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Linear Operators Preserving Generalized Numerical Ranges and Radii on Certain Triangular Algebras of Matrices

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Abstract. Let $c = (c_1, \dots, c_n)$ be such that $c_1 \geq \cdots \geq c_n$. The c-numerical range of an $n \times n$ matrix A is defined by

$$W_c(A) = \Big\{ \sum_{j=1}^{n} c_j (Ax_j, x_j) : \{x_1, \dots, x_n\} \text{ an orthonormal basis for } \mathbf{C}^n \Big\},$$

and the c-numerical radius of A is defined by $r_c(A) = \max\{|z| : z \in W_c(A)\}$. We determine the structure of those linear operators ϕ on algebras of block triangular matrices, satisfying

$$W_c\big(\phi(A)\big) = W_c(A) \text{ for all } A \quad \text{or} \quad r_c\big(\phi(A)\big) = r_c(A) \text{ for all } A.$$

1 Introduction

Let $\mathcal{M}_n$ denote the algebra of $n \times n$ matrices. Suppose $c = (c_1, \dots, c_n)$ with $c_1 \geq \cdots \geq c_n$. The c-numerical range of $A \in \mathcal{M}_n$ is defined by

$$W_c(A) = \Big\{ \sum_{j=1}^{n} c_j (Ax_j, x_j) : \{x_1, \dots, x_n\} \text{ an orthonormal basis for } \mathbf{C}^n \Big\},$$

which was introduced by Westwick [15], and the c-numerical radius of A is the quantity

$$r_c(A) = \max\{|z| : z \in W_c(A)\}.$$

When $c = (1, 0, \dots, 0)$, these concepts reduce to the classical numerical range $W(A)$ and the classical numerical radius $r(A)$, which have been studied extensively because of their connections and applications to many pure and applied areas (see *e.g.* [4, Chapter 1]). In particular, there has been considerable interest in studying the linear preservers ϕ of the classical numerical range or the classical numerical radius on $\mathcal{M}_n$, *i.e.*, those linear operators ϕ on $\mathcal{M}_n$ satisfy

$$W\big(\phi(A)\big) = W(A) \text{ for all } A \in \mathcal{M}_n \quad \text{or} \quad r\big(\phi(A)\big) = r(A) \text{ for all } A \in \mathcal{M}_n.$$

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It turns out (see [5], [11]) that a linear preserver of the classical numerical range on $\mathcal{M}_n$ must be of the form

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU$$

for some unitary $U \in \mathcal{M}_n$; and a linear preserver of the classical numerical radius must be a unit multiple of a linear preserver of the classical numerical range. In particular, linear preservers of the classical numerical range on $\mathcal{M}_n$ are Jordan isomorphisms on $\mathcal{M}_n$, *i.e.*, those algebra isomorphisms ϕ satisfying $\phi(A^2) = \phi(A)^2$ for all $A \in \mathcal{M}_n$. Researchers have extended these results in different directions. Some of them considered linear preservers of different types of generalized numerical ranges and radii, and other focused on the linear preservers of the numerical range and radius defined on different algebras (see [1], [2], [6], [10], [11], [13, Chapter 5] and the extensive list of reference therein). The purpose of this paper is to study linear preservers of c-numerical ranges and radii on certain triangular algebras of matrices. We introduce some more definitions and notations to describe our results.

Denote by $\mathcal{T}(n_1, \ldots, n_k)$ the subalgebra of $\mathcal{M}_n$ consisting of $k \times k$ block triangular matrices (A_{pq}) such that $A_{pp} \in \mathcal{M}_{n_p}$ for all p and $A_{pq} = 0$ whenever $p > q$. In this paper, we determine the structure of those linear operators ϕ on $\mathcal{T}(n_1, \ldots, n_k)$ satisfying

$$W_c\big(\phi(A)\big) = W_c(A) \quad \text{for all } A \in \mathcal{T}(n_1, \ldots, n_k)$$

or

$$r_c\big(\phi(A)\big) = r_c(A) \quad \text{for all } A \in \mathcal{T}(n_1, \ldots, n_k).$$

In Section 2, we prove some duality theorems relating the linear preservers of c-numerical ranges and radii to other preserver problems on $\mathcal{T}(n_1, \ldots, n_k)$. In Sections 3 and 4, we show that linear preservers of c-numerical ranges and c-numerical radii on the algebra $\mathcal{T}(n_1, \ldots, n_k)$ can be extended to $\mathcal{M}_n$, and then use the results on $\mathcal{M}_n$ to determine the structure of ϕ on $\mathcal{T}(n_1, \ldots, n_k)$. In addition to solving the linear preserver problems involving the c-numerical ranges and radii, the duality and extension techniques developed in this paper may be useful in studying other linear preserver problems on triangular algebras that have attracted many researchers recently, see *e.g.* [3], [8], [9].

