STABILITY OF ELLIPTICAL GALAXIES. NUMERICAL EXPERIMENTS

David Merritt
Canadian Institute for Theoretical Astrophysics
University of Toronto
Toronto, Ontario M5S 1A1
Canada

1. INTRODUCTION

The idea that dynamical instabilities might play an important role in determining the equilibrium structure of elliptical galaxies is a startling one, especially to those of us who are accustomed to associating instabilities with rapidly-rotating systems like disk galaxies. The shock is even greater when we learn that these instabilities have been taken seriously by workers in the Soviet Union for a long time. As Dr. Polyachenko points out in his talk, instabilities affecting spherical, nonrotating galaxies were being discussed by Soviet astronomers as long ago as 1972. Much of this work has recently become more accessible through the publication of an English-language version of Fridman and Polyachenko's monograph, *Physics of Gravitating Systems* (Fridman and Polyachenko 1984). In the West, it appears that only two people were prescient enough to systematically test the stability of spherical models before learning of the Soviet work (Hénon 1973; Barnes 1985). In particular, Barnes discovered independently that a spherical system composed of predominantly radial orbits evolves rapidly into a bar. Subsequent work has demonstrated that even some mildly anisotropic models can be unstable in this way.

Why did this important class of instabilities remain undetected for so long? There are probably two reasons. First, it is easy to construct stable equilibrium models for elliptical (or at least spherical) galaxies. Indeed, a number of mathematical proofs, beginning with Antonov's remarkable papers of 1960 and 1962, demonstrated (though not in a physically very intuitive way) that isotropic spherical systems are guaranteed to be stable as long as their distribution functions satisfy certain reasonable constraints. By contrast, stable disk models are notoriously difficult to construct. The second reason is more subtle, and, in retrospect, rather ironic. When an N-body experimenter wants to make a strongly anisotropic galaxy model, he generally does so by relaxing an out-of-equilibrium set of initial coordinates and velocities chosen in such a way as to give him roughly the final state he is looking for. This procedure is obviously much easier than constructing an exact equilibrium solution to the collisionless Boltzmann equation. But a model galaxy formed in this way will never be unstable, because if it were, the instability would have acted during the relaxation to produce a different (and stable) final state.

T. de Zeeuw (ed.), Structure and Dynamics of Elliptical Galaxies, 315–329.

What is ironic is that it is clear, from a careful re-reading of past work on galaxy formation, that instabilities very similar to the ones now known to afflict equilibrium systems were active in many of the published collapse simulations. It takes only a little imagination to wonder whether instabilities of the sort talked about by Polyachenko and Barnes might have played an active role during the formation of real galaxies.

This talk is divided into four parts. The first part summarizes what has been learned so far about the stability and instability of spherical equilibrium models. Unfortunately, nothing definite is known yet about the stability of triaxial models. The second part discusses how dynamical instabilities might be used to constrain the dynamics of particular well-observed galaxies. The third part describes some preliminary work on the question of whether instabilities could have played an active role during galaxy formation. The fourth part presents an efficient new algorithm for testing the stability of spherical and triaxial models.

The total number of published papers on this topic is still quite small, and this talk should be seen less as a review then as an introduction to a rapidly developing field.

2. STABILITY OF SPHERICAL MODELS

Orbits in spherical potentials are characterized by four integrals of motion, the energy E and the three components of the angular momentum $\mathbf{J}$. Since $\mathbf{J}$ is conserved, every orbit lies in a plane. According to Jeans's theorem, equilibrium models can be constructed from distribution functions f that depend on the phase-space coordinates $\mathbf{r}$ and $\mathbf{v}$ through E and $\mathbf{J}$ alone. If the model is to exhibit the same symmetries as the potential, then f must be a function of r, which means that $f = f(E, J^2)$. In fact this is not quite correct: one can imagine adding to f a term that is odd in $\mathbf{J}$ and contributes nothing to the total density. Such a term effectively specifies what fraction of the stars revolve clockwise or counterclockwise on each orbit. In what follows, however, only non-rotating models will be considered.

Consider first the case of velocity isotropy, $f = f(E)$. Most of what we know about the stability of isotropic systems was first discovered by Antonov (1960, 1962). Antonov considered systems for which $df/dE < 0$, and found a necessary and sufficient condition for stability in the form of a complicated variational principle. He went on to derive a number of simpler, *sufficient* conditions for stability. The most important of these are:

I. A spherical system with $f = f(E)$ and $df/dE < 0$ is stable to all non-radial (i.e. non-spherically-symmetric) perturbations;

II. A spherical system with $f = f(E)$, $df/dE < 0$ and $d^3\rho/d\Phi^3 \leq 0$ is stable to all perturbations. Here ρ and Φ are the density and potential, respectively.

Antonov was able to show that the family of "stellar dynamical polytropes" defined by

$$f(E) \propto (E_0 - E)^{n-3/2}\ , \quad E \leq E_0, \tag{1}$$

is stable for $n \geq 3/2$, i.e., for all values of n such that $df/dE \leq 0$. Antonov's theorems may be used to verify that many of the isotropic models that resemble real galaxies, such as Hénon's (1959) isochrone or the isotropic Michie-King (1966) models, are also stable.

https://doi.org/10.1017/S0074180900185274 Published online by Cambridge University Press

Antonov's proofs leave open the question of the stability of systems whose distribution functions do not satisfy conditions I or II. The first systematic search for unstable, isotropic models was carried out by Hénon (1973), who used a spherical N-body code to check the stability of polytropes with $n < 3/2$ (i.e., $df/dE > 0$). He found that all were stable (at least to the spherical modes permitted by his computer code), with the possible exception of the $n = 1/2$ model, which appeared to oscillate at a level slightly above the noise. (The case $n = 1/2$ turns out to be a peculiar one, since all the stars in this model have exactly the same energy; smaller values of n are not allowed.) Barnes, Goodman and Hut (1986) later showed that these polytropic models are stable to non-spherical modes as well.

