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SMOOTH DERIVATIONS ON ABELIAN C^*-DYNAMICAL SYSTEMS

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Abstract

Let $(\mathbf{A}, \mathbb{R}, \sigma)$ be an abelian C^*-dynamical system. Denote the generator of σ by δ_0 and define $\mathbf{A}_\infty = \bigcap_{n \geq 1} D(\delta_0^n)$. Further define the Lipschitz algebra

$$\mathbf{A}_{1/2} = \left\{ f \in \mathbf{A};\ \sup_{|t|>0} \|(\sigma_t f - f)/t\| < +\infty \right\}.$$

If δ is a $*$-derivation from $\mathbf{A}_\infty$ into $\mathbf{A}_{1/2}$, then it follows that δ is closable, and its closure generates a strongly continuous one-parameter group of $*$-automorphisms of $\mathbf{A}$. Related results for local dissipations are also discussed.

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1. Introduction

Let $\mathbf{A} = C_0(X)$ be an abelian C^*-algebra, let $t \in \mathbb{R} \mapsto \sigma_t$ be a strongly continuous one-parameter group of $*$-automorphisms of $\mathbf{A}$ with generator δ_0, and set $\mathbf{A}_n = D(\delta_0^n)$, $\mathbf{A}_\infty = \bigcap_{n \geq 1} D(\delta_0^n)$. Our aim is to continue the investigation [1], [2], [3] of the structure of $*$-derivations from $\mathbf{A}_\infty$ into $\mathbf{A}$. To describe the elements of this investigation, it is necessary to introduce a number of additional concepts.

First let S denote the group of homeomorphisms of X associated with σ by

$$(\sigma_t f)(\omega) = f(S_t \omega),$$

where $f \in C_0(X)$, $t \in \mathbb{R}$, and $\omega \in X$. Next define the fixed point set X_0 by

$$X_0 = \{\omega \in X;\ S_t\omega = \omega \text{ for all } t \in \mathbb{R}\}.$$

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Further, introduce the period p of each $\omega \in X$ by

$$p(\omega) = \inf\{t > 0;\ S_t\omega = \omega\},$$

and the frequency ν by $\nu(\omega) = 1/p(\omega)$. Thus X_0 corresponds to the set of points with period zero.

Now in [1] it was established that δ is a *-derivation from $\mathbf{A}_\infty$ into $\mathbf{A}$ if, and only if, $\delta = \lambda\delta_0$, where λ denotes multiplication by a real function which vanishes on X_0, is continuous over $X \setminus X_0$, and is polynomially bounded in the frequency, i.e.

$$|\lambda(\omega)| \leqslant c\left(1 + \nu(\omega)^k\right)$$

for some $c > 0$ and $k \geqslant 0$, and for all $\omega \in X \setminus X_0$. (For an earlier partial result see [4], and for an alternative derivation of the polynomial bounds see [2].) Note that the representation $\delta = \lambda\delta_0$ implies that both $\pm\delta$ are dissipative, and in particular that δ is closable. Let $\bar{\delta}$ denote its closure.

It was also established in [1] that δ maps $\mathbf{A}_\infty$ into $\mathbf{A}_n$ if, and only if, $\lambda \in D(\delta_0^n)$ and $\delta_0^m\lambda$ is polynomially bounded in the frequency for all $0 \leqslant m \leqslant n$. Note that here δ_0 is defined as the derivative at the origin of σ extended to $C(X)$ by the definition $(\sigma_t f)(\omega) = f(S_t\omega)$.

Finally, in [3] it was proved that if σ maps $\mathbf{A}_\infty$ into $\mathbf{A}_1$, then $\bar{\delta}$ is automatically the generator of a strongly continuous one-parameter group of *-automorphisms. The principal aim of this paper is to generalize and optimize this last statement. A key feature of this generalization is the Lipschitz algebra

$$\mathbf{A}_{1/2} = \left\{ f \in \mathbf{A};\ \sup_{|t| \geqslant 0} \|(\sigma_t f - f)/t\| < +\infty \right\}.$$

It is easily established that $\mathbf{A}_{1/2}$ is a *-algebra and that $\mathbf{A}_1 \subseteq \mathbf{A}_{1/2}$. In particular, $\mathbf{A}_{1/2}$ is norm dense. Our first result, in Section 2, shows that the *-derivation δ maps $\mathbf{A}_\infty$ into $\mathbf{A}_{1/2}$ if, and only if, $\delta = \lambda\delta_0$, where λ and $(\sigma_t\lambda - \lambda)/t$ are polynomially bounded in the frequency, with the latter bound uniform in t. In Section 3 we prove that these conditions are sufficient to ensure that $\bar{\delta}$ is a generator. The proof of this result is based upon a version of the Trotter-Kato theorem on semigroup convergence which is given in an appendix. In Section 4 related results for dissipations are discussed.

2. Smooth derivations

In this section we derive a characterization of derivations from $\mathbf{A}_\infty$ into $\mathbf{A}_{1/2}$. But first note that since $\|(\sigma_t f - f)/t\| \leqslant 2\|f\|/|t|$, one can equally well define $\mathbf{A}_{1/2}$ by

$$\mathbf{A}_{1/2} = \left\{ f \in \mathbf{A};\ \sup_{0 < |t| \leqslant 1} \|(\sigma_t f - f)/t\| < +\infty \right\}.$$

In fact since

$$(\sigma_t - 1)f = \sum_{m=0}^{n-1} \sigma_{mt/n}(\sigma_{t/n} - 1)f,$$

one has

$$\|(\sigma_t - 1)f/t\| \leqslant \|(\sigma_{t/n} - 1)f/(t/n)\|,$$

and hence

$$\mathbf{A}_{1/2} = \left\{f \in \mathbf{A};\ \limsup_{t \to 0} \|(\sigma_t f - f)/t\| < +\infty\right\}.$$

THEOREM 2.1. *Let σ be a strongly continuous one-parameter group of *-automorphisms of an abelian C^*-algebra $\mathbf{A} = C_0(X)$ with generator δ_0 and associated flow S on X. Let $X_0 \subseteq X$ denote the fixed points of S and $\nu(\omega)$ the frequency of the point $\omega \in X$ under the group S. Define $\mathbf{A}_\infty = \bigcap_{n \geqslant 1} D(\delta_0^n)$ and*

$$\mathbf{A}_{1/2} = \left\{f \in \mathbf{A};\ \sup_{0<|t|\leqslant 1} \|(\sigma_t f - f)/t\| < +\infty\right\}.$$

*If δ is a *-derivation from $\mathbf{A}_\infty$ into $\mathbf{A}$, then the following conditions are equivalent*:

1. $\delta(\mathbf{A}_\infty) \subseteq \mathbf{A}_{1/2}$.

2. $\delta = \lambda\delta_0|_{\mathbf{A}_\infty}$, *where λ vanishes on X_0, λ is continuous on $X \setminus X_0$, and there exist positive constants c_1, c_2 and non-negative integers k_1, k_2 such that*

$$|\lambda(\omega)| \leqslant c_1\left(1 + \nu(\omega)^{k_1}\right),$$

$$|\lambda(S_t\omega) - \lambda(\omega)| \leqslant c_2\left(1 + \nu(\omega)^{k_2}\right)|t|$$

for all $\omega \in X \setminus X_0$ and $t \in \mathbb{R}$.

PROOF. The proof is an elaboration of arguments given in [1].

