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ASYMPTOTIC STABILITY OF $\mathrm{ATT}_R\mathrm{TOR}_1^R((R/\mathfrak{A}^n), A)$

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Abstract Let R be a commutative ring. Let M respectively A denote a Noetherian respectively Artinian R-module, and $\mathfrak{a}$ a finitely generated ideal of R. The main result of this note is that the sequence of sets $(\mathrm{Att}_R\mathrm{Tor}_1^R((R/\mathfrak{a}^n), A))_{n\in\mathbb{N}}$ is ultimately constant. As a consequence, whenever R is Noetherian, we show that $\mathrm{Ass}_R\mathrm{Ext}_R^1((R/\mathfrak{a}^n), M)$ is ultimately constant for large n, which is an affirmative answer to the question that was posed by Melkersson and Schenzel in the case $i = 1$.

Keywords: attached prime; associated prime; Tor; Ext

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1. Introduction

Let R be a commutative ring with identity, $\mathfrak{a}$ an ideal in R, and M a Noetherian R-module. It follows (see [**1**]) that the sequence of sets $\mathrm{Ass}_R(M/\mathfrak{a}^n M)$ is ultimately constant for large n. Assume A is an Artinian R-module. Dual to this result, Sharp has shown in [**6**, **7**] that the sequence of sets $\mathrm{Att}_R(0 :_A \mathfrak{a}^n)$ is ultimately constant for large n. Recently, in [**3**], Melkersson and Schenzel showed, in the case where R is Noetherian, that for each i the set of prime ideals $\mathrm{Ass}_R \mathrm{Tor}_i^R((R/\mathfrak{a}^n), M)$ and $\mathrm{Att}_R \mathrm{Ext}_R^i((R/\mathfrak{a}^n), A)$ become, for n large, independent of n. They also asked whether the sets $\mathrm{Ass}_R \mathrm{Ext}_R^i((R/\mathfrak{a}^n), M)$ become stable for sufficiently large n. The aim of this note is to show that, for a finitely generated ideal $\mathfrak{a}$ of R, the sequence of sets $\mathrm{Att}_R \mathrm{Tor}_1^R((R/\mathfrak{a}^n), A)$ is ultimately constant for large n. This implies, under the Noetherian hypothesis on R, that the sequence of sets $\mathrm{Ass}_R \mathrm{Ext}_R^1((R/\mathfrak{a}^n), M)$ become stable for sufficiently large n, which is an affirmative answer to the above question in the case $i = 1$.

Throughout this note, R will denote a commutative ring with identity and $\mathfrak{a}$ a finitely generated ideal of R. Also, M (respectively A) will denote a Noetherian (respectively Artinian) R-module. We use $\mathbb{N}$ to denote the set of positive integers.

2. The results

For a positive integer n, we use $f_{n,A}$ to denote the natural homomorphism from

$$\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n+1}}, A\right) \quad \text{to} \quad \operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n}}, A\right).$$

Note that if $\mathfrak{a}$ is a finitely generated ideal of R and A is an Artinian R-module, then $\operatorname{Tor}_1^R((R/\mathfrak{a}^n), A)$ is also an Artinian R-module. We say that $x \in \mathfrak{a}$ is an A-coregular element if $xA = A$. We start with the following lemma.

Lemma 2.1. *Let $\mathfrak{a}$ contain an A-coregular element. Then*

(i) *$f_{n,A}$ is epimorphism for all $n \in \mathbb{N}$; and*

(ii) *$\operatorname{Att}_R \operatorname{Tor}_1^R((R/\mathfrak{a}^n), A) = \operatorname{Att}_R \operatorname{Tor}_1^R((R/\mathfrak{a}^{n+1}), A)$ for all sufficiently large n.*

