LINEAR FRACTIONAL TRANSFORMS OF COMPANION MATRICES

by N. J. YOUNG

(Received 25 November, 1977)

Questions about polynomials can be turned into questions about matrices by associating with the polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n \qquad (a_n \neq 0) \tag{1}$$

(over an arbitrary field) its companion matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_0/a_n & -a_1/a_n & -a_2/a_n & \cdots & -a_{n-1}/a_n \end{bmatrix} \tag{2}$$

which has p/a_n as its characteristic polynomial. This technique is often used in stability theory, as indicated in [**1**]; companion matrices also occur in the theory of the rational canonical form.

In the study of root location for polynomials a common device is the use of the linear fractional transformation

$$\phi(z) = (\alpha z + \beta)/(\gamma z + \delta) \qquad (\alpha\delta - \beta\gamma \neq 0). \tag{3}$$

If one passes from p to the polynomial

$$q(z) = (-\gamma z + \alpha)^n p \circ \phi^{-1}(z) \tag{4}$$

one obtains a new polynomial q whose roots are the images under ϕ of the roots of p. Thus the matrix

$$\phi(T) = (\alpha T + \beta I)(\gamma T + \delta I)^{-1}, \tag{5}$$

provided it is defined, has the roots of q as its eigenvalues, and with the right multiplicities. However, $\phi(T)$ is not in general the companion matrix of q, as virtually any example will show. How, then, can one obtain the companion matrix of q from $\phi(T)$? It is a remarkable fact, proved by Shane and Barnett [**1**], that there is an $n \times n$ matrix M_ϕ, depending only on n and ϕ, such that $M_\phi \phi(T) M_\phi^{-1}$ is a companion matrix for every companion matrix T. The proof in [**1**] proceeds by laborious computation of entries: indeed, the complexity is such that at one point the authors carry out the calculation for $n = 3$ and leave the reader to convince himself that the procedure works in higher dimensions. The purpose of this note is to provide a proof which is simple and at the same time shows how to obtain M_ϕ more directly than by the method of [**1**].

Glasgow Math. J. **20** (1979) 129–132.

THEOREM. *Let M_ϕ be the $n\times n$ matrix $[m_{ij}]$, where, for $i=1,2,\dots,n$,*

$$\sum_{j=1}^{n} m_{ij}z^{j-1}=(\gamma z+\delta)^{n-i}(\alpha z+\beta)^{i-1}, \tag{6}$$

ϕ being given by (3). Then M_ϕ is non-singular and, for any companion matrix T such that $\gamma T+\delta I$ is non-singular, $M_\phi\phi(T)M_\phi^{-1}$ is again a companion matrix.

In fact (6) and (3) determine M_ϕ only up to a scalar multiple, but this is clearly unimportant.

We shall use tensor product notation: if P, Q are $n\times n$ matrices then $P\otimes Q$ denotes the $n^2\times n^2$ matrix which, written in block form, has $p_{ij}Q$ as the $n\times n$ block in the (i,j) position, where $P=[p_{ij}]$. It is also convenient to introduce the notation

$$H(P)=\begin{bmatrix} I \\ P \\ P^2 \\ \vdots \\ P^{n-1}\end{bmatrix}, \tag{7}$$

so that $H(P)$ is of type $n^2\times n$.

The proof of the theorem is based on the following characterization of companion matrices.

LEMMA. *Let B be an $n\times n$ matrix whose minimal polynomial p is of degree n. An $n\times n$ matrix X is the companion matrix of p if and only if*

$$H(B)B=(X\otimes I)H(B). \tag{8}$$

The proof is immediate on writing down both sides of (8) in block form.

Proof of Theorem. Let T be a companion matrix such that $\phi(T)$ is defined, and write $B=\phi(T)$. One can see by considering the first row that the matrices $I, T,\dots,T^{n-1}$ are linearly independent, and since A and $\phi(A)$ have minimal polynomials of the same degree, for any A, it follows that B also has minimal polynomial of degree n.

Equation (8) implies in particular that

$$H(T)T=(T\otimes I)H(T)$$

and hence

$$H(T)(\alpha T+\beta I)=((\alpha T+\beta I)\otimes I)H(T). \tag{9}$$

Replace α,β by γ,δ respectively, premultiply by $(\gamma T+\delta I)^{-1}\otimes I$ and postmultiply by $(\gamma T+\delta I)^{-1}$ to obtain

$$H(T)(\gamma T+\delta I)^{-1}=((\gamma T+\delta I)^{-1}\otimes I)H(T).$$

Combining this with (9), we have

$$H(T)B=(B\otimes I)H(T). \tag{10}$$

From the definition of M_ϕ one has

$$H(B)(\gamma T+\delta I)^{n-1}=\begin{bmatrix}(\gamma T+\delta I)^{n-1}\\ (\gamma T+\delta I)^{n-2}(\alpha T+\beta I)\\ \cdots\\ (\alpha T+\beta I)^{n-1}\end{bmatrix}$$