The following definitions and notations will be used in our discussion. The standard basis for $\mathcal{M}_n$ will be denoted by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$. Suppose ϕ is a linear operator on a subspace $\mathbf{S}$ of $\mathcal{M}_n$. The dual transformation $\phi^* \colon \mathbf{S} \to \mathbf{S}$ is uniquely defined by the condition $\big(\phi(A), B\big) = \big(A, \phi^*(B)\big)$ for all $A, B \in \mathbf{S}$, where $(X, Y) = \operatorname{tr}(XY^*)$ is the usual inner product on $\mathcal{M}_n$. Furthermore, we will use $\mathcal{H}_n$ to denote the real linear space of $n \times n$ hermitian matrices. The Frobenius norm on $\mathcal{M}_n$ or $\mathcal{T}(n_1, \ldots, n_k)$ are defined and denoted by

$$\|A\|_F = (A, A)^{1/2} = \Big\{ \sum_{1 \le p,q \le n} |a_{pq}|^2 \Big\}^{1/2}.$$

We also introduce the T-norm on $\mathcal{T}(n_1, \ldots, n_k)$ defined by

$$\|(A_{pq})\|_T = \Big\{ \sum_{j=1}^{k} \|A_{jj}\|_F^2 + 2 \sum_{1 \le p < q \le k} \|A_{pq}\|_F^2 \Big\}^{1/2}.$$

It is clear that the T-norm on $\mathcal{T}(n_1,\dots,n_k)$ is induced by the inner product

$$\langle (A_{pq}),(B_{pq})\rangle = \sum_{j=1}^{k} \operatorname{tr} A_{jj}B_{jj}^* + 2\sum_{1\le p<q\le k} \operatorname{tr} A_{pq}B_{pq}^*.$$

Hence the T-norm is strictly convex, *i.e.*, $\|A+B\| < \|A\|+\|B\|$ whenever $A \ne B$ and $\|A\| = \|B\|$. The convex hull of a set S of matrices or scalars will be denoted by $\operatorname{conv} S$. Furthermore, if $n_j = n_{k-j+1}$ for all $1 \le j \le k/2$, we can define the transpose of $A \in \mathcal{T}(n_1,\dots,n_k)$ with respect to the anti-diagonal $E = E_{1n} + E_{2,n-1} + \cdots + E_{n1}$ by $A' = EA^tE$.

We shall always assume that $c = (c_1,\dots,c_n)$ with $c_1 \ge \cdots \ge c_n$ unless otherwise specified. If $c_1 = \cdots = c_n$, then $W_c(A) = \{c_1 \operatorname{tr} A\}$ for all $A \in M_n$. To avoid this trivial case, we assume that $c_1 > c_n$.

2 Duality Theorems

Let $\mathcal{U}(c)$ be the set of hermitian matrices with eigenvalues $c_1,\dots,c_n$; and let

$$\mathcal{V}(c) = \bigcup_{t\in[0,2\pi)} e^{it}\mathcal{U}(c).$$

Denote by $\pi\colon \mathcal{M}_n \to \mathcal{T}(n_1,\dots,n_k)$ the canonical (orthogonal) projection, *i.e.*, for any block matrix $(A_{pq}) \in \mathcal{M}_n$ with $A_{pp} \in \mathcal{M}_{n_p}$, the matrix $\pi(A_{pq}) \in \mathcal{T}(n_1,\dots,n_k)$ is obtained from (A_{pq}) by setting all the strictly lower triangular blocks to zeros. Let

$$\widetilde{\mathcal{U}}(c) = \pi\big(\mathcal{U}(c)\big) \quad \text{and} \quad \widetilde{\mathcal{V}}(c) = \pi\big(\mathcal{V}(c)\big).$$

In the following, we prove that ϕ on $\mathcal{T}(n_1,\dots,n_k)$ preserves the c-numerical range (respectively, radius) is equivalent to the fact that the dual transformation ϕ^* satisfies $\phi^*\big(\widetilde{\mathcal{U}}(c)\big) = \widetilde{\mathcal{U}}(c)$ (respectively, $\phi^*\big(\widetilde{\mathcal{V}}(c)\big) = \widetilde{\mathcal{V}}(c)$). We remark that the idea of studying the linear preserver problems via the dual transformations has been used by other authors (see [7], [11]).

We first establish several lemmas. The first one is easy to verify.

Lemma 2.1 *Given any $A = (A_{pq}) \in \mathcal{T}(n_1,\dots,n_k)$ with Hermitian diagonal blocks, there exists a unique $\hat{A} \in \mathcal{H}_n$ (the real space of Hermitian matrices) such that $\pi(\hat{A}) = A$ and $\|A\|_T = \|\hat{A}\|_F$. Consequently, if $B \in \widetilde{\mathcal{V}}(c)$, then there exists $|\mu| = 1$ and $\hat{B} \in \mathcal{U}(c)$ such that $B = \mu\pi(\hat{B})$ and $\|B\|_T = \|\hat{B}\|_F = \ell_2(c)$.*

Lemma 2.2 *Let $A \in \mathcal{T}(n_1,\dots,n_k)$. Then*

$$W_c(A) = \{(A,X) : X \in \widetilde{\mathcal{U}}(c)\} = \{(A,X) : X \in \operatorname{conv}\widetilde{\mathcal{U}}(c)\}$$

and

$$r_c(A) = \max\{\Re(A,X) : X \in \widetilde{\mathcal{V}}(c)\} = \max\{\Re(A,X) : X \in \operatorname{conv}\widetilde{\mathcal{V}}(c)\}.$$