As of now, Hénon's is the *only* candidate for an unstable isotropic system. It is certainly possible that other unstable isotropic models exist, especially when $df/dE > 0$. On the other hand, it seems very unlikely that any isotropic model resembling a real galaxy will ever be found to be unstable. This is because an observed density profile $\rho(r)$ implies a unique isotropic distribution function $f(E)$, and the isotropic models corresponding to many of the standard galaxy surface density profiles are known to be stable.

The situation is very different, and much more interesting, for *anisotropic* systems. A number of attempts have been made to generalize Antonov's sufficient stability criteria to systems with $f = f(E, J^2)$. These proofs (e.g. Doremus and Feix 1973; Gillon, Doremus and Baumann 1976; Sygnet et al. 1984) are mostly still controversial, and at least one (Gillon, Doremus and Baumann 1976) appears to be contradicted by the numerical experiments described below. A more fruitful approach has been to search for particular, unstable models. Hénon (1973) tested the radial stability of the "anisotropic polytropes" defined by

$$f(E, J^2) \propto J^{2m}(E_0 - E)^{n-3/2}. \quad (2)$$

Models generated from equation (2) have velocity ellipsoids with fixed axis ratios $\sigma_r^2/\sigma_t^2 = (1 + m)^{-1}$, where σ_r and σ_t are the radial and tangential components of the velocity dispersion tensor. Hénon found that the oscillatory instability that seemed to be present in the isotropic model with $n = 1/2$ became stronger as the velocity ellipsoid was made more prolate.

At about the same time that Hénon published his paper on instabilities in polytropic models, a number of Soviet workers had begun to apply techniques of perturbation theory to spherical systems. So far, two new classes of instabilities have been identified in this way. One class, associated with systems dominated by circular orbits, is probably not very relevant to elliptical galaxies. The other class, affecting systems dominated by radial orbits, almost certainly is.

2.1. Spherical Systems Dominated by Nearly-Circular Orbits

A disk galaxy in which all the stars move along exactly circular orbits is violently unstable to small-scale axisymmetric modes, i.e. clumping in rings (Toomre 1964). It is natural to ask whether a spherical galaxy composed of circular orbits is similarly unstable. The simple answer is "no": unlike a disk, the gravitational force in a sphere is due entirely to the interior mass, and this fact is sufficient to insure that a sphere will not clump into shells (Bisnovatyi-Kogan, Zel'dovich and Fridman 1968). Non-spherical perturbations can be unstable, however. For a spherical system composed purely of circular orbits, a sufficient condition for instability to

https://doi.org/10.1017/S0074180900185274 Published online by Cambridge University Press

non-spherically-symmetric modes is

$$\frac{d\rho}{dr} > 0 \quad (3)$$

(Fridman and Polyachenko 1984, Vol. 1, p. 179). Barnes, Goodman and Hut (1986) explored this type of instability numerically using N-body models generated from the anisotropic polytrope distribution function (2). For $m > 0$, equation (2) implies a preponderance of circular orbits, and a density profile that peaks at nonzero r. In the limit of large m, all the matter lies on a thin shell. Oscillatory instabilities are easy to understand for such an extreme system, since all the orbits have nearly the same period, and perturbations tend to recur naturally after a fraction of an orbital period. Barnes, Goodman and Hut (1986) found that all models with $m \gtrsim 1/2$ exhibited quadropole oscillations with rapidly increasing amplitude, and achieved stability only after a substantial rearrangement of matter.

Tangentially anisotropic models generated from equation (2) could never be mistaken for real galaxies because of their peculiar density profiles. Recently Polyachenko (1985) has suggested that systems with $d\rho/dr < 0$ might exhibit similar instabilities. This hypothesis is an important one to check. At present, however, there is no numerical evidence to suggest that this class of instability is relevant to systems that look like real elliptical galaxies.

2.2. Spherical Systems Dominated by Eccentric Orbits

If a circular-orbit model seems an unlikely one for an early-type galaxy, the opposite extreme, a galaxy consisting largely of radial orbits, seems much more natural: after all, collapse from cold initial conditions tends to build in very elongated orbits, at least at large radii. Antonov suggested as early as 1973 that a purely radial-orbit model would be unstable to clumping of particles around any radius vector. There is presently some uncertainty about the validity of Antonov's proof. Nevertheless the instability exists. It was first verified numerically by Polyachenko (1981), who used a direct-summation N-body code to follow the evolution of a 200-particle radial-orbit model with density profile $\rho \propto r^{-2}$. The initially equilibrium model evolved rapidly into a bar. The instability was rediscovered independently by Barnes (1985), who used a more sophisticated mean-field N-body code to test the stability of the anisotropic polytropes of equation (2).

Figure 1 illustrates the development of the radial-orbit instability in a model with an initial surface-density profile that is essentially identical to a de Vaucouleurs $r^{1/4}$ law; the initial velocities were chosen from a distribution function that gives increasingly radial orbits at large radii, similar to the models produced in collapse simulations (Merritt and Aguilar 1985).

