NOTE ON A STIELTJES TYPE OF INVERSION

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If $F(z)$ is an analytic function for $z \notin [-\infty, -1]$, $g(t)$ of bounded variation and real valued for $0 \leqslant t \leqslant 1$ and

$$F(z) = \int_0^1 (1 + zt)^{-1} dg(t),$$

then the Stieltjes type of inversion between $F(z)$ and $g(t)$ (cf. **1**, p. 339, Theorem 7a) is

$$\lim_{y \to 0+} \frac{-1}{\pi} \int_v^u \frac{1}{t} I_m F\left(-\frac{1}{t} + iy\right) dt = \frac{g(u+) + g(u-)}{2} - \frac{g(v+) + g(v-)}{2},$$

where $0 \leqslant v < u \leqslant 1$, $I_m F(z)$ is the imaginary part of $F(z)$ and $z = -t^{-1} + iy$.

A second type of inversion between $F(z)$ and $g(t)$ was obtained by Widder (**1**, p. 340, Theorem 7b) under the additional hypothesis that $g(t)$ is an absolutely continuous function. In the following theorem we shall establish an inversion between $F(z)$ and the right- and left-hand derivatives of $g(t)$ without the restriction that $g(t)$ be an integral.

THEOREM. *Let $F(z)$ be analytic for $z \notin [-\infty, -1]$, $g(t)$ real valued and of bounded variation on* $[0, 1]$ *and*

$$F(z) = \int_0^1 (1 + zt)^{-1} dg(t), \tag{1}$$

then

$$\lim_{y \to 0+} \frac{-1}{\pi t} I_m F\left(\frac{-1}{t} + iy\right) = \frac{g'^{+}(t) + g'^{-}(t)}{2}$$

for any t in $(0, 1)$ *at which the right- and left-hand derivatives $g'^{+}(t)$ and $g'^{-}(t)$ exist.*

Proof. Let us suppose that $g(0) = 0$, $0 < t_0 < 1$ and that $g'^{+}(t_0)$ and $g'^{-}(t_0)$ exist. If we set

$$R(t) = [(t_0 - t)^2 + (t_0 y t)^2]^{-1}$$

and $s = t_0 \pi^{-1}$, then from (1) we have

$$\frac{-1}{\pi t_0} I_m F\left(\frac{-1}{t} + iy\right) = sy \int_0^{t_0} tR(t) dg(t) + sy \int_{t_0}^1 tR(t) dg(t). \tag{2}$$

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In the first integral of this expression we can replace $g(t)$ by $[g(t_0) + g'^{-}(t_0)(t - t_0) + h(t)(t - t_0)]$, where $h(t)$ is continuous at t_0 and $h(t_0) = 0$, so that

$$sy \int_0^{t_0} tR(t)dg(t) = syg'^{-}(t_0) \int_0^{t_0} tR(t)dt + sy \int_0^{t_0} tR(t)dh(t)(t - t_0).$$

The first term on the right side of this equation can be integrated directly and we can easily verify that it approaches $2^{-1}g'^{-}(t_0)$ as y approaches $0+$. Upon using the integration by parts formula, the second term reduces to

$$- sy \int_0^{t_0} h(t)(t - t_0)[R(t) + tR'(t)]dt.$$

If J denotes the value of the last expression, then

$$|J| < \frac{2y}{\pi} \int_0^{t_0} |h(t)|R(t)dt.$$

For each $\epsilon > 0$, there exists $\gamma > 0$ such that $|h(t)| < \frac{1}{2}\epsilon t_0^2$ for $t_0 - t < \gamma$, so that

$$\frac{2y}{\pi} \int_{t_0-\gamma}^{t_0} |h(t)|R(t)dt < \frac{2\epsilon y t_0^2}{\pi} \int_0^1 R(t)dt < \epsilon$$

for $y > 0$. Since $h(t)$ is a bounded function, there exists $\gamma' > 0$ such that, for $y < \gamma'$,

$$\frac{2y}{\pi} \int_0^{t_0-\gamma} |h(t)|R(t)dt < \epsilon.$$

In order to treat the second integral appearing in (2) we replace $g(t)$ by

$$[g(t_0) + g'^{+}(t_0)(t - t_0) + k(t)(t - t_0)]$$

and proceed as above. However, in this case the integration by parts formula yields the additional term

$$\pi^{-1}y[k(1)(1 - t_0)/t_0^2(1 + y^2)]$$

which approaches zero with y. This completes the proof of the Theorem.

Remark added in the revision. I am indebted to the referee for suggesting the revised form of the Theorem. Also, as he points out, the Theorem is valid if we replace the interval of integration $[0, 1]$ by the ray $[0, \infty]$ and restrict z so that $z \notin [-\infty, 0]$.

Reference

1. D. V. Widder, *The Laplace Transform*, (Princeton, 1946).

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