Proof. Let $x \in \mathfrak{a}$ be an A-coregular element and let $n \in \mathbb{N}$. Then, using the exact sequence

$$0 \to (0 :_A x^{n+1}) \to A \xrightarrow{x^{n+1}} A \to 0,$$

we obtain a commutative square:

$$\begin{array}{ccc} \operatorname{Tor}_1^R\left(\dfrac{R}{\mathfrak{a}^{n+1}}, A\right) & \longrightarrow & \dfrac{R}{\mathfrak{a}^{n+1}} \otimes_R (0 :_A x^{n+1}) \\ \Big\downarrow f_{n,A} & & \Big\downarrow \\ \operatorname{Tor}_1^R\left(\dfrac{R}{\mathfrak{a}^{n}}, A\right) & \longrightarrow & \dfrac{R}{\mathfrak{a}^{n}} \otimes_R (0 :_A x^{n+1}) \end{array}$$

in which the rows are isomorphism and the right vertical arrow is an epimorphism. Hence $f_{n,A}$ is an epimorphism. Now, in order to prove (ii), consider the exact sequence $0 \to (0 :_A x) \to A \xrightarrow{x} A \to 0$ to deduce the commutative diagram:

$$\begin{array}{ccccccc} \operatorname{Tor}_1^R\left(\dfrac{R}{\mathfrak{a}^{n+1}}, A\right) & \xrightarrow{x} & \operatorname{Tor}_1^R\left(\dfrac{R}{\mathfrak{a}^{n+1}}, A\right) & \longrightarrow & \dfrac{R}{\mathfrak{a}^{n+1}} \otimes_R (0 :_A x) & \longrightarrow & 0 \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ \operatorname{Tor}_1^R\left(\dfrac{R}{\mathfrak{a}^{n}}, A\right) & \xrightarrow{x} & \operatorname{Tor}_1^R\left(\dfrac{R}{\mathfrak{a}^{n}}, A\right) & \longrightarrow & \dfrac{R}{\mathfrak{a}^{n}} \otimes_R (0 :_A x) & \longrightarrow & 0 \end{array}$$

in which the rows are exact and, for sufficiently large n, the right vertical arrow is an isomorphism. Hence, by (i), it is enough to show that

$$\operatorname{Att}_R \operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n+1}}, A\right) \subseteq \operatorname{Att}_R\left(\frac{R}{\mathfrak{a}^{n+1}} \otimes_R (0 :_A x)\right).$$

Assume the contrary. Let $\mathfrak{p} \in \operatorname{Att}_R(T) \setminus \operatorname{Att}_R((R/\mathfrak{a}^{n+1}) \otimes_R (0 :_A x))$, where $T = \operatorname{Tor}_1^R((R/\mathfrak{a}^{n+1}), A)$. Then there exists a submodule L of T such that $\mathfrak{p} = \sqrt{(0 :_R (T/L))}$ and $xT + L = T$. Since $x^{n+1}T = 0$, it is routine to check that $xT \subseteq L$. Therefore $L = T$, which is the required contradiction. □

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Theorem 2.2. *The set* $\mathrm{Att}_R\,\mathrm{Tor}_1^R((R/\mathfrak{a}^n),A)$ *is ultimately constant for large* n.

Proof. Let k be a positive integer. Our first aim is to show that the sequence of sets $\mathrm{Att}_R\,\mathrm{Tor}_1^R((R/\mathfrak{a}^n),(A/\mathfrak{a}^kA))$ are, for all sufficiently large n, independent of n. Let $n\in\mathbb{N}$. Then the exact sequence $0\to\mathfrak{a}^n\to R\to(R/\mathfrak{a}^n)\to 0$ induces an exact sequence

$$0\to\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n},\frac{A}{\mathfrak{a}^kA}\right)\to\left(\mathfrak{a}^n\otimes_R\frac{A}{\mathfrak{a}^kA}\right)\to\left(R\otimes_R\frac{A}{\mathfrak{a}^kA}\right)\to\left(\frac{R}{\mathfrak{a}^n}\otimes_R\frac{A}{\mathfrak{a}^kA}\right)\to 0.$$

