Characterizations of MacNeille completions and topological functors

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In this note we give external characterizations of absolutely
topological functors $U : A \to X$ (where A may be a large
category) as functors which are injective with respect to all
functors based on the category X , or equivalently, which do not
admit essential extensions over X . From this we deduce several
characterizations and properties of the MacNeille completion of
faithful functors. These results generalize the work of Horst
Herrlich [*Math. Z.* **150** (1976), 101-110] in case of small
categories, where the crucial point of this generalization lies
in the fact that we cannot make use of the existence of MacNeille
completions and we therefore have to introduce a local-initial-
completion-construction.

Introduction

Considering a partly ordered set as a faithful (and amnestic) functor
$U : A \to 1$, the MacNeille completion of this ordered set is a functor
$E : A \to B$ over the category 1 which can be characterized categorically
[3], [7]. In [7] Herrlich has generalized recently the notion of MacNeille
completions to arbitrary faithful (and amnestic) functors $U : A \to X$, and
shown that MacNeille completions may be characterized in this general
situation just as in case $X = 1$, if A is a small category. This

smallness condition is an important assumption, because the proofs make use
of the existence of MacNeille completions, and MacNeille completions do not
exist in general unless A is small [7].

Since the MacNeille completion of a faithful functor $U : A \to X$ is
the smallest initial completion of U (that is, extension of U to a
topological functor $V : B \to X$, see Section 1) which may be available,
characterizations of MacNeille completions might be useful in searching for
mild conditions for the existence of initial completions in case A is a
large category. For the importance of this problem we refer to [1], [2],
[7], [8].

By the same argument as above, Henrlich's proofs of the character-
izations of absolutely topological functors [6] and initially and finally
dense functors [7] are not transferable to functors $U : A \to X$ if A is
large.

In this note we show that nevertheless all these characterizations
hold with respect to arbitrary faithful functors, and show how these
.characterizations work practically. Moreover we distinguish absolutely
topological functors $U : A \to X$ not only relative to faithful X-based
functors (as in [7]) but also to arbitrary ones (as was suggested to the
author by J.W. Gray). Further results are proved on the uniqueness, the
existence, and some additional properties of MacNeille completions, which
are not known as yet, or which could be proved only using special
constructions of MacNeille completions.

Some historical remarks on the topic of this note seem to be
necessary. External characterizations of topologicity related to those
given here had first been given by Brümmer and Hoffmann in 1975 [5], using
different methods in their proofs. Their results were unknown to the
author when he presented the contents of this note at the Sussex Conference
on Categories in 1976 [9]. Furthermore the author learned that his
construction of a locally initial completion used in the proof of Theorem 1
had already been known to Brümmer and Hoffmann in 1975 - though not stated
in [5].

1. Preliminaries

A pair (A, U) , where $U : A \to X$ is a functor, is called a *category*

over X . (A, U) is called *concrete* if U is faithful; (A, U) is called *initially complete* if U is absolutely topological in the sense of [6]. A *functor over* X , $F : (A, U) \rightarrow (B, V)$ is any functor $F : A \rightarrow B$ with $VF \simeq U$; F is called an *extension* (of (A, U)) if F is full and faithful. An extension $E : (A, U) \rightarrow (B, V)$ is called *initial completion* (of (A, U)) if (B, V) is initially complete. Furthermore we need the following definitions.

DEFINITION 1. A category (A, U) over X is called *injective* if, for any extension $E : (B, V) \rightarrow (C, W)$ and any functor $F : (B, V) \rightarrow (A, U)$, there exists a functor $G : (C, W) \rightarrow (A, U)$ with $GE = F$. (A, U) is called *c-injective* if all involved categories are concrete. An *injective extension* is an extension $E : (A, U) \rightarrow (B, V)$ where (B, V) is injective.

DEFINITION 2. Let (B, V) be concrete. A functor $F : (A, U) \rightarrow (B, V)$ over X is called *essential*, if any functor $G : (B, V) \rightarrow (C, W)$ such that (C, W) is concrete and GF is full, is a full functor itself.

DEFINITION 3. A functor $F : (A, U) \rightarrow (B, V)$ is called *initially dense* (dually, *finally dense*), if for any B-object B there exists a V-initial source $(B, (B \xrightarrow{a} FA))$ where a ranges over some class of B-morphisms from B to objects of the form FA .

The notion of initial denseness generalizes the notion of meet-denseness in the category of ordered sets [7]. It may be illustrated by the following examples.

1. The full embedding of the category Ind of indiscrete spaces into the category Top of topological spaces is neither initially dense nor finally dense.

2. The full embedding of the category Comp of compact Hausdorff spaces into the category K of Kelley spaces is finally dense.