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Proof Note that

$$\begin{aligned} W_c(A) &= \{(A, X) : X \in \mathcal{U}(c)\} \\ &= \{ \big(A, \pi(X)\big) : X \in \mathcal{U}(c)\} \\ &= \{(A, Y) : Y \in \widetilde{\mathcal{U}}(c)\}. \end{aligned}$$

Furthermore, by the convexity of $W_c(A)$ (see [12], [15]), we have

$$\begin{aligned} W_c(A) &= \operatorname{conv} W_c(A) \\ &= \operatorname{conv}\{(A, Y) : Y \in \widetilde{\mathcal{U}}(c)\} \\ &= \{(A, Y) : Y \in \operatorname{conv} \widetilde{\mathcal{U}}(c)\}. \end{aligned}$$

The proof of the assertion on $r_c(A)$ is similar. ■

Lemma 2.3 *The extreme points of* $\operatorname{conv}\big(\widetilde{\mathcal{U}}(c)\big)$ *and* $\operatorname{conv}\big(\widetilde{\mathcal{V}}(c)\big)$ *are* $\widetilde{\mathcal{U}}(c)$ *and* $\widetilde{\mathcal{V}}(c)$ *respectively.*

Proof Let $\mathcal{L} = \widetilde{\mathcal{U}}(c)$ or $\widetilde{\mathcal{V}}(c)$. We need to prove that $\mathcal{L}$ is the set of extreme points of $\operatorname{conv} \mathcal{L}$. Clearly, $\mathcal{L}$ contains all the extreme points of $\operatorname{conv} \mathcal{L}$. Now, suppose $A \in \mathcal{L}$ is such that $A = \lambda_1 A_1 + \cdots + \lambda_m A_m$ for some $A_1, \ldots, A_m \in \mathcal{L}$, and positive numbers $\lambda_1, \ldots, \lambda_m$ with $\lambda_1 + \cdots + \lambda_m = 1$. By the last assertion of Lemma 2.1, we have $\|A\|_T = \|A_1\|_T = \cdots = \|A_m\|_T = \ell_2(c)$. Since $\|\cdot\|_T$ is strictly convex on $\mathcal{T}(n_1, \ldots, n_k)$, we see that $A = A_1 = \cdots = A_m$. Thus A is an extreme point. ■

We are now ready to prove the main results of this section.

Theorem 2.4 *Let ϕ be a linear operator on $\mathcal{T}(n_1, \ldots, n_k)$. The following conditions are equivalent.*

(a) *ϕ satisfies*

$$W_c\big(\phi(A)\big) = W_c(A) \quad \text{for all } A \in \mathcal{T}(n_1, \ldots, n_k). \tag{1}$$

(b) $\phi^*\big(\widetilde{\mathcal{U}}(c)\big) = \widetilde{\mathcal{U}}(c)$.

(c) $\phi^*\big(\operatorname{conv} \widetilde{\mathcal{U}}(c)\big) = \operatorname{conv} \widetilde{\mathcal{U}}(c)$.

Proof (c) ⇒ (b): Suppose (c) holds. Then the restriction of ϕ^* on the linear span of $\widetilde{\mathcal{U}}(c)$ is invertible, and will map the set of extreme points of $\operatorname{conv} \widetilde{\mathcal{U}}(c)$ onto itself. By Lemma 2.3, condition (b) holds.

(b) ⇒ (a): Suppose (b) holds. Then for any $A \in \mathcal{T}(n_1, \ldots, n_k)$, we have

$$\begin{aligned} W_c\big(\phi(A)\big) &= \{ \big(\phi(A), X\big) : X \in \widetilde{\mathcal{U}}(c)\} \quad \text{(by Lemma 2.2)} \\ &= \{ \big(A, \phi^*(X)\big) : X \in \widetilde{\mathcal{U}}(c)\} \\ &= \{(A, Y) : Y \in \widetilde{\mathcal{U}}(c)\} \quad \text{(by assumption)} \\ &= W_c(A) \quad \text{(by Lemma 2.2)}. \end{aligned}$$

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Thus condition (a) holds.