2 $\Rightarrow$ 1. Observation 6 in Section 3 of [1] establishes that if $f \in \mathbf{A}_\infty$ and k is a positive integer, then the function

$$\omega \in X \setminus X_0 \mapsto \nu(\omega)^k(\delta_0 f)(\omega)$$

vanishes at infinity on $X \setminus X_0$. Consequently, $\delta f = \lambda\delta_0 f$ is continuous on X and vanishes at infinity on $X \setminus X_0$. Thus $\delta(\mathbf{A}_\infty) \subseteq \mathbf{A}$. But

$$\begin{aligned} t^{-1}(\sigma_t - 1)\delta f &= (\sigma_t\lambda)t^{-1}(\sigma_t - 1)\delta_0 f + \left(t^{-1}(\sigma_t - 1)\lambda\right)\delta_0 f \\ &= (\sigma_t\lambda)t^{-1}\int_0^t ds \frac{d}{ds}\sigma_s\delta_0 f + \left(t^{-1}(\sigma_t - 1)\lambda\right)\delta_0 f, \end{aligned}$$

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and hence

$$|t^{-1}(\sigma_t\delta f - \delta f)(\omega)| \leqslant c_1|t|^{-1}\int_0^{|t|} ds\left|\sigma_s\left((1+\nu)^{k_1}\delta_0^2 f\right)(\omega)\right| + c_2\left|\left((1+\nu)^{k_2}\delta_0 f\right)(\omega)\right|.$$

Consequently the above observation implies that δ maps $\mathbf{A}_\infty$ into $\mathbf{A}_{1/2}$.

1 ⇒ 2. Since $\delta(\mathbf{A}_\infty) \subseteq \mathbf{A}$, it follows from [1], Theorem 1.2 that Condition 2 is verified with the possible exception of the uniform polynomial bound on $|(\lambda(S_t\omega) - \lambda(\omega))/t|$. But this bound follows from the hypothesis $\delta(\mathbf{A}_\infty) \subseteq \mathbf{A}_{1/2}$. We will prove this in two stages. First we prove that $\omega \mapsto (\lambda(S_t\omega) - \lambda(\omega))/t$ is uniformly bounded on sets of bounded frequencies.

Suppose $\nu(\omega) \leqslant N/2$. Then the map $t \in (-1/N, 1/N) \mapsto S_t\omega$ is injective. Now let $F_N \in C_c^\infty(-1/N, 1/N)$ be an infinitely differentiable function with compact support in $(-1/N, 1/N)$ such that $F_N = 1$ on $[-1/2N, 1/2N]$ and define G_N by setting $G_N(t) = tF_N(t)$. Hence $G_N'(t) = 1$ for $t \in [-1/2N, 1/2N]$. But it then follows that there exists a $g_N \in \mathbf{A}_\infty$, with compact support, such that $g_N(S_t\omega) = G_N(t)$ for $t \in (-1/N, 1/N)$ (see, for example, the argument used in the proof of Observation 5.2 of [2]). Consequently,

$$\begin{aligned}(\sigma_t\delta g_N - \delta g_N)(\omega)/t &= \left(\lambda(S_t\omega)G_N'(t) - \lambda(\omega)G_N'(0)\right)/t \\ &= \left(\lambda(S_t\omega) - \lambda(\omega)\right)/t\end{aligned}$$

for $t \in [-1/2N, 1/2N]$. Combining this with the estimate given at the beginning of the section, one concludes that

$$\begin{aligned}\sup_{0<|t|\leqslant 1}\left|(\lambda(S_t\omega) - \lambda(\omega))/t\right| &\leqslant \sup_{0<|t|\leqslant 1/2N}\left|(\lambda(S_t\omega) - \lambda(\omega))/t\right| \\ &\leqslant \sup_{0<|t|\leqslant 1}\left|(\sigma_t\delta g_N - \delta g_N)/t\right|,\end{aligned}$$

i.e. one has boundedness on sets of bounded frequency.

Now consider polynomial boundedness on sets of large frequency. We establish this property by adapting the argument used to prove Observation 5 in Section 3 of [1]. We argue by contradiction.

Assume there exist sequences $\omega_i \in X \setminus X_0$ and $0 < |t_i| \leqslant 1$ such that $(\lambda(S_{t_i}\omega_i) - \lambda(\omega_i))/t_i$ is not polynomially bounded in the frequency. One may assume that $\nu(\omega_i) \geqslant 1/2$ because of the boundedness property proved above. Proceeding as in Section 3 of [1], one constructs functions $f_i \in \mathbf{A}_\infty$ with compact support $\mathbf{O}_i$ such that

$$\begin{aligned}&\|f_i\|_j \leqslant 2(2\pi\nu(\omega_i))^j, \qquad j = 1, 2, \ldots, i, \\ &S_{[-1,1]}\omega_i \subseteq \mathbf{O}_i, \\ &f_i(S_t\omega_i) = \exp\{2\pi i\nu(\omega_i)t\}, \qquad t \in [-1, 1],\end{aligned}$$

and the $\mathbf{O}_i$ are mutually disjoint. Then if ρ_i is any sequence in $\mathbb{C}$ which is rapidly decreasing in the sense that

$$\lim_{i\to\infty} \nu(\omega_i)^j \rho_i = 0$$

for $j = 1, 2, \ldots,$ it follows that

$$f = \sum_{i\geq 1} \rho_i f_i$$

converges with respect to the C_n-seminorms to an $f \in \mathbf{A}_\infty$, and $f = \rho_i f_i$ on $\mathbf{O}_i$. Hence $(\delta f)(\omega) = \rho_i(\delta f_i)(\omega)$ for all $\omega \in \mathbf{O}_i$. Consequently,

$$\begin{aligned}\left(\sigma_{t_i}\delta f - \delta f\right)(\omega_i)/t_i &= \rho_i\left(\sigma_{t_i}\delta f_i - \delta f_i\right)(\omega_i)/t_i \\ &= \rho_i\left(\lambda\left(S_{t_i}\omega\right)e^{2\pi i\nu(\omega_i)t_i} - \lambda(\omega_i)\right)\nu(\omega_i)2\pi i/t_i.\end{aligned}$$

Since $\|(\sigma_t\delta f - \delta f)/t\|$ is bounded uniformly in t, it follows that the coefficient of ρ_i must be bounded by a polynomial in the frequencies $\nu(\omega_i)$. But

$$\begin{aligned}\left(\lambda(S_t\omega_i)e^{2\pi i\nu(\omega_i)t_i} - \lambda(\omega_i)\right)/t_i &= \left(\lambda\left(S_{t_i}\omega_i\right) - \lambda(\omega_i)\right)/t_i \\ &\quad + \lambda\left(S_{t_i}\omega_i\right)\left(e^{2\pi i\nu(\omega_i)t_i} - 1\right)/t_i,\end{aligned}$$

and the second term on the right hand side is bounded by a polynomial in the frequencies $\nu(\omega_i)$. Hence the first term is also polynomially bounded, which is inconsistent with the initial hypothesis. This completes the proof that

$$\left|\left(\lambda(S_t\omega) - \lambda(\omega)\right)/t\right| \leq c_2\left(1 + \nu(\omega)^{k_2}\right)$$

for all $\omega \in X \setminus X_0$ and $t \in \mathbb{R}$, and completes the proof of the theorem.

3. Generators

In this section we establish that the condition $\delta(\mathbf{A}_\infty) \subseteq \mathbf{A}_{1/2}$ is sufficient to ensure that $\bar{\delta}$ is a generator. The proof uses the Lipschitz criterion derived in Theorem 2.1 and semigroup convergence techniques.

The major part of the proof consists of a generator result for derivations $\delta = \lambda\delta_0$, where λ is a real continuous function over $X \setminus X_0$ which satisfies slightly more general bounds than those derived in Theorem 2.1. Note that the Lipschitz bounds automatically imply that λ is continuous along orbits, and the continuity across orbits plays practically no part in our proof. It is only used to ensure that δ is densely defined. (The domain $D(\delta)$ of $\delta = \lambda\delta_0$ is, by definition, those $f \in D(\delta_0)$ such that $\lambda\delta_0(f) \in \mathbf{A}$.)