What is the physical mechanism behind the radial-orbit instability? One simple interpretation (Fridman and Polyachenko 1984, Vol. 2, p. 148; Barnes, Goodman and Hut 1986) is based on the well-known Jeans (1929) instability of a uniform, self-gravitating medium. Jeans showed that any such medium is unstable to gravitational clumping on length scales

$$\lambda > \lambda_J = \sqrt{\frac{\pi}{G\rho}}\sigma \quad (4)$$

where ρ and σ are the density and the (isotropic) velocity dispersion. The *radial* velocity dispersion in a radially-anisotropic system of size R and mean density ρ must be of order $\sigma_r^2 \approx G\rho R^2$ to satisfy the virial theorem. Since the *tangential* velocity dispersion in such a system is much lower, equation (4) suggests that clumping might be expected in a tangential direction, i.e. in a cone around any radius vector. Since the growth time for the Jeans instability is not strongly dependent on λ, one might expect an unstable model to evolve on all azimuthal length scales greater than λ_J simultaneously. This means that small-scale clumping should not be sufficient to stabilize a model before a large-scale, or bar, mode has permanently destroyed its spherical symmetry. These predictions are consistent with the numerical experiments.

Figure 1. Radial-orbit instability in an anisotropic spherical model. The initial density and velocity dispersion satisfy equations (5) and (6), with anisotropy radius $r_a = 0.1 r_0$. This is an axisymmetric, mean-field N-body calculation with 5000 particles.

On the other hand, it would be a mistake to associate this class of instability too closely with the simple instability discussed by Jeans. Any system dominated by radial orbits must be very inhomogeneous. Since the time scale for growth of the Jeans instability and the orbital time scale are both roughly equal to $(G\rho)^{-1/2}$, an unstable mode would scarcely begin to grow before the particles contributing to it had moved far away from their initial positions, to regions of very different density and velocity dispersion. Furthermore, since the unperturbed trajectories are not closed, there is no guarantee that a perturbation that is initially confined to a small angular region will not quickly be spread over a much larger one.

A better description of the physical mechanism underlying the radial-orbit instability is given, in a slightly different context, by Lynden-Bell (1979). In a spherical potential, every orbit is a rosette, with an angle between apocenters that lies between π and 2π. For very eccentric orbits, this angle is close to π, and orbital precession is very slow (cf. Figure 2a). Now suppose that the potential is modified by the addition of a weak bar-like perturbation. If the minimum of the bar potential lies ahead of the orbit, the net torque will be in the same direction as the orbital motion. What Lynden-Bell showed was that for certain orbits—and, in particular, for very eccentric ones—a positive torque leads to a greater precession rate, causing the orbits to align with the bar and oscillate about it (Figure 2b). Note that in Lynden-Bell's picture, the growth rate of the instability is determined primarily by the orbital precession rate, and not by the dynamical time.

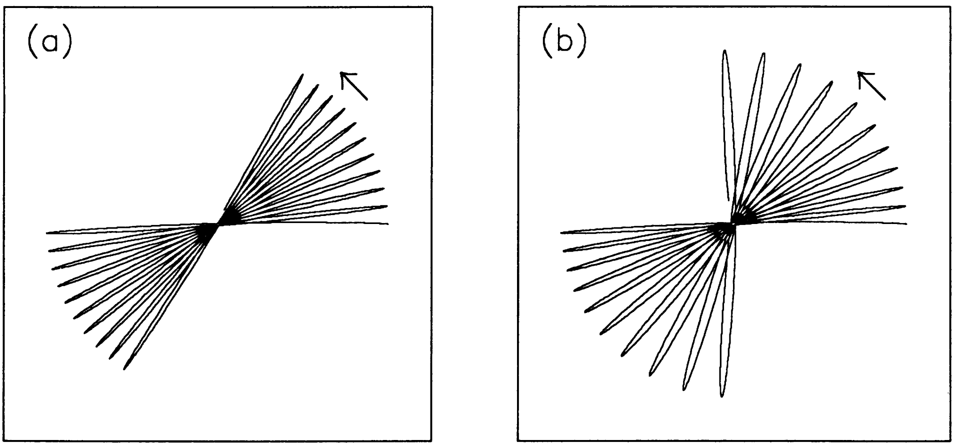

Figure 2. Attraction of eccentric orbits by a bar. (a) Spherical potential; orbit precesses at a constant rate. (b) Spherical potential plus a weak bar (oriented vertically). The orbit accelerates in the direction of the bar.

The most surprising thing about the radial-orbit instability is how *little* anisotropy is required for a model to be unstable. Barnes, Goodman and Hut (1986) found that the final ellipticity of spherical models generated from equation (2) increased smoothly as m was reduced below zero, suggesting that the instability is present even for slight departures from isotropy. Merritt and Aguilar (1985) found a similar behavior in two families of models with a density profile closer to that of real galaxies. One of these families was derived from a distribution function of the form $f(E, J^2) \propto J^{2m}g(E)$, similar to equation (2); the second was defined as the superposition of an isotropic and a purely radial model, and thus contained a finite number of particles with zero angular momentum. Note that all three of

https://doi.org/10.1017/S0074180900185274 Published online by Cambridge University Press

these distribution functions diverge as $J \to 0$. Although the ability of the N-body codes to distinguish between stable and unstable models is limited by particle noise, these numerical simulations suggest that models with velocity anisotropies as small as $\sigma_r/\sigma_t \approx 1.2$ can be bar-unstable.

This surprising conclusion has recently been verified analytically by Palmer and Papaloizou (1987; this volume). Their analysis shows that any spherical system characterized by a distribution function that diverges as fast as $J^{-\alpha}$ as J goes to zero, for all E, is guaranteed to be unstable to clumping on *all* angular scales. Since a model with $\alpha \approx 0$ has $\sigma_r/\sigma_t \approx 1$, Palmer and Papaloizou's work verifies that large anisotropies are not required for instability.

