U\to U[\underline{z}_{\Sigma_{F_\wp}}]$ with

$$\Big(\prod_{\sigma\in\Sigma_\wp}X_{-,\sigma}^{m_\sigma}\Big)u\mapsto\bigg(\Big(\prod_{\sigma\in\Sigma_\wp}\prod_{j=0}^{m_\sigma-1}(h_\sigma-j)\Big)\cdot u\bigg)z^{\underline{m}_{\Sigma_\wp}} \tag{A.3}$$

for all $u\in U$, $\underline{m}_{\Sigma_\wp}\in\mathbb{Z}_{\geqslant 0}^{|\Sigma_\wp|}$, where we let $\prod_{j=0}^{m_\sigma-1}(h_\sigma-j)$ be 1 when $m_\sigma=0$.

As in [25, (2.8)], let $I_{B(F_\wp)}^{\mathrm{GL}_2(F_\wp)}(U)\hookrightarrow(\mathrm{Ind}_{B(F_\wp)}^{\mathrm{GL}_2(F_\wp)}U)^{\mathbb{Q}_p-\mathrm{an}}$ be the closed $\mathrm{GL}_2(F_\wp)$-subrepresentation generated by $J_B((\mathrm{Ind}_{B(F_\wp)}^{\mathrm{GL}_2(F_\wp)}U)^{\mathbb{Q}_p-\mathrm{an}})$. Since U is allowable, $I_{B(F_\wp)}^{\mathrm{GL}_2(F_\wp)}(U)$ is moreover a local closed subrepresentation of $(\mathrm{Ind}_{B(F_\wp)}^{\mathrm{GL}_2(F_\wp)}U)^{\mathbb{Q}_p-\mathrm{an}}$ (cf. [25, Proposition 2.8.10]). Note M is equipped with a natural action of $\mathrm{U}(\mathfrak{t}_{\Sigma_\wp})$ induced by

$$\mathrm{U}(\mathfrak{t}_{\Sigma_\wp})\longrightarrow\mathcal{O}(\widehat{T}_{\Sigma_\wp})\longrightarrow A.$$

Let $\bar{\mathfrak{t}} \subseteq \mathfrak{t}$ be the Lie algebra of $Z_1' \subset T(F_\wp)$, thus $\mathrm{U}(\bar{\mathfrak{t}}_{\Sigma_\wp})$ is the polynomial E-algebra generated by $\{h_\sigma\}_{\sigma\in\Sigma_\wp}$.

LEMMA 15. *Suppose M is $U(\bar{\mathfrak{t}}_{\Sigma_\wp})$-torsion free, then*

$$I_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}(U) \xrightarrow{\sim} (\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U)^{\mathbb{Q}_p-\mathrm{an}}.$$

Proof. By results in [25, Section 2], it is sufficient to prove (A.2) is surjective. Indeed, any local closed subrepresentation V of $(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U)^{\mathbb{Q}_p-\mathrm{an}}$ is determined by its fiber V_e (cf. [25, Definition 2.4.2, Proposition 2.4.7 (iii)]). By [25, Lemma 2.8.8, Proposition 2.8.10], we have a commutative $\mathrm{U}(\mathfrak{g}_{\Sigma_\wp})$-invariant commutative diagram

$$\begin{array}{ccc} \mathrm{U}(\mathfrak{g}_{\Sigma_\wp}) \otimes_{\mathrm{U}(\mathfrak{b}_{\Sigma_\wp})} U & \xrightarrow{(A.2)} & \mathcal{C}^{\mathbb{Q}_p-\mathrm{pol}}(N, U) \\ \downarrow & & \downarrow \\ I_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}(U)_e & \longrightarrow & (\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U)_e^{\mathbb{Q}_p-\mathrm{an}} \end{array}$$

where the left vertical map is induced by the $\mathrm{U}(\mathfrak{b}_{\Sigma_\wp})$-invariant map $U \hookrightarrow I_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)}(U)$, and the right vertical injective map has dense image. So if (A.2) is surjective, we deduce the bottom map has dense image, thus is an isomorphism, from which the lemma would follow.

Since M is $\mathrm{U}(\bar{\mathfrak{t}}_{\Sigma_\wp})$-torsion free, for each $0 \neq x \in \mathrm{U}(\bar{\mathfrak{t}}_{\Sigma_\wp})$, $M \xrightarrow{x} M$ is injective; by taking the dual, we see the map $U \xrightarrow{x} U$ is surjective, so (A.3) (hence (A.2)) is surjective. □

By [25, Theorem 0.13], we have thus (note the modulus character δ is smooth, hence does not change the Lie algebra action)

PROPOSITION 13. *Let V be a very strongly admissible locally $\mathbb{Q}_p$-analytic representation of* $\mathrm{GL}_2(F_\wp)$ *and suppose M is $U(\bar{\mathfrak{t}}_{\Sigma_\wp})$-torsion free, then we have a natural bijection*

$$\mathrm{Hom}_{T(F_\wp)}(U, J_B(V)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(F_\wp)}\left((\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} U \otimes_E \delta^{-1})^{\mathbb{Q}_p-\mathrm{an}}, V\right)^{\mathrm{bal}},$$

where 'bal' *denotes the balanced maps (cf.* [25, Definition 0.8]*).*

Now we go back to the setting after Lemma 7: let $\mathcal{U}$ be a strictly quasi-Stein neighborhood of z in $\mathcal{V}(K^p, w)_{\overline{\rho}}$ such that for any $z' \in \mathcal{U}(\overline{E})$, z' does not have companion points. By Lemma 14, the representation $\mathcal{M}(\mathcal{U})_b^\vee$ is allowable (where $\mathcal{M} := \mathcal{M}(K^p, w)_{\overline{\rho}}$).