Although the following result could be partly deduced from Theorems 2.6 and 2.10 of [3], we give an almost independent proof based on resolvent convergence arguments. But to apply this technique it is convenient to use a density result from [3] which essentially allows one to avoid high frequencies.

First, for each $N \geqslant 0$, introduce the closed set $X^{(N)} = \{\omega \in X;\ \nu(\omega) \geqslant N\}$. Second define $\mathbf{A}^{(N)} \subseteq \mathbf{A}$ by

$$\mathbf{A}^{(N)} = \left\{ f \in \mathbf{A};\ f(S_t\omega) = f(\omega) \text{ for all } t \in \mathbb{R},\ \omega \in X^{(N)} \right\}.$$

It follows immediately that each $\mathbf{A}^{(N)}$ is a C^*-subalgebra of $\mathbf{A}$, and if $N \leqslant M$, then $\mathbf{A}^{(N)} \subseteq \mathbf{A}^{(M)}$. But one deduces from Lemma 2.7 of [3] that

$$\mathbf{A} = \overline{\bigcup_{N \geqslant 0} \mathbf{A}^{(N)}},$$

where the bar denotes norm closure. Finally each $\mathbf{A}^{(N)}$ is σ-invariant, and hence one can deduce information about the system $(\mathbf{A}, \mathbb{R}, \sigma)$ by examining the subsystems $(\mathbf{A}^{(N)}, \mathbb{R}, \sigma)$.

THEOREM 3.1. *Let σ be a strongly continuous one-parameter group of *-automorphisms of an abelian C^*-algebra $\mathbf{A} = C_0(X)$ with generator δ_0 and associated flow S on X. Let $X_0 \subseteq X$ denote the fixed points of S and $\nu(\omega)$ the frequency of the point $\omega \in X$ under the group S. Define $\mathbf{A}_\infty = \bigcap_{n \geqslant 1} D(\delta_0^n)$.*

If λ is a real continuous function over $X \setminus X_0$ satisfying

$$C_{1/2}\colon \begin{cases} |\lambda(\omega)| \leqslant K_1(\nu(\omega)), \\ |\lambda(S_t\omega) - \lambda(\omega)| \leqslant |t| K_2(\nu(\omega)), \qquad t \in \mathbb{R}, \end{cases}$$

*where $K_i\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ are positive non-decreasing functions, then the closure $\bar{\delta}$ of the derivation $\delta = \lambda\delta_0$ is the generator of a strongly continuous one-parameter group of *-automorphisms of* $\mathbf{A}$. *Moreover $D(\delta) \cap \mathbf{A}_\infty$ is a core of $\bar{\delta}$.*

REMARK. This result differs from similar statements in [3] in two respects. First the boundedness assumption on λ is much stronger than that of Theorem 2.6 of [3] which established the existence of a unique generator extension of $\lambda\delta_0$. Second the differentiability assumption, $\lambda \in D(\delta_0)$, which was necessary in Theorem 2.12 of [3] in order to identify the unique generator extension of $\lambda\delta_0$ with its closure, is not necessary in the present context. The advantage of the present result is that it suffices for the discussion of derivations from $\mathbf{A}_\infty \to \mathbf{A}_{1/2}$, it is considerably easier to prove than the analogous results of [3], and it has potential extensions to non-abelian systems.

PROOF. Fix $N \geqslant 0$ and consider the dynamical system $(\mathbf{A}^{(N)}, \mathbb{R}, \sigma)$. Here $\mathbf{A}^{(N)}$ is the C^*-subalgebra of $\mathbf{A}$ introduced before the proposition, and we identify σ with its restriction to $\mathbf{A}^{(N)}$. We also identify δ_0 and its restriction but explicitly indicate its domain $\mathbf{A}^{(N)} \cap D(\delta_0)$.

Now if $f \in \mathbf{A}^{(N)} \cap D(\delta_0)$, then $(\delta_0 f)(\omega) = 0$ whenever $\nu(\omega) \geqslant N$. Hence $\mathbf{A}^{(N)} \cap D(\delta_0) \subseteq \mathbf{A}^{(N)} \cap D(\delta)$, and one has

$$|(\lambda\delta_0 f)(\omega)| \leqslant K_1(N) |(\delta_0 f)(\omega)|,$$

$$|((\sigma_t\lambda - \lambda)\delta_0 f)(\omega)| \leqslant |t| K_2(N) |(\delta_0 f)(\omega)|.$$

Thus on the range $R(\delta_0)$ of δ_0, restricted to $\mathbf{A}^{(N)}$, the function λ is uniformly bounded and satisfies a uniform Lipschitz condition. Alternatively stated, one has

$$\|\lambda\|_N \leqslant K_1(N),$$

$$\|\sigma_t\lambda - \lambda\|_N \leqslant K_2(N)|t|,$$

where $\|\cdot\|_N$ denotes the usual operator norm calculated on the range $R(\delta_0)$ of δ_0 restricted to $\mathbf{A}^{(N)}$. This reduces the proof of the propositions on $\mathbf{A}^{(N)}$ to the case that K_1 and K_2 are uniformly bounded. We will handle this by regularizing λ and then using a convergence argument.

First, for $\alpha > 0$, define the regularization λ_α by

$$\lambda_\alpha = \frac{1}{\alpha^2}\int_0^\alpha ds \int_0^\alpha dt\, \sigma_{s+t}\lambda.$$

Then $\lambda_\alpha \in D(\delta_0^2)$, and

$$\delta_0\lambda_\alpha = \frac{1}{\alpha^2}\int_0^\alpha ds(\sigma_{s+\alpha}\lambda - \sigma_s\lambda).$$

Therefore,

$$\|\lambda_\alpha\|_N \leqslant K_1(N), \qquad \|\delta_0\lambda_\alpha\|_N \leqslant K_2(N),$$

and

$$\|\lambda_\alpha - \lambda\|_N \leqslant \frac{1}{\alpha^2}\int_0^\alpha ds \int_0^\alpha dt \|\sigma_{s+t}\lambda - \lambda\|_N \leqslant \alpha K_2(N).$$

In particular, $\lambda_\alpha \to \lambda$ uniformly on the range of δ_0, as $\alpha \to 0$.

Second, for $\beta > 0$, define $H_{\alpha,\beta}$ on $\mathbf{A}^{(N)} \cap D(\delta_0^2)$ by

$$H_{\alpha,\beta} = \lambda_\alpha\delta_0 - \beta\delta_0^2.$$

It follows easily that $H_{\alpha,\beta}$ is the generator of a C_0-semigroup $\tau^{\alpha,\beta}$ of contractions on $\mathbf{A}^{(N)}$. To establish this, note that $-\delta_0^2$ is the generator of a contraction semigroup, the Gaussian semigroup associated with σ. Moreover,

$$\|\delta_0 f\| \leqslant b\|\delta_0^2 f\| + \|f\|/b$$

for all $f \in D(\delta_0^2)$ and $b > 0$, by application of Taylor's theorem and the triangle inequality to the function $t \mapsto \sigma_t f$. Hence

$$\|\lambda_\alpha\delta_0 f\| \leqslant b\beta\|\delta_0^2 f\| + \left(K_1(N)^2/\beta b\right)\|f\|$$

for all $f \in D(\delta_0^2)$ and $b > 0$. But $H_{\alpha,\beta}$ is dissipative (see, for example, Lemma 4.1 of [3]), and hence $H_{\alpha,\beta}$ generates a C_0-semigroup of contractions $\tau^{\alpha,\beta}$ by perturbation theory.