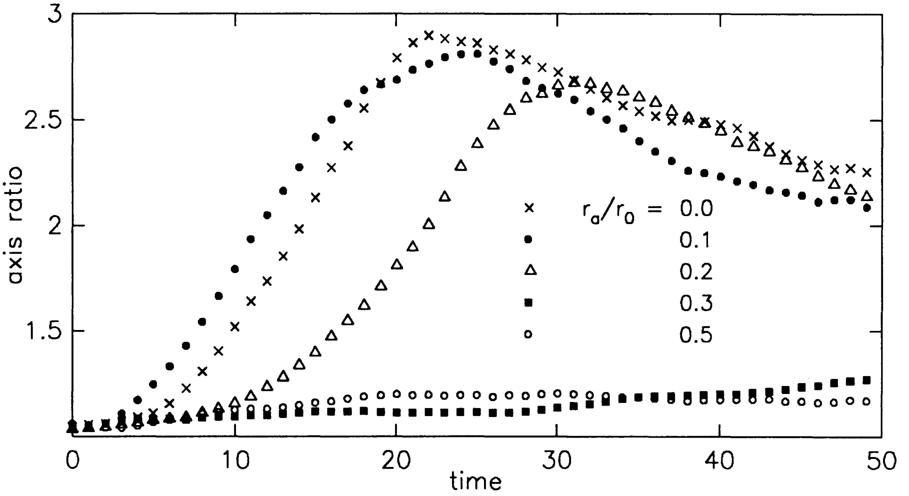

Figure 3. Evolution of the ellipticity of a family of anisotropic spherical models. r_a/r_0 is the ratio of anisotropy to half-mass radii (cf. eqs. [5], [6]). From a set of mean-field, 5000-particle N-body simulations by Merritt and Aguilar (1985).

Less is known about the stability of anisotropic models whose distribution functions do not diverge as $J \to 0$. Only one such family has been analyzed numerically. (Stability boundaries for two other non-singular families have been calculated by a normal-mode technique [Fridman and Polyachenko 1984, Vol. 1., p. 207], but there is some reason to doubt the accuracy of these calculations, which have not been checked numerically; see Polyachenko [1985] and Sec. 5 below.) Merritt and Aguilar (1985) tested the stability of a family of models with density profile

$$\rho(r) = \rho_0(r/r_0)^{-2}(1+r/r_0)^{-2}, \tag{5}$$

suggested by Jaffe (1983) as an approximation to real galaxies, and a distribution function of the form $f = f(E + J^2/2r_a^2)$; here r_0 is the half-mass radius, and r_a is a free parameter that determines the degree of velocity anisotropy through the relation

$$\frac{\sigma_r^2}{\sigma_t^2} = 1 + \frac{r^2}{r_a^2}. \tag{6}$$

These models obtain their anisotropy essentially by excluding all high-angular-momentum orbits from the region $r > r_a$. Merritt and Aguilar (1985) found that

https://doi.org/10.1017/S0074180900185274 Published online by Cambridge University Press

this family undergoes a sudden transition to instability when r_a is reduced below $\sim 0.3r_0$ (Figure 3). Models near the stability boundary have an average anisotropy $\langle\sigma_r^2\rangle^{1/2}/\langle\sigma_t^2\rangle^{1/2} \approx 1.6$. Thus, at least for this family, instability does seem to require a sizable anisotropy.

The obvious next question is whether models with reasonable (i.e. non-divergent) distribution functions, and small overall anisotropies, can be bar-unstable. If the answer is "yes", then the stability of virtually *every* radially anisotropic model will have to be verified before it can be used to describe an equilibrium galaxy. A good starting point for such a study might be the families of models derived by Dejonghe (1986).

3. CONSTRAINING THE DYNAMICS OF OBSERVED GALAXIES

No galaxy can remain in an unstable state for more than a few crossing times. In the case of disk galaxies, this fact is often used to infer the presence of heavy halos, massive bulges, or some other source of "rigid" gravity capable of inhibiting the tendency to bar formation. The situation is obviously very different for elliptical galaxies. The bar instability discussed above will destroy the spherical symmetry of an initially spherical model; but since elliptical galaxies are often very elongated, there is no reason to suppose that this instability did not act at some time in the past. In fact, it will be argued below that the radial-orbit instability may have been partly responsible for producing the flattenings of observed galaxies.

Nevertheless the existence of instabilities means that a theorist has less freedom than he might otherwise have had in constructing equilibrium models for particular galaxies. Consider the simplest case, a spherical galaxy with constant mass-to-light ratio. Complete knowledge of the surface-brightness profile $\mu(r)$ is sufficient to yield a unique space density $\rho(r)$. The number of different distribution functions $f(E, J^2)$ consistent with a given $\rho(r)$ is very large; the only constraint derives from the fact that a very radial distribution of orbits implies a density profile that diverges as fast as

$$\rho \propto r^{-2}|\ln r|^{-1/3} \tag{7}$$

at small radii (Agekyan 1961). Since real galaxies appear to have cores, they cannot be constructed out of purely radial orbits. However one can come very close; the "maximally anisotropic" models corresponding to observed galaxy luminosity profiles are almost completely radial (Richstone and Tremaine 1984; Merritt 1985). This is where instabilities can play a useful role: a sequence of models with fixed density profile will often become bar-unstable before the limit of maximum anisotropy is reached.