$$=(M_\phi\otimes I)H(T). \tag{11}$$

Since B commutes with $(\gamma T+\delta I)$, we have

$$H(B)B=(M_\phi\otimes I)H(T)B(\gamma T+\delta I)^{-n+1},$$

and now (10) yields

$$\begin{aligned}H(B)B&=(M_\phi\otimes I)(B\otimes I)H(T)(\gamma T+\delta I)^{-n+1}\\ &=(M_\phi B\otimes I)H(T)(\gamma T+\delta I)^{-n+1}.\end{aligned} \tag{12}$$

Suppose for the moment that M_ϕ is non-singular: then (11) tells us that

$$H(T)(\gamma T+\delta I)^{-n+1}=(M_\phi^{-1}\otimes I)H(B),$$

and, combining this with (12), we find that

$$H(B)B=(M_\phi BM_\phi^{-1}\otimes I)H(B).$$

An application of the lemma now shows that $M_\phi BM_\phi^{-1}$ is a companion matrix, as was claimed.

It remains to show that M_ϕ is non-singular. This is most neatly accomplished by proving that $\phi\to M_\phi$ is a representation of GL(2). To see this write the definition (6) of M_ϕ in the form

$$M_\phi\begin{bmatrix}1\\ z\\ \vdots\\ z^{n-1}\end{bmatrix}=(\gamma z+\delta)^{n-1}\begin{bmatrix}1\\ \phi(z)\\ \vdots\\ \phi(z)^{n-1}\end{bmatrix}.$$

Then, if $\phi'(z)=(\alpha' z+\beta')(\gamma' z+\delta')^{-1}$, we have

$$M_{\phi'}M_\phi\begin{bmatrix}1\\ z\\ \vdots\\ z^{n-1}\end{bmatrix}=(\gamma z+\delta)^{n-1}(\gamma'\phi(z)+\delta')^{n-1}\begin{bmatrix}1\\ \phi'\circ\phi(z)\\ \vdots\\ \phi'\circ\phi(z)^{n-1}\end{bmatrix}$$

$$=\{(\gamma'\alpha+\delta'\gamma)z+\gamma'\beta+\delta'\delta\}^{n-1}\begin{bmatrix}1\\ \phi'\circ\phi(z)\\ \vdots\\ \phi'\circ\phi(z)^{n-1}\end{bmatrix}=M_{\phi'\circ\phi}\begin{bmatrix}1\\ z\\ \vdots\\ z^{n-1}\end{bmatrix}.$$

https://doi.org/10.1017/S0017089500003839 Published online by Cambridge University Press

Thus $M_{\phi'}M_{\phi}=M_{\phi'\circ\phi}$, and it is easy to see that if $\phi(z)=z$, then $M_{\phi}=I$. It follows that M_{ϕ} is non-singular, and $M_{\phi}^{-1}=M_{\phi^{-1}}$ (to make this precise we should normalize by requiring $\alpha\delta-\beta\gamma=1$). This completes the proof of the theorem, and also shows how to obtain M_{ϕ}^{-1} easily: the entries in the ith row of M_{ϕ}^{-1} are the coefficients in the polynomial

$$(\alpha\delta-\beta\gamma)^{-n+1}(-\gamma z+\alpha)^{n-i}(\delta z-\beta)^{i-1}.$$

We can re-state our conclusions in the following way. Write $\tilde{\phi}(X)=M_{\phi}\phi(X)M_{\phi}^{-1}$, whenever $\phi(X)$ is defined. The mapping $\phi\to\tilde{\phi}$ is an isomorphism of GL(2) onto a group of (non-linear) transformations of the space of $n\times n$ matrices which leaves invariant the set of companion matrices. When the field in question is **C** the elements of the image group are bianalytic transformations.

Notice a special case: if $\phi(z)=1/z$, then it is easy to see that M_{ϕ} is the matrix with ones on the principal cross-diagonal and zeros everywhere else. It follows that if T is the companion matrix of a polynomial p for which $p(0)\neq 0$ then T^{-1} is the matrix obtained by reversing the order of both rows and columns in the companion matrix of the polynomial $z^n p(1/z)$. This fact can easily be checked directly.

REFERENCE

1. B. A. Shane and S. Barnett, On the bilinear transformation of companion matrices, *Linear Algebra and Appl.* **9** (1974), 175–184.

University of Glasgow
University Gardens
Glasgow
Scotland
G12 8QW

https://doi.org/10.1017/S0017089500003839 Published online by Cambridge University Press