It follows from general semigroup theory that $\|(I + \varepsilon H_{\alpha,\beta})^{-1}\| \leqslant 1$ for all $\varepsilon > 0$, and we use this to prove the strong convergence of $(I + \varepsilon H_{\alpha,\beta})^{-1}$ in the limit $\beta \to 0$, then $\alpha \to 0$. For this proof, note that

$$\left\{\left(I + \varepsilon H_{\alpha,\beta_1}\right)^{-1} - \left(I + \varepsilon H_{\alpha,\beta_2}\right)^{-1}\right\}f$$
$$= \varepsilon(\beta_1 - \beta_2)\left(I + \varepsilon H_{\alpha,\beta_2}\right)^{-1}\delta_0^2\left(I + \varepsilon H_{\alpha,\beta_1}\right)^{-1}f$$

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and

$$\left\{\left(I+\varepsilon H_{\alpha_1,\beta}\right)^{-1}-\left(I+\varepsilon H_{\alpha_2,\beta}\right)^{-1}\right\}f$$
$$=\varepsilon\left(I+\varepsilon H_{\alpha_2,\beta}\right)^{-1}\left(\lambda_{\alpha_1}-\lambda_{\alpha_2}\right)\delta_0\left(I+\varepsilon H_{\alpha_1,\beta}\right)^{-1}f$$

for all $f\in\mathbf{A}^{(N)}$. Hence

$$\left\|\left\{\left(I+\varepsilon H_{\alpha,\beta_1}\right)^{-1}-\left(I+\varepsilon H_{\alpha,\beta_2}\right)^{-1}\right\}f\right\|\leqslant\varepsilon|\beta_1-\beta_2|\left\|\delta_0^2\left(I+\varepsilon H_{\alpha,\beta_1}\right)^{-1}f\right\|$$

and

$$\left\|\left\{\left(I+\varepsilon H_{\alpha_1,\beta}\right)^{-1}-\left(I+\varepsilon H_{\alpha_2,\beta}\right)^{-1}\right\}f\right\|\leqslant\varepsilon\left\|\lambda_{\alpha_1}-\lambda_{\alpha_2}\right\|_N\left\|\delta_0\left(I+\varepsilon H_{\alpha_1,\beta}\right)^{-1}f\right\|.$$

Now since all the resolvents $(I+\varepsilon H_{\alpha,\beta})^{-1}$ are contractions, it suffices to prove the strong convergence on a dense subspace of $\mathbf{A}^{(N)}$ such as $\mathbf{A}^{(N)}\cap D(\delta_0^2)$. But it follows from the first of these estimates that one has convergence as $\beta\to 0$ if $\|\delta_0^2(I+\varepsilon H_{\alpha,\beta})^{-1}f\|$ is bounded uniformly in β for $f\in\mathbf{A}^{(N)}\cap D(\delta_0^2)$. Then it follows from the second estimate that one has convergence as $\beta\to 0$ and then $\alpha\to 0$ if, in addition, $\|\delta_0(I+\varepsilon H_{\alpha,\beta})^{-1}f\|$ is bounded uniformly in α and β for $f\in\mathbf{A}^{(N)}\cap D(\delta_0^2)$. Hence we next examine these boundedness properties.

First, if $f\in\mathbf{A}^{(N)}\cap D(\delta_0)$, then

$$\delta_0\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f=\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}\left\{\delta_0 f+\left[I+\varepsilon H_{\alpha,\beta},\delta_0\right]\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\}$$
$$=\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}\left\{\delta_0 f-\varepsilon\left(\delta_0\lambda_\alpha\right)\delta_0\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\}.$$

Hence, setting $K_2=K_2(N)$, we obtain

$$\left\|\delta_0\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\|\leqslant\|\delta_0 f\|+\varepsilon K_2\left\|\delta_0\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\|,$$

and for $\varepsilon K_2<1$, one has the bound

$$\left\|\delta_0\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\|\leqslant\|\delta_0 f\|(1-\varepsilon K_2)^{-1},$$

which is uniform in α and β. Similarly, if $f\in\mathbf{A}^{(N)}\cap D(\delta_0^2)$, one finds that

$$\left\|\delta_0^2\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\|\leqslant\left\|\delta_0^2 f\right\|+2\varepsilon\left\|\delta_0\lambda_\alpha\right\|_N\left\|\delta_0^2\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\|$$
$$+\varepsilon\left\|\delta_0^2\lambda_\alpha\right\|_N\left\|\delta_0\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\|.$$

Thus, if $2\varepsilon K_2<1$, one has the bound

$$\left\|\delta_0^2\left(I+\varepsilon H_{\alpha,\beta}\right)^{-1}f\right\|\leqslant\left[\left\|\delta_0^2 f\right\|+\varepsilon\left\|\delta_0^2\lambda_\alpha\right\|_N\|\delta_0 f\|(1-\varepsilon K_2)^{-1}\right](1-2\varepsilon K_2)^{-1},$$

which is uniform in β.

Therefore, if $0 < \varepsilon < (2K_2)^{-1}$, we have established the existence of the strong limit

$$R_\varepsilon = \lim_{\alpha\to 0} \lim_{\beta\to 0} \left(I + \varepsilon H_{\alpha,\beta}\right)^{-1}.$$

Next consider the convergence of $(I + \varepsilon H_{\alpha,\beta})^{-1}$ as $\varepsilon \to 0$. Given $f \in \mathbf{A}^{(N)}$ and $\kappa > 0$, one can choose $g \in \mathbf{A}^{(N)} \cap D(\delta_0^2)$ such that $\|f - g\| < \kappa\|f\|/2$. Hence, setting $K_1 = K_1(N)$, we have

$$\begin{aligned}\left\|\left\{\left(I + \varepsilon H_{\alpha,\beta}\right)^{-1} - I\right\}f\right\| &\leqslant \kappa\|f\| + \varepsilon\|H_{\alpha,\beta}g\| \\ &\leqslant \kappa\|f\| + \varepsilon\|\lambda_\alpha\|_N\|\delta_0 g\| + \varepsilon\beta\|\delta_0^2 g\| \\ &\leqslant \kappa\|f\| + \varepsilon K_1\|\delta_0 g\| + \varepsilon\beta\|\delta_0^2 g\|.\end{aligned}$$

Consequently,

$$\lim_{\varepsilon\to 0} \sup_{0<\alpha,\beta<1} \left\|\left\{\left(I + \varepsilon H_{\alpha,\beta}\right)^{-1} - I\right\}f\right\| \leqslant \kappa\|f\|.$$

Since $\kappa > 0$ was arbitrary, this proves that $(I + \varepsilon H_{\alpha,\beta})^{-1}$ converges strongly to the identity as $\varepsilon \to 0$, uniformly in α and β.

Therefore it follows from the version of the Trotter-Kato convergence theorem given in the appendix that there exists a C_0-contraction semigroup τ on $\mathbf{A}^{(N)}$ with generator H such that $\tau_t^{\alpha,\beta} \to \tau_t$ in the limit $\beta \to 0$, then $\alpha \to 0$, uniformly for t in finite intervals of $\mathbb{R}_+$, and $(I + \varepsilon H_{\alpha,\beta})^{-1} \to (I + \varepsilon H)^{-1}$ as $\beta \to 0$ then $\alpha \to 0$, uniformly for $\varepsilon > 0$. We will argue that $H = \overline{\lambda\delta_0|}_{\mathbf{A}^{(N)} \cap D(\delta_0^2)}$.