In practice, more information is often available than just $\mu(r)$. But even an exact determination of the projected velocity dispersion profile $\sigma_p(r)$ still leaves a formally infinite set of possible distribution functions (Dejonghe 1987). In the case of the most thoroughly modelled elliptical galaxy, M87, theorists have so far been content to stop after finding just one or two distribution functions consistent with the observed profiles (Newton and Binney 1984; Richstone and Tremaine 1985). It would be interesting to construct a *family* of models for M87, each member of which has the *same* surface brightness and velocity dispersion profiles, and see whether some members can be ruled out on the basis of instability.

A first step in this direction would be to test the stability of the handful of published models of M87. These models appear to be excellent candidates for instability since their central regions are dominated by radial orbits. Unfortunately the fraction of the total mass contained within the strongly anisotropic region is only $\sim 10^{-3}$, making these models very difficult to treat with a standard N-body code. Preliminary calculations (Merritt 1986), based on a mean-field code with 10^4 particles, suggest that the model of Newton and Binney (1984) is unstable.

4. THE RADIAL-ORBIT INSTABILITY DURING GALAXY FORMATION

Polyachenko (1981) was the first to make the point that, if an elliptical galaxy is to avoid forming in an unstable state, the instability must somehow manifest itself during the formation process. He later verified (Polyachenko 1985) that collapse starting from very cold and spherical initial conditions can be bar-unstable. In retrospect, it appears that quite a few people narrowly missed discovering the radial-orbit instability in this way. For instance, Aarseth and Binney (1978), in their study of collapse from flattened initial conditions, found that "[the] flattest final configuration...started from the least flattened initial configuration!" Polyachenko's work demonstrates that there need be no simple relation between the initial and final ellipticities of a galaxy that forms via collapse, as long as the collapse is sufficiently "strong" that it produces a significant fraction of nearly-radial orbits which can then clump into a bar.

It is easy to estimate roughly how hot a spherical, proto-galactic cloud must be to avoid making a bar. Simulations of radial collapse (e.g. van Albada 1982) show that galaxies formed in this way tend to have isotropic "cores" and radially-anisotropic "envelopes." The total squared angular momentum of such a galaxy–defined as the sum of the squared angular momenta of all the stars–is roughly $M^2 r_c^2 \sigma^2$, where r_c is the radius of the isotropic "core" and σ is the central velocity dispersion. In a spherical collapse, angular momentum is conserved; thus

$$M^2 r_c^2 \sigma^2 \approx M^2 R_i^2 \sigma_i^2 \approx 2MR_i^2 T_i \approx 8MR^2 T_i \tag{8}$$

where the subscript i refers to the unrelaxed state, T is the kinetic energy, and R is the radius. Energy conservation further requires

$$M\sigma^2 \approx 2|W_i| \tag{9}$$

where W is the potential energy. Combining relations (8) and (9),

$$\frac{r_c^2}{R^2} \approx \frac{4T_i}{|W_i|}. \tag{10}$$

The family of equilibrium models studied by Merritt and Aguilar (1985) is bar-unstable when the "anisotropy radius" r_a (cf. equation [6]) is less than ~ 0.3 times the half-mass radius. Equating r_a with r_c and $r_{1/2}$ with R gives

$$\frac{2T_i}{|W_i|} \gtrsim 0.05 \tag{11}$$

to insure that the galaxy which is formed will not be bar-unstable.

This admittedly crude estimate turns out to be roughly correct: Merritt and Aguilar (1985) find, for a particular set of smooth and spherical initial conditions, that collapses with $2T_i/|W_i| \lesssim 0.1$ are bar-unstable. This is a potentially important result because, as van Albada (1982) and McGlynn (1984) have shown, collisionless collapse from smooth initial conditions is *only* capable of producing objects resembling real galaxies if the initial state is roughly this cold.

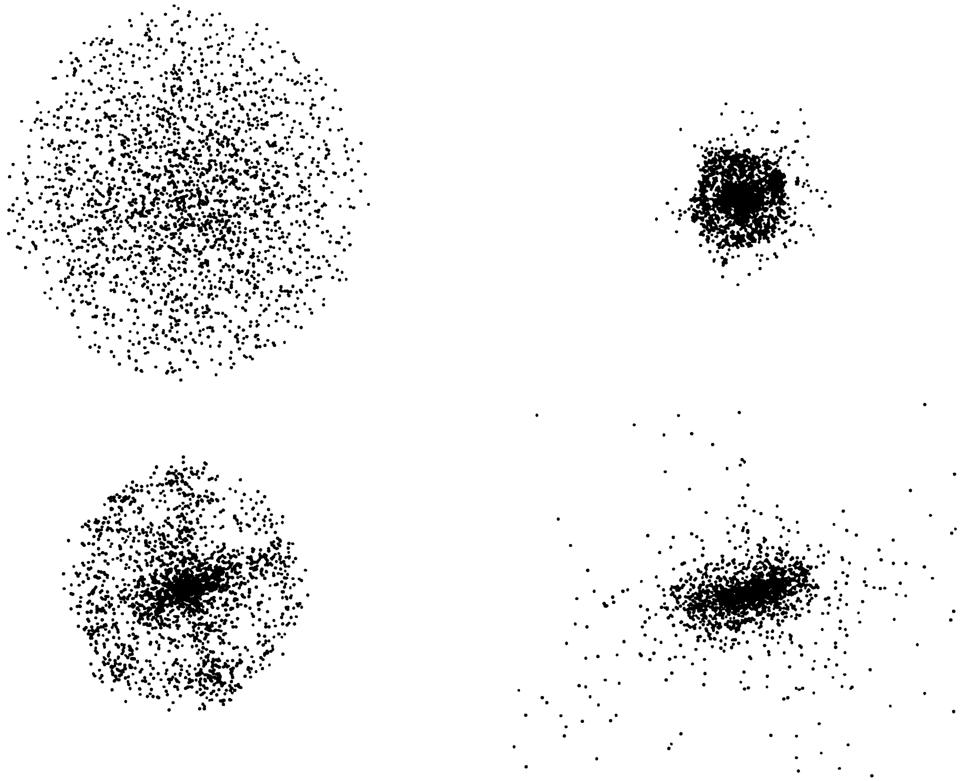

Figure 4. Collapse of a cold, initially oblate cloud, as seen from the direction of the initial symmetry axis. This is a 2500-particle, direct-summation N-body simulation (from M. Duncan).