(a) $\Rightarrow$ (c): Suppose (a) holds. Then for any $A \in \mathcal{T}(n_1, \ldots, n_k)$, one has

$$\begin{aligned}
\{(A, X) : X \in \operatorname{conv} \widetilde{\mathcal{U}}(c)\} &= W_c(A) \quad \text{(by Lemma 2.2)} \\
&= W_c\big(\phi(A)\big) \quad \text{(by assumption)} \\
&= \big\{ \big(\phi(A), X\big) : X \in \operatorname{conv} \widetilde{\mathcal{U}}(c)\big\} \quad \text{(by Lemma 2.2)} \\
&= \big\{ \big(A, \phi^*(X)\big) : X \in \operatorname{conv} \widetilde{\mathcal{U}}(c)\big\} \\
&= \big\{ (A, Y) : Y \in \phi^*\big(\operatorname{conv} \widetilde{\mathcal{U}}(c)\big) \big\}.
\end{aligned}$$

Thus, every support plane $\mathcal{P} = \{X \in \mathcal{T}(n_1, \ldots, n_k) : \Re(A, X) \leq r\}$ of $\operatorname{conv} \widetilde{\mathcal{U}}(c)$ is a support plane of $\phi^*\big(\operatorname{conv} \widetilde{\mathcal{U}}(c)\big)$, and vice versa. Since a compact convex set is the intersection of the closed half spaces defined by its support planes, we see that the two compact convex sets $\operatorname{conv} \widetilde{\mathcal{U}}(c)$ and $\phi^*\big(\operatorname{conv} \widetilde{\mathcal{U}}(c)\big)$ are equal. ■

Theorem 2.5 *Let ϕ be a linear operator on $\mathcal{T}(n_1, \ldots, n_k)$. The following conditions are equivalent.*

(a) ϕ satisfies

$$r_c\big(\phi(A)\big) = r_c(A) \quad \textit{for all } A \in \mathcal{T}(n_1, \ldots, n_k). \tag{2}$$

(b) $\phi^*\big(\widetilde{\mathcal{V}}(c)\big) = \widetilde{\mathcal{V}}(c)$.

(c) $\phi^*\big(\operatorname{conv} \widetilde{\mathcal{V}}(c)\big) = \operatorname{conv} \widetilde{\mathcal{V}}(c)$.

Proof (c) $\Rightarrow$ (b): Suppose (c) holds. Then the restriction of ϕ^* on the linear span of $\widetilde{\mathcal{V}}(c)$ is invertible, and will map the set of extreme points of $\operatorname{conv} \widetilde{\mathcal{V}}(c)$ onto itself. By Lemma 2.3, condition (b) holds.

(b) $\Rightarrow$ (a): Suppose (b) holds. Then for any $A \in \mathcal{T}(n_1, \ldots, n_k)$, we have

$$\begin{aligned}
r_c\big(\phi(A)\big) &= \max\big\{ \Re\big(\phi(A), X\big) : X \in \widetilde{\mathcal{V}}(c)\big\} \quad \text{(by Lemma 2.2)} \\
&= \max\big\{ \Re\big(A, \phi^*(X)\big) : X \in \widetilde{\mathcal{V}}(c)\big\} \\
&= \max\{\Re(A, Y) : Y \in \widetilde{\mathcal{V}}(c)\} \quad \text{(by assumption)} \\
&= r_c(A) \quad \text{(by Lemma 2.2)}.
\end{aligned}$$

Thus, condition (a) holds.

(a) $\Rightarrow$ (c): Suppose (c) holds. For any $A \in \mathcal{T}(n_1, \ldots, n_k)$, we have

$$\begin{aligned}
\max\{\Re(A, X) : X \in \operatorname{conv} \widetilde{\mathcal{V}}(c)\} &= r_c(A) \quad \text{(by Lemma 2.2)} \\
&= r_c\big(\phi(A)\big) \quad \text{(by assumption)}
\end{aligned}$$

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$$= \max\left\{ \Re\big(\phi(A), X\big) : X \in \operatorname{conv} \widetilde{\mathcal{V}}(c) \right\}$$
$$\text{(by Lemma 2.2)}$$
$$= \max\left\{ \Re\big(A, \phi^*(X)\big) : X \in \operatorname{conv} \widetilde{\mathcal{V}}(c) \right\}$$
$$= \max\left\{ \Re(A, Y) : Y \in \phi^*\left(\operatorname{conv} \widetilde{\mathcal{V}}(c)\right) \right\}.$$

Thus, every support plane $\mathcal{P} = \{X \in \mathcal{T}(n_1, \ldots, n_k) : \Re(A, X) \le r\}$ of $\operatorname{conv} \widetilde{\mathcal{V}}(c)$ is a support plane of $\phi^*\left(\operatorname{conv} \widetilde{\mathcal{V}}(c)\right)$, and vice versa. Since a compact convex set is the intersection of the closed half spaces defined by its support planes, we see that the two compact convex sets $\operatorname{conv} \widetilde{\mathcal{V}}(c)$ and $\phi^*\left(\operatorname{conv} \widetilde{\mathcal{V}}(c)\right)$ are equal. ■

3 Generalized Numerical Range Preservers

The purpose of this section is to prove the following theorems.