Let $f \in D(H)$ and set $g = (I + \varepsilon H)f$. Now choose $g_\gamma \in \mathbf{A}^{(N)}$ such that $\|g_\gamma - g\| \to 0$ as $\gamma \to 0$. Next set $f_{\alpha,\beta,\gamma} = (I + \varepsilon H_{\alpha,\beta})^{-1}g_\gamma$. Since $D(H_{\alpha,\beta}) = D(\delta_0^2)$, one has $f_{\alpha,\beta,\gamma} \in \mathbf{A}^{(N)} \cap D(\delta_0^2)$. But

$$\lim_{\gamma\to 0} \lim_{\alpha\to 0} \lim_{\beta\to 0} f_{\alpha,\beta,\gamma} = f.$$

Moreover,

$$(I + \varepsilon\lambda\delta_0)f_{\alpha,\beta,\gamma} - g_\gamma = \left\{\varepsilon(\lambda - \lambda_\alpha)\delta_0 - \varepsilon\beta\delta_0^2\right\}f_{\alpha,\beta,\gamma}$$

and

$$\begin{aligned}\left\|\left\{(\lambda - \lambda_\alpha)\delta_0 - \beta\delta_0^2\right\}f_{\alpha,\beta,\gamma}\right\| \leqslant{}& \|\lambda - \lambda_\alpha\|_N\left\|\delta_0\left(I + \varepsilon H_{\alpha,\beta}\right)^{-1}g_\gamma\right\| \\ &+ \beta\left\|\delta_0^2\left(I + \varepsilon H_{\alpha,\beta}\right)^{-1}g_\gamma\right\|.\end{aligned}$$

Therefore, using the estimates derived above, one finds that

$$\lim_{\gamma\to 0} \lim_{\alpha\to 0} \lim_{\beta\to 0} \left(I + \varepsilon\lambda\delta_0\right)f_{\alpha,\beta,\gamma} = g = (I + \varepsilon H)f.$$

This establishes that the closure $\bar{\delta}$ of $\lambda\delta_0|_{\mathbf{A}^{(N)}\cap D(\delta_0^2)}$ is an extension of H. But since H is a generator, it has no proper dissipative extensions, and hence $H = \bar{\delta}$.

The foregoing argument applies equally well to $-\lambda$, and consequently both $\pm\bar{\delta}$ are generators of C_0-contraction semigroups. Hence, by a simple standard argument, $\bar{\delta}$ in fact generates a C_0-group of isometries τ. Finally, as λ is real, δ is a *-derivation, and τ must be a group of *-automorphisms of $\mathbf{A}^{(N)}$. Next we extend τ from $\mathbf{A}^{(N)}$ to $\mathbf{A}$.

For each $N \geqslant 0$, we have constructed a group of *-automorphisms, which we now denote by $\tau^{(N)}$, of $\mathbf{A}^{(N)}$ and the generator of $\tau^{(N)}$ is $H^{(N)} = \overline{\lambda\delta_0|}_{\mathbf{A}^{(N)}\cap D(\delta_0^2)}$. But if $N \leqslant M$, then $\mathbf{A}^{(N)} \subseteq \mathbf{A}^{(M)}$, and $\tau^{(N)} \subseteq \tau^{(M)}$ because $H^{(N)} \subseteq H^{(M)}$. Thus, defining τ on $\bigcup_{N\geqslant 0}\mathbf{A}^{(N)}$ by setting $\tau = \tau^{(N)}$ on $\mathbf{A}^{(N)}$, one can then extend τ to a C_0-group of *-automorphisms of $\mathbf{A}$ by continuity, because

$$\mathbf{A} = \overline{\bigcup_{N\geqslant 0}\mathbf{A}^{(N)}}.$$

The generator H of τ is by construction a closed extension of $\lambda\delta_0$ restricted to $\mathbf{D} = \bigcup_{N\geqslant 0}\mathbf{A}^{(N)} \cap D(\delta_0^2)$, but in fact $H = \overline{\lambda\delta_0|}_{\mathbf{D}}$. To prove this, let $f \in D(H)$ and set $g = (1 + \varepsilon H)f$. By density there exists a sequence $g_N \in \mathbf{A}^{(N)}$ such that $\|g_N - g\| \to 0$, and, since $H^{(N)} = H|_{\mathbf{A}^{(N)}}$ is a generator, there exists a sequence $f_N \in D(H^{(N)})$ such that $g_N = (1 + \varepsilon H^{(N)})f_N = (1 + \varepsilon H)f_N$. Then it follows that $f_N = (1 + \varepsilon H)^{-1}g_N \to f$. But as $\mathbf{A}^{(N)} \cap D(\delta_0^2)$ is a core of $H^{(N)}$, one concludes that $\bigcup_{N\geqslant 0}\mathbf{A}^{(N)} \cap D(\delta_0^2)$ is a core of H.

Finally, if $f \in \mathbf{A}^{(N)} \cap D(\delta_0^2)$, and if $h \in C_0(\mathbb{R})$ is a positive, infinitely differentiable function with support in $[-1, 1]$ and total integral one, then, defining

$$\begin{aligned} f_n &= n\int dt\, h(nt)\sigma_t f \\ &= \int dt\, h(t)\sigma_{t/n} f, \end{aligned}$$

one has $f_n \in \mathbf{A}_\infty^{(N)} = \mathbf{A}^{(N)} \cap \mathbf{A}_\infty$. But $f_n \to f$ and $\delta_0 f_n \to \delta_0 f$ by strong continuity of σ. Moreover, λ is bounded on the range of δ_0 restricted to $\mathbf{A}^{(N)}$, so $\delta f_n = \lambda\delta_0 f_n \to \lambda\delta_0 f = \delta f$. Therefore $\bigcup_{N\geqslant 0}\mathbf{A}_\infty^{(N)}$ is a core of H.

Combining Theorems 2.1 and 3.1, one obtains the result stated in the abstract.

COROLLARY 3.2. *Let σ be a strongly continuous one-parameter group of *-automorphisms of an abelian C^*-algebra $\mathbf{A}$ with generator δ_0. Define $\mathbf{A}_\infty = \bigcap_{n\geqslant 1} D(\delta_0^n)$ and*

$$\mathbf{A}_{1/2} = \left\{ f \in \mathbf{A};\ \sup_{0<|t|\leqslant 1} \|(\sigma_t f - f)/t\| < +\infty \right\}.$$

*If δ is a *-derivation from $\mathbf{A}_\infty$ into $\mathbf{A}_{1/2}$, then δ is closable, and its closure $\bar{\delta}$ is the generator of a strongly continuous one-parameter group of *-automorphisms of* **A**.

PROOF. It follows from Theorem 2.1 that $\delta = \lambda\delta_0$, where λ satisfies the condition $C_{1/2}$ of Theorem 3.1 with K_1 and K_2 polynomials, thereby ensuring that $\mathbf{A}_\infty \subseteq D(\delta)$. The corollary is then a direct consequence of Theorem 3.1.

4. Local dissipations

An operator H: $\mathbf{A}_\infty \to \mathbf{A}$ is defined to be *local* if $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$ for all $f \in \mathbf{A}_\infty$, and to be a *dissipation* if

$$H(\bar{f}f) \leqslant H(\bar{f})f + \bar{f}H(f)$$

for all $f \in \mathbf{A}_\infty$. In [1] it was demonstrated that H is a local dissipation if, and only if, it has the form

$$Hf = \lambda_0 f + \lambda_1\delta_0 f - \lambda_2\delta_0^2 f, \qquad f \in \mathbf{A}_\infty,$$

where λ_0 is bounded and continuous on X, where λ_1, λ_2 vanish on X_0 and are polynomially bounded and continuous on $X \setminus X_0$, and where λ_0, $\lambda_2 \geqslant 0$. Moreover, if H maps $\mathbf{A}_\infty$ into $\mathbf{A}_n$, then the $\lambda_i \in \mathbf{A}_n$, $\delta_0^j\lambda_0$ is bounded, and $\delta_0^j\lambda_1$, $\delta_0^j\lambda_2$ are polynomially bounded, for $j \leqslant n$.