How the radial-orbit instability manifests itself in more general collapses, which are unlikely to be smooth, spherical, or even very "cold", is an open question, and one that will be very difficult to answer numerically. All that is known so far (Aguilar, Merritt and Duncan, this volume) is that initial conditions that are close to oblate are susceptible to bar-formation in much the same way that spherical initial conditions are (cf. Figure 4). It would be hasty to conclude from this that oblate galaxies are impossible to make, however, since other processes, such as mergers or tidal-torquing, naturally produce oblate systems.

5. NUMERICAL TECHNIQUES

Broadly speaking, algorithms for evaluating the stability of equilibrium models can be divided into two classes: *linear* techniques that are based on the linearized (i.e. small-perturbation) form of the collisionless Boltzmann equation; and *nonlinear* techniques that are based on the actual equations of motion. Most of the results discussed above were obtained using N-body codes, which are fully nonlinear and relatively easy to program. Linear techniques, on the other hand, are far more accurate and efficient when searching for the exact boundary between stability and instability. The standard method for solving the linearized equations (cf. Kalnajs 1977) consists of expanding an initial perturbation in terms of a complete set of basis functions, computing the linear response density, and requiring the potential generated by the response to be equal to that imposed. In this way one obtains the "normal modes" of the system as well as the frequencies at which they grow.

In practice, normal mode calculations can be very difficult, and only a handful of stellar dynamical models (mostly disks) have been analyzed in this way. A feeling for the difficulties involved may be gotten by comparing the normal mode calculations of Fridman and Polyachenko (1984, Vol. 1, p. 219) for the anisotropic polytropes of equation (2), to those of Palmer and Papaloizou (1987) for the same family of models. The two sets of authors arrive at rather different conclusions: Fridman and Polyachenko find a clear-cut stability boundary at $m \approx -0.3$ (corresponding to $\sigma_r/\sigma_t \approx 1.2$), whereas Palmer and Papaloizou find instability for all $m < 0$ ($\sigma_r/\sigma_t > 1$) . The N-body experiments based on this family (Barnes, Goodman and Hut 1986) are not sufficiently accurate to decide which result is more correct, since the instability growth rate becomes very small near the stability boundary, and any evolution is swamped by noise due to the finite number of particles.

It would clearly be useful to develop a new technique that has the computational simplicity of an N-body code, and the (potential) accuracy of a normal mode analysis. Recently S. Tremaine and I have begun to investigate such a technique. The linearized, collisionless Boltzmann equation may be written

$$\frac{D_0 f_1}{Dt} = \frac{\partial \Phi_1}{\partial \mathbf{x}} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \tag{12}$$

where f_0 and f_1 are the equilibrium and perturbed distribution functions, Φ_1 is the perturbed potential, and $D_0 f_1/Dt$ denotes the rate of change of f_1 along an unperturbed trajectory defined by Φ_0:

$$\frac{D_0 f_1}{Dt} = \frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} - \frac{\partial \Phi_0}{\partial \mathbf{x}} \cdot \frac{\partial f_1}{\partial \mathbf{v}}.$$

Equation (12) is the starting point for any calculation of the linear response (cf. Polyachenko, this volume). One way of understanding this equation is to note that it gives the change with time, evaluated along an *unperturbed* orbit, of the perturbed phase-space density. Thus

$$f_1(\mathbf{x}, \mathbf{v}, t) = f_1(\mathbf{x}_0, \mathbf{v}_0, 0) + \int_0^t \frac{\partial \Phi_1}{\partial \mathbf{x}} \cdot \frac{\partial f_0}{\partial \mathbf{v}} dt, \tag{13}$$

where $(\mathbf{x}, \mathbf{v})$ are the coordinates at time t of a particle initially at $(\mathbf{x}_0, \mathbf{v}_0)$, and the integral is understood to extend along the trajectory containing the initial and final points. If we imagine dividing up phase space at time zero into a set of N regions with volumes ΔV_i, Liouville's theorem states that motion along the unperturbed trajectories will leave these volumes unchanged. Equation (13) then predicts the change with time of the perturbed mass m_i associated with each volume element ΔV_i:

$$m_i(t) = m_i(0) + \Delta V_i \int_0^t \frac{\partial \Phi_1}{\partial \mathbf{x}} \cdot \frac{\partial f_0}{\partial \mathbf{v}} dt. \tag{14}$$

Equation (14) is a prescription for a Monte-Carlo integration of the linearized equations. Its implementation requires only (a) the unperturbed equations of motion, i.e. $\nabla\Phi_0$; (b) the first velocity derivatives of the unperturbed distribution function f_0; and (c) an algorithm for evaluating the potential Φ_1 generated by a set of mass points m_i. Since this technique assigns *all* of the particles to the perturbation, none are "wasted" in reproducing the underlying equilibrium as in a standard N-body code. Also, since the technique is derived from the linearized equations, the solution is guaranteed to remain in the linear regime at all times. This fact makes the search for normal modes much easier than in a standard N-body code (cf. Sellwood 1983).