3. The full embedding of Comp into the category LocComp of locally compact Hausdorff spaces is initially and finally dense.

REMARK 1. If (B, V) is concrete, then $F : (A, U) \rightarrow (B, V)$ is initially dense, iff any source $(B, (B \xrightarrow{a} FA))$, where a ranges over

all B-morphisms from B to all objects of the form FA , is V-initial.

REMARK 2. Any dense (codense) functor $F : (A, U) \to (B, V)$ (in the usual sense) is finally dense (initially dense) if B is initially complete.

REMARK 3. $1_A : (A, U) \to (A, U)$ is initially and finally dense.

REMARK 4. If (A, U) is $(c-)$injective, then any extension of (A, U) has a left inverse, and any essential extension of (A, U) is an equivalence.

REMARK 5. Let $E : (A, U) \to (B, V)$ and $E' : (B, V) \to (C, W)$ be extensions; if $E'E$ is essential, then E' is essential.

2. A characterization of initially complete categories

Let us first point out the connection between the notions of essentiality and denseness, which will be useful in the following.

LEMMA 1. *Let* (B, V) *be concrete; then any full and essential functor* $E : (A, U) \to (B, V)$ *is initially and finally dense.*

Proof. Let us assume that there exists a source
$$\left(B_0, \ \left[B_0 \xrightarrow{a} EA\right] \mid A \in \text{ob } A, \ a \in B\left(B_0, EA\right)\right)$$ which fails to be V-initial. We construct a functor $G : (B, V) \to (C, W)$ such that GE is full, but G is not. For this let C be the category with the same objects as B and the following sets of morphisms (which are thought to be disjoint):

$$C\left(B, B_0\right) = \left\{VB \xrightarrow{f} VB_0 \in \text{mor } X \ \Big/ \ f \ \downarrow \ Va \ \text{ for all } \ B_0 \xrightarrow{a} EA\right\} \ ,$$

where $f \downarrow Va$ iff there is a B-morphism $g : B \to EA$ such that the following diagram commutes:

$$
\begin{array}{ccc}
VB & \xrightarrow{\quad f \quad} & VB_0 \\
 & \searrow{\scriptstyle Vg} & \Big\downarrow{\scriptstyle Va} \\
 & & VEA \quad .
\end{array}
$$

For $B' = B_0$ we define

$$C(B, B') = V\big(B(B, B')\big) \cup \{Vb \cdot f \mid f \in C\big(B, B_0\big), b \in B\big(B_0, B'\big)\} \ .$$

Composition in C is the composition in X. Then it is a (long but) straightforward calculation to see that C is a category. With the following definitions we get a concrete category (C, W) over X and a functor $G : (B, V) \to (C, W)$ over X :

$$WB := VB \ , \quad Wf := f \ , \text{ and } \ GB := B \ , \quad Gf := Vf \ .$$

The definition of C implies the equation $C(EA, EA') = V\big(B(EA, EA')\big)$. Hence GE is full, because E is full by assumption. Furthermore it is an immediate consequence of the construction that G is not full.   //

   LEMMA 2. *Let* (B, V) *be concrete; then any initially and finally dense functor* $E : (A, U) \to (B, V)$ *is essential.*

   Proof. Let $G : (B, V) \to (C, W)$ be a functor over X , where (C, W) is concrete and GE is full, and let $f : GB \to GB'$ be a C-morphism. Consider the family of morphisms $GEA \xrightarrow{Ga} GB \xrightarrow{f} GB' \xrightarrow{Ga'} GEA'$, where A and A' range over ob A independently and a (respectively a') over all morphisms $EA \to B$ (respectively $B' \to EA'$). Apply W to these morphisms and use the equivalence $WG \simeq V$ to deduce from the fullness of GE and the denseness properties of E , that there is a B-morphism f $: B \to B'$ with $Gf = f$.   //

   These lemmas yield the following proposition, which generalizes a similar result of [7] on small categories.

   PROPOSITION 1. *Let* $E : (A, U) \to (B, V)$ *be an extension, where* (B, V) *is a concrete category over* X . E *is essential iff* E *is initially and finally dense.*

   The next proposition will yield a useful property of MacNeille completions; for a similar result see [8].

   PROPOSITION 2. *Full and essential functors preserve initial and final structures.*

   For the proof use the following lemma and its dual.