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LEMMA 16. *(35) is balanced.*

Proof. By definition (cf. [25, Definition 0.8]), we need to prove the kernel of the composition

$$\mathrm{U}(\mathfrak{g}_{\Sigma_\wp}) \otimes_{\mathrm{U}(\mathfrak{b}_{\Sigma_\wp})} \mathcal{M}(\mathcal{U})_b^\vee \longrightarrow \mathrm{U}(\mathfrak{g}_{\Sigma_\wp}) \otimes_{\mathrm{U}(\mathfrak{b}_{\Sigma_\wp})} J_B\Big(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho}, \mathbb{Q}_p-\mathrm{an}}\Big) \longrightarrow \widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho}, \mathbb{Q}_p-\mathrm{an}}$$

contains the kernel of (A.2) (for $U = \mathcal{M}(\mathcal{U})_b^\vee$).

It is thus sufficient to prove for $0 \neq v \in \mathcal{M}(\mathcal{U})_b^\vee$, if there exists $\underline{m}_{\Sigma_\wp} \in \mathbb{Z}_{\geqslant 0}^{|\Sigma_\wp|}$ such that $\big(\prod_{\sigma\in\Sigma_\wp}\prod_{j=1}^{m_\sigma-1}(h_\sigma - j)\big)v = 0$ (where we let $\prod_{j=1}^{m_\sigma-1}(h_\sigma - j)$ be 1 if $m_\sigma = 0$), then $\big(\prod_{\sigma\in\Sigma_\wp} X_{-,\sigma}^{m_\sigma}\big)v = 0$, where the latter v is viewed as a vector in $\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho}, \mathbb{Q}_p-\mathrm{an}}$ (via (35)). We view $h := \prod_{\sigma\in\Sigma_\wp}\prod_{j=1}^{m_\sigma-1}(h_\sigma - j)$ as a global section of $\widehat{T}_{\Sigma_\wp}$, and put $\widehat{T}_{\Sigma_\wp}(\underline{m}_{\Sigma_\wp})$ to be the closed analytic subvariety of $\widehat{T}_{\Sigma_\wp}$ defined by h (see [4, Section 9.5]). Let $\mathcal{U}(\underline{m}_{\Sigma_\wp}) := \mathcal{U} \times_{\widehat{T}_{\Sigma_\wp}} \widehat{T}_{\Sigma_\wp}(\underline{m}_{\Sigma_\wp})$ (which is also the closed rigid subspace of $\mathcal{U}$ defined by h, thus in particular is also strictly quasi-Stein), $\mathcal{M}' := \mathcal{M}|_{\mathcal{U}(\underline{m}_{\Sigma_\wp})}$, $M' := \mathcal{M}'(\mathcal{U}(\underline{m}_{\Sigma_\wp}))$. One can check that $(M')_b^\vee = (\mathcal{M}(\mathcal{U})_b^\vee)^{h=0}$. By Lemma 13, the generalized eigenvectors are dense in $(M')_b^\vee$, it is thus enough to prove $(\prod_{\sigma\in\Sigma_\wp} X_{-,\sigma}^{m_\sigma})v = 0$ for generalized eigenvectors (of $T(F_\wp) \times \mathcal{H}^p$) $v \in (M')_b^\vee$. Let $0 \neq v \in (M')_b^\vee$ be a generalized (χ', λ')-eigenvector (enlarge E if necessary), where $\chi' = \chi_1' \otimes \chi_2'$ is a continuous character of $T(F_\wp)$, since $hv = 0$, there exists $\sigma \in \Sigma_\wp$ such that $m_\sigma \geqslant 1$, $n_\sigma := k_{\chi_1',\sigma} - k_{\chi_2',\sigma} \in \{0, \ldots, m_\sigma - 1\}$ and hence $(h_\sigma - n_\sigma)v = 0$. However, as in [24, Proposition 4.4.4] (see also [20, Lemma 6.3.15]), one can prove $X_{-,\sigma}^{n_\sigma+1}$ induces a map

$$J_B\big(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho}, \mathbb{Q}_p-\mathrm{an}}\big)^{h_\sigma=n_\sigma}[T(F_\wp) = \chi', \mathcal{H}^p = \lambda'] \xrightarrow{X_{-,\sigma}^{n_\sigma+1}} J_B\big(\widetilde{H}^1_{\text{ét}}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho}, \mathbb{Q}_p-\mathrm{an}}\big)^{h_\sigma=-n_\sigma-2}[T(F_\wp) = (\chi')^c_\sigma, \mathcal{H}^p = \lambda']$$

while the latter space is zero since any point in $\mathcal{U}$ does not have companion points. So $X_{-,\sigma}^{n_\sigma+1}v = 0$, the lemma follows. □

By results in Section 4.3, $\mathcal{M}(\mathcal{U})$ is $\mathcal{O}(\mathcal{W}_{1,\Sigma_\wp})$-torsion free thus $\mathrm{U}(\overline{\mathfrak{t}}_{\Sigma_\wp})$-torsion free. By Proposition 13, we get

COROLLARY 7. *The map (35) induces an $\mathcal{H}^p$-invariant morphism of locally analytic representations of* $\mathrm{GL}_2(F_\wp)$

$$\big(\mathrm{Ind}_{\overline{B}(F_\wp)}^{\mathrm{GL}_2(F_\wp)} \mathcal{M}(\mathcal{U})_b^\vee[\delta^{-1}]\big)^{\mathbb{Q}_p-\mathrm{an}} \longrightarrow \widetilde{H}^1_{\acute{e}t}(K^p, E)^{Z_1=\mathcal{N}^{-w}}_{\overline{\rho}, \mathbb{Q}_p-\mathrm{an}}.$$

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