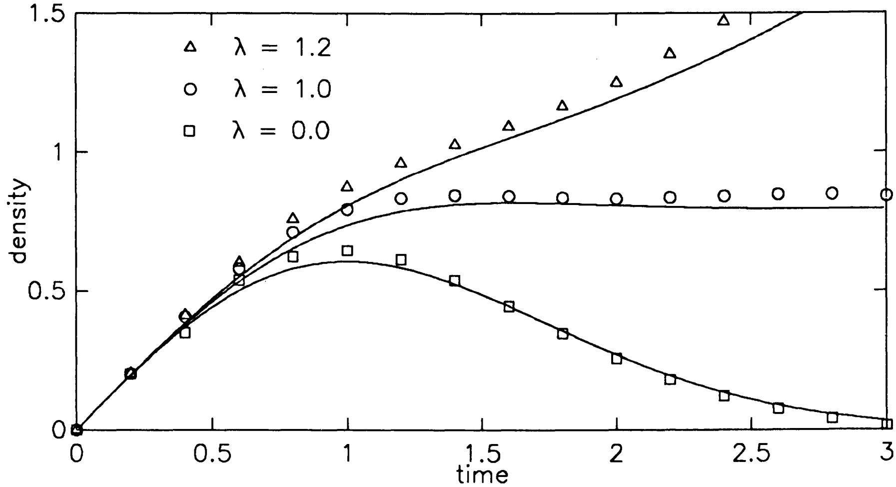

Figure 5. Evolution of the perturbed density of an initially homogeneous, Maxwellian medium. The parameter $\lambda = k_J/k$ is the ratio of the wavelength of the perturbation to the Jeans wavelength. Solid lines: exact solutions; symbols: Monte Carlo integrations.

Figure 5 shows a linearized Monte-Carlo calculation of the response of an infinite, homogeneous, Maxwellian distribution of particles to a perturbation of the form

$$\Phi_{ext}(\mathbf{x}, t) = \delta(t) \cos(\mathbf{k} \cdot \mathbf{x}),$$

i.e. an impulsive plane wave. The exact solution to the linearized equations is easy to obtain in this case (A. Toomre, private communication), and is also shown in Figure 5. The Monte-Carlo technique reproduces the exact solution quite well with only 1000 particles.

This Monte-Carlo technique can easily be applied to systems with any symmetry, as long as the equilibrium distribution function can be specified sufficiently smoothly. It should be a useful tool for evaluating the stability of both spherical and triaxial models.

REFERENCES

Aarseth, S. J., and Binney, J., 1978. *Mon. Not. R. Astron. Soc.*, **185**, 227.
Agekyan, T. A., 1961. *Leningrad University Bulletin*, **1**, 152.
Antonov, V. A., 1960. *Astr. Zh.*, **37**, 918 (translated in *Soviet Astron.*, **4**, 859).
Antonov, V. A., 1962. *Vestnik Leningrad Univ.*, **19**, 96 (translated in *Princeton Plasma Physics Lab. Rept. PPl-Trans-1*).
Antonov, V. A., 1973. In: *The Dynamics of Galaxies and Star Clusters*, p. 139, ed. G. B. Omarov, Nauka, Alma Ata (in Russian).
Barnes, J., 1985. In: *Dynamics of Star Clusters, IAU Symposium No. 114*, p. 297, ed. J. Goodman and P. Hut, Reidel, Dordrecht, The Netherlands.
Barnes, J., Goodman, J., and Hut, P., 1986. *Astrophys. J.*, **300**, 112.
Bisnovatyi-Kogan, G. S., Zel'dovich, Ya. B., and Fridman, A. M., 1968. *Doklady Akademii Nauk SSSR*, **182**, 794 (translated in *Soviet Physics-Doklady*, **13**, 969).
Dejonghe, H., 1986. *Physics Reports*, **133**, 217.
Dejonghe, H., 1987. *Mon. Not. R. Astron. Soc.*, **224**, 13.
Doremus, J. P., and Feix, M. R., 1973. *Astron. Astrophys.*, **29**, 401.
Fridman, A. M., and Polyachenko, V. L., 1984. *Physics of Gravitating Systems*, volumes 1 and 2, Springer-Verlag, New York.
Gillon, D., Doremus, J. P., and Baumann, G., 1976. *Astron. Astrophys.*, **48**, 467.
Hénon, M., 1959. *Annales d'Astrophysique*, **22**, 126.
Hénon, M., 1973. *Astron. Astrophys.*, **24**, 229.
Jaffe, W., 1983. *Mon. Not. R. Astron. Soc.*, **202**, 995.
Jeans, J. H., 1929. *Astronomy and Cosmogony, 2nd ed.*, Cambridge Univ. Press, Cambridge, England.
Kalnajs, A. J., 1977. *Astrophys. J.*, **212**, 637.
King, I. R., 1966. *Astron. J.*, **71**, 64.
Lynden-Bell, D., 1979. *Mon. Not. R. Astron. Soc.*, **187**, 101.
McGlynn, T. A., 1984. *Astrophys. J.*, **281**, 13.
Merritt, D., 1985. *Astron. J.*, **90**, 1027.
Merritt, D., 1986. In preparation.
Merritt, D., and Aguilar, L. A., 1985. *Mon. Not. R. Astron. Soc.*, **217**, 787.
Newton, A. J., and Binney, J., 1984. *Mon. Not. R. Astron. Soc.*, **210**, 711.
Palmer, P. L., and Papaloizou, J., 1987. This volume, p. 515.
Polyachenko, V. L., 1981. *Pis'ma Astron. Zh.*, **7**, 142 (translated in *Soviet Astron. Lett.*, **7**, 79).
Polyachenko, V. L., 1985. *Astronomicheskii Tsirkuliar*, No. 1405 (in Russian).
Richstone, D. O., and Tremaine, S., 1984. *Astrophys. J.*, **286**, 27.
Richstone, D. O., and Tremaine, S., 1985. *Astrophys. J.*, **296**, 370.
Sellwood, J. A., 1983. *J. Comp. Phys.*, **50**, 337.
Sygnet, J. F., Des Forets, G., Lachieze-Rey, M., and Pellat, R., 1984. *Astrophys. J.*, **276**, 737.
Toomre, A., 1964. *Astrophys. J.*, **139**, 1217.
van Albada, T. S., 1982. *Mon. Not. R. Astron. Soc.*, **201**, 939.