Theorem 3.1 *A linear operator ϕ on $\mathcal{T}(n_1, \ldots, n_k)$ satisfies*

$$W_c\big(\phi(A)\big) = W_c(A) \quad \textit{for all } A \in \mathcal{T}(n_1, \ldots, n_k)$$

if and only if there exists a unitary matrix $V \in \mathcal{T}(n_1, \ldots, n_k)$ such that one of the following conditions holds:

(a) $\phi(A) = V^*A^+V$ for all $A \in \mathcal{T}(n_1, \ldots, n_k)$;
(b) $c_j + c_{n-j+1}$ are equal for all j and

$$\phi(A) = 2(\operatorname{tr} A)I/n - V^*A^+V \quad \textit{for all } A \in \mathcal{T}(n_1, \ldots, n_k);$$

(c) $\sum_{j=1}^n c_j = 0$ and there is a linear functional f on $\mathcal{T}(n_1, \ldots, n_k)$ such that

$$\phi(A) = f(A)I + V^*A^+V \quad \textit{for all } A \in \mathcal{T}(n_1, \ldots, n_k);$$

(d) $c_j + c_{n-j+1} = 0$ for all j and there is a linear functional f on $\mathcal{T}(n_1, \ldots, n_k)$ such that

$$\phi(A) = f(A)I - V^*A^+V \quad \textit{for all } A \in \mathcal{T}(n_1, \ldots, n_k);$$

where (i) A^+ denotes A or (ii) $n_j = n_{k-j+1}$ for all $1 \le j \le k/2$ and A^+ denotes $A' = EA^tE$, the transpose of A taken with respect to the anti-diagonal $E = E_{1n} + E_{2,n-1} + \cdots + E_{n1}$.

Theorem 3.2 *A linear operator ϕ on $\mathcal{T}(n_1, \ldots, n_k)$ satisfies*

$$\phi\big(\widetilde{\mathcal{U}}(c)\big) = \widetilde{\mathcal{U}}(c)$$

if and only if there exists unitary matrix $V \in \mathcal{T}(n_1, \ldots, n_k)$ such that one of the following conditions holds:

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(a) $\phi(A) = V^*A^+V$ *for all* $A \in \mathcal{T}(n_1, \dots, n_k)$;
(b) $c_j + c_{n-j+1}$ *are equal for all* j *and*

$$\phi(A) = 2(\operatorname{tr} A)I/n - V^*A^+V \quad \textit{for all } A \in \mathcal{T}(n_1, \dots, n_k);$$

(c) $\sum_{j=1}^n c_j = 0$ *and there is an* $F \in \mathcal{T}(n_1, \dots, n_k)$ *such that*

$$\phi(A) = (\operatorname{tr} A)F + V^*A^+V \quad \textit{for all } A \in \mathcal{T}(n_1, \dots, n_k);$$

(d) $c_j + c_{n-j+1} = 0$ *for all* j *and there is an* $F \in \mathcal{T}(n_1, \dots, n_k)$ *such that*

$$\phi(A) = (\operatorname{tr} A)F - V^*A^+V \quad \textit{for all } A \in \mathcal{T}(n_1, \dots, n_k);$$

where (i) A^+ *denotes* A, *or (ii)* $n_j = n_{k-j+1}$ *for all* $1 \le j \le k/2$ *and* A^+ *denotes* $A' = EA^tE$, *the transpose of* A *taken with respect to the anti-diagonal* $E = E_{1n} + E_{2,n-1} + \cdots + E_{n1}$.

By Theorem 2.4, the linear transformations in Theorem 3.1 and Theorem 3.2 are dual to each others. One can readily check the duality relation. In particular, the transformations in parts (a) and (b) of the two theorems are self-adjoint, and hence their descriptions are the same. In the following, we will prove Theorem 3.2 and then Theorem 3.1 follows immediately.

Proof of Theorem 3.2 The "if" part can be checked easily. We consider the "only if" part in the following. First of all, consider $c' \in \mathbf{R}^n$ and a linear operator ψ on $\mathcal{T}(n_1, \dots, n_k)$ such that

(i) If $\sum_{j=1}^n c_j = 0$ then $c' = c + (1, \dots, 1)$ and

$$\psi(A) = \phi(A) + \left(\frac{\operatorname{tr} A}{n}\right)\bigl(I - \phi(I)\bigr);$$

(ii) If $\sum_{j=1}^n c_j = \gamma \neq 0$ then $c' = (nc)/\gamma$ and $\psi = \phi$.

It is easy to check that $\sum_{j=1}^n c'_j = n$ and ψ satisfies $\psi\bigl(\widetilde{\mathcal{U}}(c')\bigr) = \widetilde{\mathcal{U}}(c')$. Moreover, ψ satisfies one of conditions (a)–(d) if and only if ϕ does, where $F = \bigl(I - \phi(I)\bigr)/n$ in (c) or (d). Thus, we may replace ϕ by ψ and replace c by c' so that $\sum_{j=1}^n c_j = n$, and prove one of conditions (a)–(d) holds.

From this point, we assume that $\sum_{j=1}^n c_j = n$. We divide the rest of the proof into 3 assertions.

Assertion 1 There exists $\Phi \colon \mathcal{M}_n \to \mathcal{M}_n$ satisfying $\pi \circ \Phi = \phi \circ \pi$ and $\Phi\bigl(\mathcal{U}(c)\bigr) = \mathcal{U}(c)$.