It follows that $\mathrm{Tor}_1^R((R/\mathfrak{a}^n),(A/\mathfrak{a}^kA))\cong\mathfrak{a}^n\otimes_R(A/\mathfrak{a}^kA)$ for sufficiently large n, since, for large n,

$$\frac{A}{\mathfrak{a}^kA}\cong\frac{R}{\mathfrak{a}^n}\otimes_R\frac{A}{\mathfrak{a}^kA}.$$

Now, by using [**4**, Proposition 5.2], we can deduce that

$$\mathrm{Att}_R\,\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n},\frac{A}{\mathfrak{a}^kA}\right)=\mathrm{Supp}_R(\mathfrak{a}^n)\cap\mathrm{Att}_R\,\frac{A}{\mathfrak{a}^kA},$$

which stabilizes for large n. Let $k\in\mathbb{N}$ be such that $\mathfrak{a}^kA=\mathfrak{a}^{k+1}A$. Hence there exists $t_1\in\mathbb{N}$ such that

$$\mathrm{Att}_R\,\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n},\frac{A}{\mathfrak{a}^kA}\right)=\mathrm{Att}_R\,\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{t_1}},\frac{A}{\mathfrak{a}^kA}\right)\tag{2.1}$$

for all $n\geqslant t_1$. On the other hand, by [**2**, Theorem 2] and the above lemma, there exists $t_2\in\mathbb{N}$ such that

$$\mathrm{Att}_R\,\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n},\mathfrak{a}^kA\right)=\mathrm{Att}_R\,\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{t_2}},\mathfrak{a}^kA\right)\tag{2.2}$$

for all $n\geqslant t_2$. Set $t:=\max\{t_1,t_2\}$. Let $n\in\mathbb{N}$ be such that $n\geqslant t$. Then we claim that

$$\mathrm{Att}_R\,\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n},A\right)\subseteq\mathrm{Att}_R\,\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n+1}},A\right).$$

To see this, consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n+1}},\mathfrak{a}^kA\right) & \longrightarrow & \mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n+1}},A\right) & \longrightarrow & \mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n+1}},\frac{A}{\mathfrak{a}^kA}\right) & \longrightarrow & 0\\
\downarrow{\scriptstyle f_{n,\mathfrak{a}^kA}} & & \downarrow{\scriptstyle f_{n,A}} & & & & \\
\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n}},\mathfrak{a}^kA\right) & \xrightarrow{h} & \mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n}},A\right) & \longrightarrow & \mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n}},\frac{A}{\mathfrak{a}^kA}\right) & \longrightarrow & 0
\end{array}$$

in which the rows are exact and the left vertical map is an epimorphism. Let $\mathfrak{p}\in\mathrm{Att}_R\,\mathrm{Tor}_1^R((R/\mathfrak{a}^n),A)$. Then there exists a submodule N of $\mathrm{Tor}_1^R((R/\mathfrak{a}^n),A)$ such that $\mathfrak{p}=\sqrt{(0:_R(\mathrm{Tor}_1^R((R/\mathfrak{a}^n),A))/N)}$. If $\mathfrak{p}\in\mathrm{Att}_R\,\mathrm{Tor}_1^R((R/\mathfrak{a}^n),(A/\mathfrak{a}^kA))$, then we have nothing to do any more. So, suppose that $\mathfrak{p}\notin\mathrm{Att}_R\,\mathrm{Tor}_1^R((R/\mathfrak{a}^n),(A/\mathfrak{a}^kA))$. Thus

$$h\left(\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n},\mathfrak{a}^kA\right)\right)+N=\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n},A\right).$$

Since, by the above lemma, $f_{n,\mathfrak{a}^k A}$ is an epimorphism, it is easy to see that

$$\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n}, A\right) = N + f_{n,A}\left(\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^{n+1}}, A\right)\right).$$