   LEMMA 3. *Let* $E : (A, U) \to (B, V)$ *be full and finally dense. Any* B-*source* $\left[EA_0, \left[EA_0 \xrightarrow{Eg_i} EA_i\right]\right]$ *is* V-*initial, if the* A-*source*

$$\left[A_0, \left(A_0 \xrightarrow{\ g_i\ } A_i \right) \right] \quad \text{is} \quad U\text{-initial}.$$

Consequently, if $\left[A_0, \left(A_0 \xrightarrow{\ g_i\ } A_i \right) \right]$ is a U-initial lifting of $\left[X, \left(X \xrightarrow{\ f_i\ } UA_i \right) \right]$, then $\left[EA_0, \left(EA_0 \xrightarrow{\ Eg_i\ } EA_i \right) \right]$ is a V-initial lifting of $\left[X, \left(X \xrightarrow{\ f_i\ } UA_i \simeq VEA_i \right) \right]$.

Proof. Let $\left[A_0, \left(A_0 \xrightarrow{\ g_i\ } A_i \right) \right]$ be a U-initial source and h an X-morphism such that

$$VB \xrightarrow{\ h\ } VEA_0 \xrightarrow{\ VEg_i\ } VEA_i = V\left(B \xrightarrow{\ h_i\ } EA_i \right) .$$

Then for each B-morphism $a : EA \to B$ there is an A-morphism $h_i' : A \to A_i$ such that $VEg_i \cdot h \cdot Va = VEh_i'$. Hence the X-morphism $h_a : UA \to UA_0$ corresponding to $h \cdot Va$ by the equivalence $U \simeq VE$ is of the form $h_a = U\mathfrak{h}_a$, since $\left[A_0, \left(A_0 \xrightarrow{\ g_i\ } A_i \right) \right]$ is U-initial. Now the equation $VE\mathfrak{h}_a = hVa$ implies the existence of a B-morphism $\mathfrak{h} : B \to EA_0$ with $V\mathfrak{h} = h$, since E is finally dense.   //

The following theorem is essentially known for small categories [7].

THEOREM 1. Let (A, U) be a category over X . The following assertions are equivalent:

(i) (A, U) is initially complete;

(ii) (A, U) is concrete and injective;

(iii) (A, U) is c-injective;

(iv) up to equivalence, 1_A is the only essential extension of (A, U) .

Proof. For the implication (i) $\Rightarrow$ (ii) use Herrlich's proof [7] with the obvious modification, and the fact that topological functors are faithful.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$. Obvious by the remarks of Section 1.

For the crucial implication $(iv) \Rightarrow (i)$, let us assume that there exists a source $\left[X, \left(X \xrightarrow{f_i} UA_i\right) \mid i \in I\right]$ in X which does not admit an U-initial lifting. We define a category B as follows, where the morphism sets again are thought to be disjoint:

$$\text{ob } B := \text{ob } A \cup \{X\} \;;$$

and for $B, C \in \text{ob } A$,

$B(B, C) := U\big(A(B, C)\big)$,

$B(B, X) := \{f : UB \to X \mid f \downarrow f_i \text{ for all } i \in I\}$,

$B(X, C) := \{h : X \to UC \mid f \downarrow h \text{ for all } f \in B(B, X), B \in \text{ob } A\}$,

$B(X, X) := \{g : X \to X \mid f \downarrow g \text{ for all } f \in B(B, X), B \in \text{ob } A,$
$\text{and } g \downarrow f_i \text{ for all } i \in I\}$.

Composition in B is the composition in X . To show that B is a category, one uses the following observations:

(i) $f_i \in B\big(X, A_i\big)$;

(ii) $B(X, X) = \{g : X \to X \mid f \downarrow g \text{ for all } f \in B(B, X), B \in \text{ob } A\}$
$= \{g : X \to X \mid g \downarrow f_i \text{ for all } i \in I\}$.

We here use $g \downarrow f_i$ in the following obvious sense: there exists $g \in B\big(X, A_i\big)$ such that $f_i g = g$.

With the same definitions as in the proof of Lemma 1 we get a faithful functor $V : B \to X$ and a functor $E : (A, U) \to (B, V)$ which obviously is an extension. It is an immediate consequence of the definition of B that E is initially and finally dense. Furthermore, it is clear that B admits a V-initial lifting of the source $\left[X, \left(X \xrightarrow{f_i} UY_i = VY_i\right)\right]$; hence E can not be an equivalence.   //

3. Characterizations of MacNeille completions

In this section all categories over X are assumed to be concrete. A

MacNeille completion is, by definition, an initial completion which, in addition, is initially and finally dense. As is shown by Herrlich [7], each initial completion $E : (A, U) \rightarrow (B, V)$ has a factorization $(A, U) \xrightarrow{E^*} (B^*, V^*) \rightarrow (B, V)$, where E^* is a MacNeille completion. Moreover we can show, axiomatically, that MacNeille completions are unique up to equivalence.