Since $\sum_{j=1}^n c_j = n$, $\mathcal{U}(c)$ is a spanning set of $\mathcal{M}_n$ (see [14]). Since π is surjective, it follows that $\widetilde{\mathcal{U}}(c)$ is a spanning set of $\mathcal{T}(n_1, \dots, n_k)$. By the fact that $\phi\bigl(\widetilde{\mathcal{U}}(c)\bigr) = \widetilde{\mathcal{U}}(c)$, we see that ϕ is invertible on $\mathcal{T}(n_1, \dots, n_k)$.

Let $\{A_1, \dots, A_l\} \subseteq \mathcal{U}(c)$ be a basis for $\mathcal{M}_n$. Then $\{\pi(A_1), \dots, \pi(A_l)\} \subseteq \widetilde{\mathcal{U}}(c)$ and $\{\phi(\pi(A_1)), \dots, \phi(\pi(A_l))\} \subseteq \widetilde{\mathcal{U}}(c)$ are spanning sets of $\mathcal{T}(n_1, \dots, n_k)$. By Lemma 2.1, for each $j = 1, \dots, l$, there exists a unique $B_j \in \mathcal{U}(c)$ be such that

$$\pi(B_j) = \phi(\pi(A_j)) \in \widetilde{\mathcal{U}}(c).$$

Define the linear map Φ on $\mathcal{M}_n$ so that

$$\Phi(A_j) = B_j \quad \text{for all } j = 1, \dots, l.$$

One easily checks that $\pi \circ \Phi$ and $\phi \circ \pi$ are linear maps from $\mathcal{M}_n$ to $\mathcal{T}(n_1, \dots, n_k)$ satisfying

$$\pi\Phi(A_j) = \pi(B_j) = \phi\pi(A_j)$$

for $j = 1, \dots, l$. It follows that

$$\pi\Phi(X) = \phi\pi(X) \quad \text{for all } X \in \mathcal{M}_n. \tag{3}$$

By our construction, we have $\Phi(\mathcal{U}(c)) \subseteq \Phi(\mathcal{H}_n) \subseteq \mathcal{H}_n$. Let $A \in \mathcal{U}(c)$. Then $\pi\Phi(A) = \phi(\pi(A)) \in \widetilde{\mathcal{U}}(c)$ by (3), *i.e.*, $\pi\Phi(A) = \pi(B)$ for some $B \in \mathcal{U}(c)$. By Lemma 2.1 and the fact that $\Phi(A) \in \mathcal{H}_n$, we have $\Phi(A) = B \in \mathcal{U}(c)$. Hence $\Phi(\mathcal{U}(c)) \subseteq \mathcal{U}(c)$.

Suppose $B \in \mathcal{U}(c)$. Since $\phi(\widetilde{\mathcal{U}}(c)) = \widetilde{\mathcal{U}}(c)$, there exists $A \in \mathcal{U}(c)$ such that $\pi(B) = \phi(\pi(A)) = \pi(\Phi(A))$ by (3). By Lemma 2.1 and the fact that $\Phi(A) \in \mathcal{H}_n$, we have $\Phi(A) = B$. Thus $\Phi(\mathcal{U}(c)) \supseteq \mathcal{U}(c)$. Combining the result in the last paragraph, we have $\Phi(\mathcal{U}(c)) = \mathcal{U}(c)$. The proof of Assertion 1 is complete.

Since we have modified c so that $\sum_{j=1}^n c_j \neq 0$, we can apply Theorem 2.3 in [7] to conclude that there exists a unitary U such that

(i) Φ is of the form $A \mapsto U^*A^-U$; or
(ii) $c_j + c_{n-j+1}$ are equal for all j and Φ is of the form $A \mapsto 2(\operatorname{tr} A)I/n - U^*A^-U$;

where A^- denotes A or A^t.

Note that in both (i) and (ii), Φ preserves the Frobenius norm on $\mathcal{M}_n$. Next, we prove

Assertion 2 The mapping Φ satisfies $\Phi(\mathcal{T}(n_1, \dots, n_k)) = \mathcal{T}(n_1, \dots, n_k)$.

If (ii) holds, we consider $\Psi(A) = 2(\operatorname{tr} A)I/2 - \Phi(A) = U^*A^-U$ instead. So we assume that (i) holds.

Given $A = (A_{pq}) \in \mathcal{T}(n_1, \dots, n_k)$, we write $A = A_D + A_N$ with $A_D = \operatorname{diag}(A_{11}, \dots, A_{kk})$. Note that for any $A \in \widetilde{\mathcal{U}}(A)$ we have $\phi(A) = \pi \circ \Phi(A) = \pi(U^*A^-U)$. Thus

$$\|\phi(A)\|_F^2 = \|\pi(U^*A^-U)\|_F^2 \le \|U^*A^-U\|_F^2 = \|A\|_F^2, \tag{4}$$