In [3] it was conjectured, in analogy with results for derivations, that if H: $\mathbf{A}_\infty \to \mathbf{A}_2$ is a local dissipation, then its closure generates a C_0-semigroup of positive contractions. This conjecture was verified in the special case that λ_1 is bounded by $\lambda_2^{1/2}$. The general conjecture is, however, false, as we next demonstrate with a specific example. Subsequently we extend the positive results on dissipations obtained in [3] to local dissipations H: $\mathbf{A}_\infty \to \mathbf{A}_{3/2}$, where $\mathbf{A}_{3/2}$ is defined by

$$\mathbf{A}_{3/2} = \{ f \in \mathbf{A}_1;\ \delta_0 f \in \mathbf{A}_{1/2} \}.$$

Then we discuss some other possible characterizations of $\mathbf{A}_{3/2}$.

First consider the example $\mathbf{A} = C_0(\mathbb{R})$ and σ the group of translations. (I am indebted to Charles Batty for help in constructing this example.) Thus $\delta_0 = d/dx$ and $\mathbf{A}_\infty = C_0^\infty(\mathbb{R})$. Now let $C_c^\infty(\mathbb{R})$ denote the infinitely differentiable functions with compact support and H: $C_c^\infty(\mathbb{R}) \to C_0(\mathbb{R})$ the operator defined by

$$H = -x^2\frac{d^2}{dx^2} - (1 - x^2)\frac{d}{dx}.$$

Then H is a dissipation, but if g is defined by

$$\begin{aligned} g(x) &= x^{-2}\exp\{-x^{-1}\}, \qquad x > 0, \\ &= 0, \qquad x \leqslant 0, \end{aligned}$$

one readily computes that

$$\int dx\, g(x)((1 + H)f)(x) = 0$$

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for all $f \in C_c^\infty(\mathbb{R})$. Thus the closure of H is definitely not the generator of a contraction semigroup on $C_0(\mathbb{R})$.

This example is almost a counterexample to the conjecture in [3]. It fails only because the coefficients of H are unbounded at infinity, and hence H is not defined on all of $\mathbf{A}_\infty$. One can, however, convert this example into a genuine counterexample on $C(\mathbb{T})$ by the change of variable $x \in \mathbb{R} \mapsto y \in \mathbb{T}$, where $y = \mathrm{Tan}^{-1} x$. One then obtains the more complicated expression

$$H = (1/4)\left(-\mathrm{Sin}^2 2y \frac{d^2}{dy^2} + 2(\mathrm{Sin}\, 2y(1 - \mathrm{Cos}\, 2y) - 2\,\mathrm{Cos}\, 2y)\frac{d}{dy}\right),$$

which has bounded continuous coefficients and can be defined on all of $C^\infty(\mathbb{T})$. But by the above calculation $\overline{R(1 + H)} \neq C(\mathbb{T})$, and hence $\overline{H}$ is not a generator.

Next we turn to the examination of local dissipations H: $\mathbf{A}_\infty \to \mathbf{A}_{3/2}$, and we begin by remarking that, by an extension of the proof of Theorem 2.1, one can establish that the coefficients λ_0, λ_1, $\lambda_2 \in D(\delta_0)$ satisfy the condition

$$C_{3/2}: \begin{cases} |\lambda(\omega)| \leqslant L_1(\nu(\omega)) \\ |(\delta_0\lambda)(\omega)| \leqslant L_2(\nu(\omega)) \\ |(\delta_0\lambda)(S_t\omega) - (\delta_0\lambda)(\omega)| \leqslant |t| L_3(\nu(\omega)) \end{cases}$$

where the L_i are polynomials. Hence the basic problem is to show that the closure of $H = \lambda_1\delta_0 - \lambda_2\delta_0^2$ on $\mathbf{A}_\infty$, where $\lambda_2 \geqslant 0$, and where λ_1, λ_2 satisfy $C_{3/2}$ on $X \setminus X_0$, is a generator. (The term λ_0 causes no problems, since it is positive and bounded, and H is closable because it is automatically dissipative [3].)

Now the closure of $\lambda_1\delta_0$ generates a group of *-automorphisms by Theorem 3.1, or Theorem 3.1 of [3], and we next argue that the closure of $-\lambda_2\delta_0^2$ generates a positive contraction semigroup. Then if λ_1 is bounded by $\lambda_2^{1/2}$, the generator result follows for the sum $H = \lambda_1\delta_0 - \lambda_2\delta_0^2$ by perturbation theory.

Proposition 4.1. *Let* $(\mathbf{A}, \mathbb{R}, \sigma)$ *be an abelian* C^**-dynamical system. Denote the generator of* σ *by* σ_0 *and set* $\mathbf{A}_\infty = \bigcap_{n \geqslant 1} D(\delta_0^n)$.

If λ *is a non-negative continuous function over the spectrum* X *of* $\mathbf{A}$ *which satisfies condition* $C_{3/2}$ *above, then the closure* $\overline{H}$ *of* $H = -\lambda\delta_0^2$ *generates a positive* C_0*-contraction semigroup. Moreover,* $\mathbf{A}_\infty \cap D(H)$ *is a core of* $\overline{H}$.

Remarks. 1. For this result it suffices that the functions L_i which occur in Condition $C_{3/2}$ are positive, and finite-valued. If the L_i are polynomials, then $\mathbf{A}_\infty \subseteq D(H)$; and hence $\mathbf{A}_\infty$ is a core of $\overline{H}$.

2. It follows from the first part of the proof that, since λ is non-negative, the bound on $\delta_0\lambda$ in Condition $C_{3/2}$ follows from the other two bounds, and in fact one can assume that $L_2^2 \leqslant 2L_1L_3$.

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PROOF. The key to the proof is the observation that

$$0 \leqslant \sigma_t\lambda = \lambda + t(\delta_0\lambda) + \int_0^t ds(\sigma_s - 1)(\delta_0\lambda).$$

Therefore

$$0 \leqslant \lambda(\omega) + t(\delta_0\lambda)(\omega) + (t^2/2)L_3(\nu(\omega))$$

for all $t \in \mathbb{R}$, and hence

$$|(\delta_0\lambda)(\omega)|^2 \leqslant 2\lambda(\omega)L_3(\nu(\omega)).$$

But this implies that $\lambda^{1/2} \in D(\delta_0)$, and so

$$|(\delta_0\lambda^{1/2})(\omega)|^2 = |(\delta_0\lambda)(\omega)|^2/2\lambda(\omega) \leqslant L_3(\nu(\omega))/2.$$