DISCUSSION

Lauer: According to Gerhard, even a small black hole will destroy box orbits. Is it possible to have a black hole large enough to prevent formation of a bar, but small enough to still permit anisotropies large enough to explain cusps and rises in velocity dispersion? Can a black hole prevent formation of a bar?

Merritt: Box orbits are not a prerequisite for velocity anisotropy, so the destruction of box orbits does not necessarily imply the elimination of cusps or high central velocities. There are three ways that a central black hole might suppress instabilities. 1) A black hole fixes part of the gravitational field. 2) Reflection of radial orbits around a black hole would probably reduce the growth rate of a bar mode. 3) Adding a black hole to a galaxy with a known central velocity dispersion entails a modification of the orbital distribution near the center so that the dispersion remains constant. This modification will usually go in the direction of more circular orbits, which increases stability.

de Zeeuw: What are the smallest axis ratios one can obtain via the radial–orbit instability?

Duncan: The initially oblate collapse that Dave showed was run using a direct summation N–body code and the results agree well with the results of the quadrupolar code. Secondly, I have run oblate and prolate configurations with initial flattenings as large as 6:1. The final states are always triaxial with largest axial ratios of 2.5 to 1. This is consistent with suggestions that flatter non-rotating systems are unstable.

Villumsen: Some years ago I made some collapse calculations from non–spherical rotating initial conditions. The most extreme results were triaxial E8 systems with low v/σ. They also had beautiful $r^{1/4}$ profiles.

King: You said that an isotropic core would stabilize an anisotropic envelope. In the case $(\sigma_r/\sigma_t)^2 = 1 + (r/r_a)^2$, how big a core do you need for stability? One reason why I ask is that Meylan's poster paper at this meeting (p. 449) shows that in Omega Centauri r_a is 2 or 3 times the core radius. Is Omega Centauri unstable?

Casertano: Josh Barnes and I have run N–body simulations for a King model with a concentration parameter ~ 1.5. We can see instability (on a scale of ~ 5 half–mass dynamical times) if $r_a \leq 2r_c$; no instability can be seen of $r_a \geq 4r_c$.

Merritt: Aguilar and I found that models with r_a greater than about 0.3 times the half–mass radius were stable. Meylan's inferred value for the anisotropy radius of Omega Centauri is roughly equal to its half–mass radius, so his models are probably stable.

Djorgovski: Can you tell whether the instabilities will occur if the stars move in a pre–existing, perhaps spherical, probably isothermal dark halo?

Merritt: I believe that Dr. Polyachenko addresses this question in his presentation.

Richstone: The next question one might ask is what the dynamical appearance of a galaxy is after the instability has run its course. In particular, suppose you take a model of M87 with $\sigma_r >> \sigma_T$ near $r = 0$ and it makes a bar. If you observe the bar, does it still have a large σ_{obs} as $r \to 0$? If so, maybe you don't need to put the black hole back in.

Merritt: The answer will depend on the direction from which the bar is observed. As seen from the long axis, the galaxy might well appear circular, with a high central velocity dispersion.

Palmer: In some of the models for which we calculated growth rates for the unstable modes, we found very little difference between the growth rates of the $n = 2$, 4, 6 & 8 terms in the spherical harmonic expansion of the potential. How well do you believe that your code can describe the evolution of the instability in these models?

Merritt: Mean–field codes will, of course, have difficulty following the growth of very small–scale modes. However, even in the models you analyzed, the largest scale ($n = 2$) modes were always the fastest growing ones. Furthermore, J. Barnes has shown that both mean–field and "exact" N–body codes give similar results for a number of unstable models.

Duncan: The diagram that Dave showed contains several spoke–like structures just as the bar is forming. They are real, but the bar–mode instability is dominant and one bar wins in a rather short time.

Palmer: Recently I have been simulating systems with very radial orbits using a direct–summation code. I find higher instabilities as well as the original $l = 2$ bar instability; however, in the long run these higher–l features seem to become weaker, leaving only the bar–deformation.

Binney: Andrew May and I have a note coming out (1986, *Mon. Not. R. astr. Soc.*, **221**, 13P) in which we point out that there is a natural method of testing for the stability of any model whose distribution function can be written in terms of the action integrals, $f = f(J_i)$: One distorts the model's potential and asks whether the density over– or under–responds. We used this method to determine the stability of anisotropic isochrone spheres with less than $1/500$ of the computer time required by a typical N–body simulation.

Merritt: The method is a very clever one and should be explored. Your comparison of computing times neglects the fact that the same N–body code can be used for a variety of models. Also, the adiabatic deformation technique is inherently approximate, whereas the accuracy of N–body tests can always be improved, e.g., by increasing the number of particles.

David Merritt.

https://doi.org/10.1017/S0074180900185274 Published online by Cambridge University Press