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and hence

$$\begin{aligned}\|A_D\|_F^2 + \|A_N\|_F^2 = \|A\|_F^2 &\geq \|\phi(A)\|_F^2 \\ &= \frac{1}{2}\|\phi(A)\|_T^2 + \frac{1}{2}\|\phi(A)_D\|_F^2 \quad \text{(by the definition of } T\text{-norm)} \\ &= \frac{1}{2}\|A\|_T^2 + \frac{1}{2}\|\phi(A)_D\|_F^2 \quad \text{(by the fact that } A, \phi(A) \in \widetilde{\mathcal{U}}(c)) \\ &= \frac{1}{2}\|A_D\|_F^2 + \|A_N\|_F^2 + \frac{1}{2}\|\phi(A)_D\|_F^2.\end{aligned}$$

It follows that

$$\|A_D\|_F^2 \geq \|\phi(A)_D\|_F^2. \tag{5}$$

Since ϕ is invertible on $\mathcal{T}(n_1, \ldots, n_k)$ and satisfies $\phi^{-1}\big(\widetilde{\mathcal{U}}(c)\big) = \widetilde{\mathcal{U}}(c)$, we can apply a similar argument to ϕ^{-1} and conclude that

$$\|B_D\|_F^2 \geq \|\phi^{-1}(B)_D\|_F^2 \quad \text{for all } B \in \widetilde{\mathcal{U}}(c). \tag{6}$$

Substituting $\phi^{-1}(B) = A$ in (6), we have

$$\|\phi(A)_D\|_F^2 \geq \|A_D\|_F^2 \quad \text{for all } A \in \widetilde{\mathcal{U}}(c). \tag{7}$$

By (5) and (7), we have

$$\|\phi(A)_D\|_F^2 = \|A_D\|_F^2 \quad \text{for all } A \in \widetilde{\mathcal{U}}(c).$$

Thus, the inequality in (4) is an equality. Clearly, it is equivalent to saying that

$$\Phi(A) = \pi(U^* A^- U) \in \mathcal{T}(n_1, \ldots, n_k)$$

for any $A \in \widetilde{\mathcal{U}}(c)$. Since $\widetilde{\mathcal{U}}(c)$ is a spanning set of $\mathcal{T}(n_1, \ldots, n_k)$, every $X \in \mathcal{T}(n_1, \ldots, n_k)$ is a linear combination of elements in $\widetilde{\mathcal{U}}(c)$. Thus, we see that $U^* X^- U \in \mathcal{T}(n_1, \ldots, n_k)$ whenever $X \in \mathcal{T}(n_1, \ldots, n_k)$. Hence $\Phi\big(\mathcal{T}(n_1, \ldots, n_k)\big) \subseteq \mathcal{T}(n_1, \ldots, n_k)$. Since Φ is invertible, we have $\Phi\big(\mathcal{T}(n_1, \ldots, n_k)\big) = \mathcal{T}(n_1, \ldots, n_k)$. The proof of Assertion 2 is complete.

Assertion 3 Let U be the unitary matrix in (i) or (ii) of Assertion 1. If $A^- = A$, then $U \in \mathcal{T}(n_1, \ldots, n_k)$; if $A^- = A^t$, then $n_j = n_{k-j+1}$ for all $1 \leq j \leq k/2$ and $EU \in \mathcal{T}(n_1, \ldots, n_k)$. Consequently, $\phi(X) = \Phi(X)$ is of the asserted form.

Let $V = U$ or EU depending on whether $A^- = A$ or A^t. Since $\Phi\big(\mathcal{T}(n_1, \ldots, n_k)\big) = \mathcal{T}(n_1, \ldots, n_k)$, the mapping

$$\rho\colon \mathcal{T}(m_1, \ldots, m_k) \to \mathcal{T}(n_1, \ldots, n_k)$$

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defined by $\rho(A) = V^*AV$ is a linear isomorphism, where

$$(m_1, \dots, m_k) = \begin{cases} (n_1, \dots, n_k) & \text{if } A^- = A, \\ (n_k, \dots, n_1) & \text{if } A^- = A^t. \end{cases}$$

We prove by induction on k that we always have $n_j = m_j$ for $j = 1, \dots, k$, and $V = V_{11} \oplus \cdots \oplus V_{nn} \in \mathcal{T}(n_1, \dots, n_k)$.

The statement is clear if $k = 1$. Assume the statement is true for $k - 1$. Write

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & \hat{V}_{22} \end{pmatrix},$$

where V_{11} is an $m_1 \times m_1$ matrix. For all $m_1 \times m_1$ matrix B_{11}, let $B = (B_{11}\ 0 \cdots\ 0)V^*$. Then

$$V^* \begin{pmatrix} B \\ 0 \end{pmatrix} V = (V_1^* B_{11}\ 0 \cdots\ 0) \in \mathcal{T}(n_1, \dots, n_k).$$

Since the m_1 column vectors of V_1^* are linear independent, there exists B_{11} such that the first column of $V_1^* B_{11}$ has m_1 nonzero entries. Thus $n_1 \geq m_1$. Using the same argument for ρ^{-1}, we get $m_1 \geq n_1$. Hence $n_1 = m_1$. Note that now $V_{12}^* B_{11} = 0$ for all B_{11} hence $V_{12} = 0$. As a result, $V = V_{11} \oplus \hat{V}_{22}$. Applying the induction assumption to $\hat{V}_{22}$ and

$$\rho_1 \colon \mathcal{T}(m_2, \dots, m_k) \to \mathcal{T}(n_2, \dots, n_k)$$

defined by $\rho_1(A) = \hat{V}_{22}^* A \hat{V}_{22}$, we have $n_j = m_j$ for $j = 2, \dots, k$, and $\hat{V}_{22} = V_{22} \oplus \cdots \oplus V_{kk} \in \mathcal{T}(n_2, \dots, n_k)$. The result follows. ■