Hence $\mathfrak{p} \in \operatorname{Att}_R \operatorname{Tor}_1^R((R/\mathfrak{a}^{n+1}), A)$ and the claim follows. Now, use the exact sequence

$$\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n}, \mathfrak{a}^k A\right) \to \operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n}, A\right) \to \operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n}, \frac{A}{\mathfrak{a}^k A}\right) \to 0,$$

in conjunction with (2.1) and (2.2), to deduce that the set $\operatorname{Att}_R \operatorname{Tor}_1^R((R/\mathfrak{a}^n), A)$ contained in the finite set

$$\operatorname{Att}_R \operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^t}, \mathfrak{a}^k A\right) \cup \operatorname{Att}_R \operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^t}, \frac{A}{\mathfrak{a}^k A}\right).$$

The proof now follows from the above claim. □

Let E be the injective hull of the direct sum of all the $R/\mathfrak{m}$, with $\mathfrak{m}$ a maximal ideal of R. In the following corollary, we denote the Matlis duality functor $\operatorname{Hom}_R(\cdot, E)$ by $D(\cdot)$.

Corollary 2.3. *Let R be Noetherian and let $\mathfrak{a}$ be an ideal of R. Then*

$$\operatorname{Ass}_R \operatorname{Ext}_R^1\left(\frac{R}{\mathfrak{a}^n}, M\right)$$

is ultimately constant for large n.

Proof. By [**5**, Theorem 1.6(2)], $D(M)$ is an Artinian R-module. Hence, by the above theorem, the sequence of sets $\operatorname{Att}_R \operatorname{Tor}_1^R((R/\mathfrak{a}^n), D(M))$ are, for all sufficiently large n, independent of n. Now the result follows from the isomorphism

$$\operatorname{Tor}_1^R\left(\frac{R}{\mathfrak{a}^n}, D(M)\right) \cong D\left(\operatorname{Ext}_R^1\left(\frac{R}{\mathfrak{a}^n}, M\right)\right)$$

and [**8**, 2.1]. □

The following remark will provide a direct proof of the above corollary. Its idea was sketched by the referee.

Remark 2.4. Let $H_{\mathfrak{a}}^0(M)$ denote the 0th local cohomology of M with respect to $\mathfrak{a}$. Then the short exact sequence $0 \to \mathfrak{a}^n \to R \to (R/\mathfrak{a}^n) \to 0$ provides, for large n, an isomorphism

$$\operatorname{Hom}_R(\mathfrak{a}^n, H_{\mathfrak{a}}^0(M)) \cong \operatorname{Ext}_R^1(R/\mathfrak{a}^n, H_{\mathfrak{a}}^0(M)).$$

That is, $\operatorname{Ass}_R \operatorname{Ext}_R^1(R/\mathfrak{a}^n, H_{\mathfrak{a}}^0(M))$ becomes ultimately constant. Now the short exact sequence $0 \to H_{\mathfrak{a}}^0(M) \to M \to M' \to 0$ provides, for large n, an exact sequence

$$0 \to \operatorname{Ext}_R^1\left(\frac{R}{\mathfrak{a}^n}, H_{\mathfrak{a}}^0(M)\right) \to \operatorname{Ext}_R^1\left(\frac{R}{\mathfrak{a}^n}, M\right) \to \operatorname{Ext}_R^1\left(\frac{R}{\mathfrak{a}^n}, M'\right).$$

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So it will be enough to prove that the claim in case M admits an M-regular element $x \in \mathfrak{a}$. Under this additional circumstance there is an isomorphism

$$\mathrm{Ext}_R^1\left(\frac{R}{\mathfrak{a}^n}, M\right) \cong \mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}^n}, \frac{M}{x^n M}\right)$$

for all $n \geqslant 1$. Because x is M-regular it follows that

$$\mathrm{Ass}_R\, \mathrm{Ext}_R^1\left(\frac{R}{\mathfrak{a}^n}, M\right) = \mathrm{Ass}\left(\frac{M}{xM}\right) \cap V(\mathfrak{a}),$$

which is independent of n.

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