Hence by Theorem 3.1 of [3], or Theorem 3.1 in the previous section, the closure $\bar{\delta}$ of $\delta = \lambda^{1/2}\delta_0$ generates a C_0-group of *-automorphisms τ of $\mathbf{A}$. Consequently, $-\bar{\delta}^2$ generates a positive C_0-contraction semigroup ρ, the convolution semigroup associated with τ, defined by

$$\rho_t f = (\pi t)^{-1/2}\int_{-\infty}^{\infty} ds\, e^{-s^2/t}\tau_s f.$$

Next let $\mathbf{A}^{(N)}$ denote the C^*-subalgebra of $\mathbf{A}$ spanned by those $f \in \mathbf{A}$ which satisfy $f(S_t\omega) = f(\omega)$ for all $t \in \mathbb{R}$ and all ω in the closed set $X^{(N)} = \{\omega: \nu(\omega) \geqslant N\}$. The groups σ and τ leave each $\mathbf{A}^{(N)}$ invariant, and hence ρ also leaves the $\mathbf{A}^{(N)}$ invariant. Next, following [3], we observe that

$$H = -\lambda\delta_0^2 = -(\lambda^{1/2}\delta_0)^2 + (\delta_0\lambda^{1/2})(\lambda^{1/2}\delta_0),$$

and if $f \in \mathbf{A}^{(N)} \cap D((\lambda^{1/2}\delta_0)^2)$, then

$$\|(\delta_0\lambda^{1/2})\lambda^{1/2}\delta_0 f\| \leqslant L_3(N)^{1/2}\|\lambda^{1/2}\delta_0 f\|/2^{1/2}$$
$$\leqslant b\|(\lambda^{1/2}\delta_0)^2 f\| + (L_3(N)/2)\|f\|/b$$

for all $b > 0$. Now H is dissipative, and hence $\overline{H}$ generates a C_0-semigroup κ of contractions by perturbation theory. Since H is a dissipation, κ is also positive by [3], Proposition 4.3. Now κ is defined on each $\mathbf{A}^{(N)}$ consistently, i.e. if $N \leqslant M$, then the restriction of κ from $\mathbf{A}^{(M)}$ to $\mathbf{A}^{(N)}$ agrees with the direct definition of κ on $\mathbf{A}^{(N)}$. Finally, $\mathbf{A} = \overline{\bigcup_{N \geqslant 1}\mathbf{A}^{(N)}}$, and hence κ extends to a positive C_0-contraction semigroup on $\mathbf{A}$ by continuity.

The core property follows by the same arguments used to derive the analogous property in Theorem 3.1.

We conclude with some comments on the definition of $\mathbf{A}_{3/2}$. There are various alternatives to the choice that we have used. But the following propositions show that the obvious ones coincide, even for non-abelian $\mathbf{A}$.

PROPOSITION 4.2. *Let* $(\mathbf{A}, \mathbb{R}, \sigma)$ *be a* C^**-dynamical system and let* δ *denote the generator of* σ. *Then, for each* $A \in \mathbf{A}$, *the following conditions are equivalent*:

1. $\sup_{|t|>0}\|(\sigma_t - 1)^2 A/t^2\| < +\infty$,
2. $\sup_{|t|>0}\sup_{|s|>0}\|(\sigma_t - 1)(\sigma_s - 1)A/ts\| < +\infty$,
3. $A \in D(\delta)$ *and* $\sup_{|t|>0}\|(\sigma_t - 1)\delta(A)/t\| < +\infty$.

Moreover, if these conditions are satisfied, then the three suprema are equal.

PROOF. 3 $\Rightarrow$ 2. This follows from the triangle inequality once one observes that

$$\frac{(\sigma_s - 1)A}{s} = -\frac{1}{2}\int_0^s dr\,\sigma_r\delta(A).$$

2 $\Rightarrow$ 1. This is obvious.

1 $\Rightarrow$ 3. First we prove that $A \in D(\delta)$. Set

$$a = \sup_{|t|>0}\left\|(\sigma_t - 1)^2 A/t^2\right\|.$$

Then note that

$$\left(\frac{\sigma_t - 1}{t} - \frac{\sigma_{t/2} - 1}{t/2}\right)A = \frac{(\sigma_{t/2} - 1)}{t/2}\left(\frac{(\sigma_{t/2} + 1)}{2} - 1\right)A$$
$$= \frac{(\sigma_{t/2} - 1)^2}{t/2}A/2.$$

Replacing t by $t_m = t/2^m$ and summing from $m = 0$ to $m = n - 1$, one finds that

$$\left(\frac{(\sigma_t - 1)}{t} - \frac{(\sigma_{t_n} - 1)}{t_n}\right)A = \sum_{m=0}^{n}\frac{(\sigma_{t_{m+1}} - 1)^2}{t_{m+1}}A/2.$$

Therefore

$$(*)\qquad \left\|\left(\frac{(\sigma_t - 1)}{t} - \frac{(\sigma_{t_n} - 1)}{t_n}\right)A\right\| \leqslant (a/2)\sum_{m=0}^{\infty}|t_{m+1}|$$
$$\leqslant (a/2)|t|\sum_{m\geqslant 0}\frac{1}{2^{m+1}}$$
$$= (a/2)|t|.$$

Consequently, for all m, $n \geqslant 0$, one has

$$\left\|\left(\frac{(\sigma_{t_n} - 1)}{t_n} - \frac{(\sigma_{t_m} - 1)}{t_m}\right)A\right\| \leqslant a|t|,$$

and, replacing t by t_p, one concludes that

$$\left\|\left(\frac{(\sigma_{t_n} - 1)}{t_n} - \frac{(\sigma_{t_m} - 1)}{t_m}\right)A\right\| \leqslant a|t|/2^p$$

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for all $m, n \geqslant p$. This proves that

$$\lim_{n\to\infty} \frac{(\sigma_{t_n} - 1)}{t_n} A = A_t$$

exists, but in principle it could depend upon t. Nevertheless,

$$\frac{(\sigma_{s_n+t_n} - 1)}{s_n + t_n} A = \frac{t_n}{s_n + t_n} \sigma_{s_n} \frac{(\sigma_{t_n} - 1)}{t_n} A + \frac{s_n}{s_n + t_n} \frac{(\sigma_{s_n} - 1)}{s_n} A,$$

and hence

$$(s + t) A_{s+t} = s A_s + t A_t,$$

i.e. the function $t \in \mathbb{R} \mapsto tA_t \in \mathbf{A}$ is additive. But $(*)$ implies that

$$\|tA_t\| \leqslant 2\|A\| + (a/2)t^2,$$

and consequently, by classical reasoning, the function $t \in \mathbb{R} \mapsto tA_t \in \mathbf{A}$ must be linear, i.e. $A_t = A_1$ is independent of t. Now referring to $(*)$ once again, one sees that

$$\lim_{t\to 0} \|(\sigma_t - 1)A/t - A_1\| = 0,$$

i.e. $A \in D(\delta)$ and $\delta(A) = A_1$.

Finally, using the foregoing identification and the estimate $(*)$, one has

$$\left\|(\sigma_t - 1)^2 A/t^2 - (\sigma_t - 1)\delta(A)/t\right\| \leqslant a.$$

Hence the supremum in Condition 3 is finite. This establishes that $1 \Rightarrow 3$.

Now consider the last statement of the proposition. Let a_1, a_2, a_3 denote the values of the suprema occurring in Conditions 1, 2, and 3 respectively. Then by elementary reasoning $a_1 \leqslant a_2 \leqslant a_3$. But using $(\sigma_t - 1) = (\sigma_{t/2} - 1)(\sigma_{t/2} + 1)$, one deduces that

$$\begin{aligned}\left\|(\sigma_t - 1)^2 A/t^2\right\| &\leqslant \left\|(\sigma_t - 1)(\sigma_{t_n} - 1)A/tt_n\right\| \\ &\leqslant \left\|(\sigma_{t_n} - 1)^2 A/t_n^2\right\| \leqslant a_1,\end{aligned}$$

where we have once again used the notation $t_n = t/2^n$. Therefore, taking the limit as $n \to \infty$, one finds that

$$\left\|(\sigma_t - 1)^2 A/t^2\right\| \leqslant \|(\sigma_t - 1)\delta(A)/t\| \leqslant a_1.$$

Finally, taking the supremum over t gives $a_1 \leqslant a_3 \leqslant a_1$, and hence $a_1 = a_2 = a_3$.

Appendix-semigroup convergence

In Section 3 we make several applications of a version of the Trotter-Kato theorem on semigroup convergence. The resolvent formulation of this result is given in [5, Chapter IV], but it can also be stated in terms of the semigroup. The complete result, for contraction semigroups, is summarized in the following proposition.

