ON STRICT MONOTONICITY OF CONTINUOUS SOLUTIONS OF CERTAIN TYPES OF FUNCTIONAL EQUATIONS

J. Aczel

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1. It is a commonplace that F is continuous on the cartesian square of the range of f if f is continuous and satisfies

$$(1) \qquad f(x + y) = F(f(x), f(y)),$$

say, for all real x, y (cf. e.g. [2]). A.D. Wallace has kindly called my attention to the fact, that this is trivial only if f is (constant or) strictly monotonic and asked for a simple proof of the strict monotonicity of f. The following could serve as such: if on an interval f is continuous, nonconstant and satisfies (1), then f is strictly monotonic there. In fact, if f were not strictly monotonic, then there would exist two values s_1 and s_2 such that $f(s_1) = f(s_2)$, but then (see figure) there exist also two t_1, t_2 arbitrarily near to each other (i.e., $|t_2 - t_1|$ arbitrarily small) so that $f(t_1) = f(t_2)$. But then, from (1) with $x = t - t_1$, $y = t_2$ resp. $x = t - t_1$, $y = t_1$

$$f(t + (t_2 - t_1)) = f((t - t_1) + t_2) = F(f(t - t_1), f(t_2)) =$$

$$F(f(t - t_1), f(t_1)) = f((t - t_1) + t_1) = f(t)$$

i.e., f is periodic with the period $t_2 - t_1$. But then f, being a continuous function with arbitrarily small periods, is constant, against the supposition, and this proves our assertion.

A similar argument was used in [3] (cf. [2], [6], [4]) to

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prove that all continuous nonconstant solutions of

(2) $$f(x + y) = F(x, f(y))$$

are strictly monotonic, and also the equation

(3) $$f(y + zf(y)) = f(y)f(z)$$

was partly handled in [5] (cf. [2_2]) with the aid of this argument but only for $f(t) \not\equiv 0$. This restriction becomes natural if, in order to get the form of (3) more similar to that of (1) or (2) we write in (3) $x = zf(y)$, $z = x/f(y)$ and get (cf. [1])

(4) $$f(x + y) = f(x/f(y))f(y).$$

The restriction of non-nullity can be removed altogether if we denote $f^*(t) = 1/f(t)$ in (4) and get

(5) $$f^*(x + y) = f^*(xf^*(y))\, f^*(y)$$

for which again an argument similar to that applied to (1) can be used in order to get the result that all solutions of (5) non-constant and continuous on an interval are strictly monotonic.

2. Now, "an idea applied once is a trick, an idea applied twice is a method" ([7]), and here we see an idea applied (at least) three times, so there might be a point in stating it as a method or giving a broad class of functional equations for which it can be applied.

This we do by proving the following

THEOREM. If on an interval f is continuous and satisfies a functional equation of the form

(6) $$\begin{aligned} f(x + y) = F(x, f(x), f(y), f(G(x, f(x), f(y))), f(H(x, f(x), f(y))), \ldots, \\ f(I(x, f(x), f(y), f(K(x, f(x), f(y))), f(L(x, f(x), f(y))), \ldots)), \ldots) \end{aligned}$$

(F, G, H, ... defined on this interval for their first and on the range of f for their remaining variables) then f is either constant or strictly monotonic there. (The form (6) indicates that, on the right hand side, any combination of x, $f(x)$, $f(y)$ can be put again into f and so on, only y does not figure outside of $f(y)$.)

Proof. If f is not strictly monotonic, then there exist s_1, s_2 so that $f(s_1) = f(s_2)$ and then (see figure) also t_1, t_2 with arbitrarily small $|t_2 - t_1|$ such that $f(t_1) = f(t_2)$ and so from (6) with $x = t - t_1$, $y = t_2$ and $x = t - t_1$, $y = t_1$ respectively

$$\begin{aligned}
&f(t + (t_2 - t_1)) = f((t - t_1) + t_2) = F(t - t_1, f(t - t_1) f(t_2),\\
&f(G(t - t_1, f(t - t_1), f(t_2))), f(H(t - t_1, f(t - t_1), f(t_2))), \ldots,\\
&f(I(t - t_1, f(t - t_1), f(t_2), f(K(t - t_1, f(t - t_1), f(t_2))),\\
&f(L(t - t_1, f(t - t_1), f(t_2))), \ldots)), \ldots) = F(t - t_1, f(t - t_1), f(t_1),\\
&f(G(t - t_1, f(t - t_1), f(t_1))), f(H(t - t_1, f(t - t_1), f(t_1))), \ldots,\\
&f(I(t - t_1, f(t - t_1), f(t_1), f(K(t - t_1, f(t - t_1), f(t_1))),\\
&f(L(t - t_1, f(t - t_1), f(t_1))), \ldots)), \ldots) = f((t - t_1) + t_1) = f(t),
\end{aligned}$$

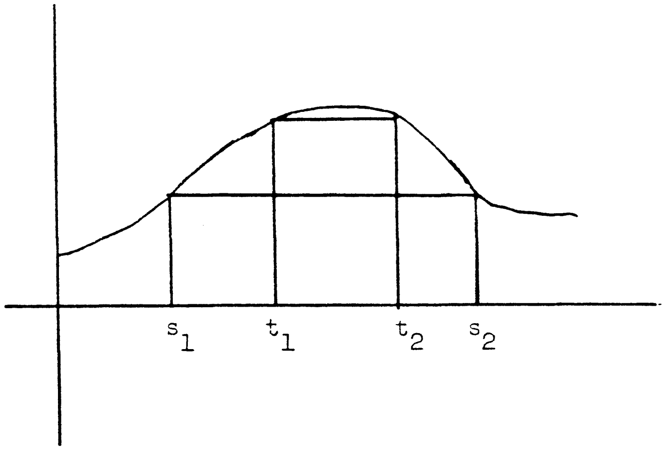
so that f is periodic with arbitrarily small periods and continuous, and therefore constant, q.e.d.

Equations (1), (2), (4), (5) are evidently of the form (6).

The same proof shows, that f is either constant or strictly monotonic on an interval if it is continuous there and satisfies an equation of the form

$$\begin{aligned}
&f(x+y) = F(x, g(x), f(y), h(G(x, i(x), f(y))), j(H(x, k(x), f(y))), \ldots,\\
&m(I(x, n(x), f(y), p(J(x, q(x), f(y))), r(K(x, s(x), f(y))), \ldots)), \ldots).
\end{aligned}$$

Observe, that no regularity suppositions were made for $g, h, i, j, k, m, n, p, q, r, s$ and for $F, G, H, I, J, K, L, \ldots$ (in either of the theorems).



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University of Waterloo