PROPOSITION 3. *If $E : (A, U) \rightarrow (B, V)$ and $E' : (A, U) \rightarrow (B', V')$ are MacNeille completions of (A, U), then there is an equivalence $G : (B, V) \overset{\sim}{\rightarrow} (B', V')$ such that $GE = E'$.*

Proof. The desired G exists, since (B', V') is injective. G is an extension, because E is essential, and G is essential, since E' is essential. Now use Remark 4.   //

As in the case of small categories we now get the following characterization of MacNeille completions.

THEOREM 2. *If $E : (A, U) \rightarrow (B, V)$ is an extension, then the following assertions are equivalent:*

(i)   *E is a MacNeille completion of (A, U);*

(ii) *E is essential and injective;*

(iii) *E is the smallest injective extension of (A, U) (that is, if $E' : (A, U) \rightarrow (C, W)$ is an injective extension, then there is an extension G of (B, V) such that $GE = E'$);*

(iv) *E is a minimal injective extension of (A, U) (that is, if E has a factorization $E = GE'$ over an injective extension E' such that G is an extension, then G is an equivalence);*

(v)   *E is a maximal essential extension.*

Moreover any of these conditions imply

(vi) *E is the largest essential extension.*

Proof. By the preceding remarks and results most implications are obvious. For the implications *(iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (ii)*, observe that in both cases E is equivalent to the MacNeille completion contained in it.

Note that the extensions are not really "ordered". //

REMARK. The implication *(vi) ⇒ (ii)* is valid, for example, if MacNeille completions exist (see [3]).

COROLLARY 1. *If E : (A, U) → (B, V) is a MacNeille completion, then E preserves initial and final structures.*

COROLLARY 2. *A MacNeille completion of a concrete category (A, U) is an equivalence if and only if (A, U) is initially complete.*

COROLLARY 3. *If (A, U) → (B, V) is an extension and if the MacNeille completion of (B, V) exists, then the MacNeille completion of (A, U) exists and admits an extension to the MacNeille completion of (B, V).*

COROLLARY 4. *Let E : (A, U) → (B, V) be an essential extension. If one of these categories has a MacNeille completion, then so has the other, and both MacNeille completions are equivalent.*

4. Applications and remarks

The characterization of absolutely topological functors $U : A → X$ as concrete and injective pairs (A, U) can be used to get very easily some of the standard results. For example the well known duality property, or the fact that full topological functors are equivalences [6], are immediate consequences. It also works practically as a checking criterion for initial completeness: the category Comp fails to be initially complete, because Comp ↪ LocComp is a proper essential extension; moreover both categories have (topological) MacNeille completions in Top and these are equivalent

It should be mentioned that all assertions of this note remain valid with respect to the "strong" notion of initial completeness [7] (that is, the U-initial liftings are to be exact), provided we replace "faithful" by "faithful and amnestic", "extension" by "full embedding", and "equivalence" by "isomorphism". Similarly one can consider fibre small categories over X only, and so one gets statements as, for example, "(A, U) is a Top-category over X iff (A, U) is concrete, fibre small, and injective relative to all (fibre small) categories over X ".

Finally we remark that the notion of essentiality seems to be fruitful

with respect to questions on the existence of initial completions, although the notion of injectivity looks more elegant.

References

[1] Philippe Antoine, "Étude élémentaire des catégories d'ensembles structurés" (Première partie), *Bull. Soc. Math. Belg.* 18 (1966), 142-164.

[2] Philippe Antoine, "Étude élémentaire des catégories d'ensembles structurés" (Deuxième partie), *Bull. Soc. Math. Belg.* 18 (1966), 387-414.

[3] B. Banaschewski and G. Bruns, "Categorical characterization of the MacNeille completion", *Arch. Math. (Basel)* 18 (1967), 369-377.

[4] G.C.L. Brümmer, "Topological functors and structure functors", *Categorical topology*, 109-135 (Proceedings of the Conference held at Mannheim, 1975. Lecture Notes in Mathematics, 540. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[5] G.C.L. Brümmer and R.-E. Hoffmann, "An external characterization of topological functors", *Categorical topology*, 136-151 (Proceedings of the Conference held at Mannheim, 1975. Lecture Notes in Mathematics, 540. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[6] Horst Herrlich, "Topological functors", *General Topology and Appl.* 4 (1974), 125-142.

[7] Horst Herrlich, "Initial completions", *Math. Z.* 150 (1976), 101-110.

[8] H. Herrlich and L.D. Nel, "Cartesian closed topological hulls", *Proc. Amer. Math. Soc.* 62 (1977), 215-222.

[9] Hans-E. Porst, "On the MacNeille completion of faithful functors", *Category theory meeting*, Isle of Thorns, Sussex, England (Mimeographed Abstracts, 1976).

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