PROPOSITION. *Let S^α be a net of C_0-contraction semigroups on a Banach space* **B** *and denote the generator of S^α by H_α. The following conditions are equivalent.*

1. *The strong limit of S_t^α exists for all small $t > 0$, and for each $a \in$* **B**,

$$\lim_{t \to 0+} \|(S_t^\alpha - I)a\| = 0$$

uniformly in α.

2. *The strong limit of $(I + \varepsilon H_\alpha)^{-1}$ exists for all small $\varepsilon > 0$, and for each $a \in$* **B**,

$$\lim_{\varepsilon \to 0+} \left\|\left((I + \varepsilon H_\alpha)^{-1} - I\right)a\right\| = 0$$

uniformly in α.

Moreover, these conditions imply that there exists a C_0-contraction semigroup S, with generator H, such that $S_t^\alpha \to S_t$ uniformly for t in any finite interval of $\mathbb{R}_+$, and such that $(I + \varepsilon H_\alpha)^{-1} \to (I + \varepsilon H)^{-1}$ uniformly for $\varepsilon > 0$.

PROOF. Assume that Condition 1 holds and let S_t denote the strong limit of S_t^α. Since $\|S_t^\alpha\| \leqslant 1$ for all $t \geqslant 0$, it readily follows that the strong limit exists for all $t \geqslant 0$ and that $S_s S_t = S_{s+t}$ for all $s,t \geqslant 0$. But $S_0 = I$, and one automatically has $\|S_t\| \leqslant 1$. Therefore, to conclude that S is a C_0-contraction semigroup, it remains to prove continuity at the origin. But given $a \in$ **B** and $\varepsilon > 0$, one can choose t_0 such that

$$\|(S_t^\alpha - I)a\| < \varepsilon/2$$

for all $0 \leqslant t \leqslant t_0$, uniformly in α. Then for $0 < t \leqslant t_0$ fixed, one can choose α_0 such that

$$\|(S_t^\alpha - S_t)a\| < \varepsilon/2$$

for $\alpha > \alpha_0$. Therefore, by the triangle inequality,

$$\|(S_t - I)a\| < \varepsilon,$$

and this is valid for any $0 \leqslant t \leqslant t_0$.

Now since S^α converges to the C_0-contraction semigroup S, it follows from the usual Trotter-Kato theorem (see, for example, [7],Theorem 3.1.26) that $(I + \varepsilon H_\alpha)^{-1}$ converges strongly to $(I + \varepsilon H)^{-1}$, where H is the generator of S, and the convergence is uniform in ε. But

$$\left[(I + \varepsilon H_\alpha)^{-1} - I\right]a = \int_0^\infty dt\, e^{-t}(S_{\varepsilon t}^\alpha - I)a,$$

and hence, for any $M > 0$, one estimates that

$$\left\|\left[(I + \varepsilon H_\alpha)^{-1} - I\right]a\right\| \leqslant 2e^{-M}\|a\| + \sup_{|t| < \varepsilon M} \|(S_t^\alpha - I)a\|.$$

It follows immediately that $(I + \varepsilon H_\alpha)^{-1} \to I$ as $\varepsilon \to 0$ uniformly in α.

This establishes that Condition 2, and also the last statement of the proposition, follow from Condition 1.

Next assume that Condition 2 holds. Then the existence of S and H and the identification

$$(I + \varepsilon H)^{-1} = \lim_{\alpha} (I + \varepsilon H_\alpha)^{-1}$$

follow from [5, Chapter IX, Theorem 2.17]. The proof can be summarized as follows.

Let R_ε denote the strong limit of $(I + \varepsilon H_\alpha)^{-1}$. Since $\|(I + \varepsilon H_\alpha)^{-1}\| \leqslant 1$, and since the resolvent relation

$$\varepsilon_1 (I + \varepsilon_1 H_\alpha)^{-1} - \varepsilon_2 (I + \varepsilon_2 H_\alpha)^{-1} = (\varepsilon_1 - \varepsilon_2)(I + \varepsilon_1 H_\alpha)^{-1}(I + \varepsilon_2 H_\alpha)^{-1}$$

is valid, it follows that

$$\|R_\varepsilon\| \leqslant 1$$

and that

$$\varepsilon_1 R_{\varepsilon_1} - \varepsilon_2 R_{\varepsilon_2} = (\varepsilon_1 - \varepsilon_2) R_{\varepsilon_1} R_{\varepsilon_2}.$$

But given $a \in \mathbf{B}$ and $\kappa > 0$, one can choose $\varepsilon_0 \geqslant 0$ such that

$$\left\| \left((I + \varepsilon H_\alpha)^{-1} - I \right) a \right\| < \kappa/2$$

for $0 < \varepsilon \leqslant \varepsilon_0$ uniformly in α. Then, for $0 < \varepsilon \leqslant \varepsilon_0$ fixed, one can choose α_0 such that

$$\left\| \left((I + \varepsilon H_\alpha)^{-1} - R_\varepsilon \right) a \right\| < \kappa/2$$

for $\alpha > \alpha_0$. Hence, by the triangle inequality,

$$\|(R_\varepsilon - I) a\| < \kappa,$$

and this is valid for any $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Consequently,

$$\lim_{\varepsilon \to 0} R_\varepsilon = I.$$

It immediately follows that $R_\varepsilon = (I + \varepsilon H)^{-1}$, where H is the generator of a C_0-contraction semigroup S. Then $S^\alpha \to S$ by the usual Trotter-Kato convergence theorem, and this implies the last statement of the proposition.

It remains to prove that $S_t^\alpha \to I$ as $t \to 0$ uniformly in α. Now given $a \in \mathbf{B}$, set $a_\alpha = (I + \kappa H_\alpha) a^{-2}$. Then $a_\alpha \in D(H_\alpha^2)$,

$$a_\alpha - a = \left((I + \kappa H_\alpha)^{-1} + I \right) \left((I + \kappa H_\alpha)^{-1} - I \right) a,$$

and

$$H_\alpha^2 a_\alpha = \frac{1}{\kappa^2} \left((I + \kappa H_\alpha)^{-1} - I \right)^2 a.$$

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In particular, $\|a_\alpha - a\| \leqslant 2\|((I + \kappa H_\alpha)^{-1} - I)a\|$, and $\|H_\alpha^2 a_\alpha\| \leqslant 4\|a\|/\kappa^2$. But by a standard estimate (see, for example, [5, Chapter IX, Section 1.2]), we have

$$\left\|\left(S_t^\alpha - \left(I + \frac{t}{n}H_\alpha\right)^{-n}\right)a_\alpha\right\| \leqslant \frac{t^2}{2n}\|H_\alpha^2 a_\alpha\| \leqslant 2t^2\|a\|/n\kappa^2.$$

Therefore

$$\begin{aligned}\|(S_t^\alpha - I)a\| \leqslant& \left\|\left(\left(I + \frac{t}{n}H_\alpha\right)^{-n} - I\right)a\right\| \\ &+ \left\|\left(S_t^\alpha - \left(I + \frac{t}{n}H_\alpha\right)^{-n}\right)a_\alpha\right\| + 2\|a - a_\alpha\| \\ \leqslant& \, n\left\|\left(\left(I + \frac{t}{n}H_\alpha\right)^{-1} - I\right)a\right\| + 2t^2\|a\|/n\kappa^2 \\ &+ 4\left\|\left((I + \kappa H_\alpha)^{-1} - I\right)a\right\|.\end{aligned}$$

Hence, by first choosing κ and then t, one deduces that $S_t^\alpha \to I$ uniformly in α as $t \to 0$. This completes the proof.

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