4 Generalized Numerical Radius Preservers

It is known (see *e.g.* [14]) that if $c = (c_1, \dots, c_n)$ is such that $c_1 \geq \cdots \geq c_n$ with $\sum_{j=1}^n c_j \neq 0$ and $c_1 > c_n$, then r_c is a norm on $\mathcal{M}_n$. Evidently, under the same assumption, r_c will also be a norm on $\mathcal{T}(n_1, \dots, n_k)$. The purpose of this section is to characterize the isometries for r_c when it is a norm on $\mathcal{T}(n_1, \dots, n_k)$. Again, we approach the problem by studying the dual transformations and prove the following result.

Theorem 4.1 *Let $c = (c_1, \dots, c_n)$ with $c_1 + \cdots + c_n \neq 0$. Then a linear operator ϕ on $\mathcal{T}(n_1, \dots, n_k)$ satisfies*

$$\phi\big(\widetilde{\mathcal{V}}(c)\big) = \widetilde{\mathcal{V}}(c)$$

if and only if there exists $\mu \in \mathbf{C}$ with $|\mu| = 1$ such that $\phi\big(\widetilde{\mathcal{U}}(c)\big) = \widetilde{\mathcal{U}}(c)$.

Proof Choose $A \in \widetilde{\mathcal{U}}(c)$, then $\phi(A) = \mu B$ for some $B \in \widetilde{\mathcal{U}}(c)$ and $|\mu| = 1$. We want to show that if $X \in \widetilde{\mathcal{U}}(c)$ then $\phi(X) = \mu Y$ for some $Y \in \widetilde{\mathcal{U}}(c)$.

Let $A_1 = A$ and $\{A_1, A_2, \ldots, A_l\} \subseteq \widetilde{\mathcal{U}}(c)$ be a basis of $\mathcal{T}(n_1, \ldots, n_k)$. Let $B_j \in \widetilde{\mathcal{U}}(c)$ and $|\mu_j| = 1$ be such that $\phi(A_j) = \mu_j B_j$ for $j = 1, \ldots, l$. Consider the linear functional $\sigma\colon \mathcal{T}(n_1, \ldots, n_k) \to \mathbf{C}$ defined by

$$\sigma(a_1 A_1 + \cdots + a_l A_l) = \sum_{j=1}^{l} a_j \mu_j \in \mathbf{C}.$$

Then there exists $J \in \mathcal{T}(n_1, \ldots, n_k)$ such that

$$(J, X) = \sigma(X) \quad \text{for all } X \in \mathcal{T}(n_1, \ldots, n_k).$$

For any $X = a_1 A_1 + \cdots + a_l A_l \in \widetilde{\mathcal{U}}(c)$, there exist $Y \in \widetilde{\mathcal{U}}(c)$ and $|\alpha| = 1$ such that

$$\alpha Y = \phi(X) = a_1 \mu_1 B_1 + \cdots + a_l \mu_l B_l.$$

Taking the traces of the two sides, we see that

$$\alpha \operatorname{tr} Y = \sum_{j=1}^{l} a_j \mu_j \operatorname{tr} B_j.$$

Since $\operatorname{tr} B_1 = \cdots = \operatorname{tr} B_l = \operatorname{tr} Y = c_1 + \cdots + c_n \neq 0$,

$$\alpha = \sum_{j=1}^{l} a_j \mu_j = (J, X).$$

Using this equality and Lemma 2.2, we see that $W_c(J) = \{(J, X) : X \in \widetilde{\mathcal{U}}(c)\}$ is a set of complex numbers of modulus 1. By the convexity of $W_c(J)$ (see [12], [15]), we deduce that $W_c(J)$ is a singleton. Hence for any $X \in \widetilde{\mathcal{U}}(c)$, $(J, X) = (J, A) = \mu$ and $\phi(X) = (J, X)Y = \mu Y$ as asserted. ∎

By Theorem 2.5, we can determine the structure of the c-numerical radii preservers.

Theorem 4.2 *Let $c = (c_1, \ldots, c_n)$ with $c_1 + \cdots + c_n \neq 0$. Then a linear operator ϕ on $\mathcal{T}(n_1, \ldots, n_k)$ satisfies*

$$r_c\bigl(\phi(A)\bigr) = r_c(A) \quad \textit{for all } A \in \mathcal{T}(n_1, \ldots, n_k)$$

if and only if there exists $\mu \in \mathbf{C}$ with $|\mu| = 1$ such that $\mu\phi$ preserves the c-numerical range.

It would be interesting to characterize the linear preservers of r_c even if it is not a norm on $\mathcal{T}(n_1, \ldots, n_k)$.

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