Canad. J. Math. Vol. **58** (2), 2006 pp. 401–418

On Pointwise Estimates of Positive Definite Functions With Given Support

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Abstract. The following problem has been suggested by Paul Turán. Let Ω be a symmetric convex body in the Euclidean space $\mathbb{R}^d$ or in the torus $\mathbb{T}^d$. Then, what is the largest possible value of the integral of positive definite functions that are supported in Ω and normalized with the value 1 at the origin? From this, Arestov, Berdysheva and Berens arrived at the analogous pointwise extremal problem for intervals in $\mathbb{R}$. That is, under the same conditions and normalizations, the supremum of possible function values at z is to be found for any given point $z \in \Omega$. However, it turns out that the problem for the real line has already been solved by Boas and Kac, who gave several proofs and also mentioned possible extensions to $\mathbb{R}^d$ and to non-convex domains as well.

Here we present another approach to the problem, giving the solution in $\mathbb{R}^d$ and for several cases in $\mathbb{T}^d$. Actually, we elaborate on the fact that the problem is essentially one-dimensional and investigate non-convex open domains as well. We show that the extremal problems are equivalent to some more familiar ones concerning trigonometric polynomials, and thus find the extremal values for a few cases. An analysis of the relationship between the problem for $\mathbb{R}^d$ and that for $\mathbb{T}^d$ is given, showing that the former case is just the limiting case of the latter. Thus the hierarchy of difficulty is established, so that extremal problems for trigonometric polynomials gain renewed recognition.

1 Extremal Problems for Positive Definite Functions, Periodic and Not

Let us denote $\mathbb{T}^d := \left[-\frac{1}{2}, \frac{1}{2}\right)^d \subset \mathbb{R}^d$ with the usual modified topology of periodicity, that is, take the topology of $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. For $\Omega \subseteq \mathbb{T}^d$ any open domain[1], we put

$$\mathcal{F}^*(\Omega) := \{f\colon \mathbb{T}^d \to \mathbb{R} : \operatorname{supp} f \subseteq \Omega,\ f(0) = 1,\ f \text{ positive definite}\},\tag{1}$$

and, analogously, when $\Omega \subseteq \mathbb{R}^d$ is any open set,

$$\mathcal{F}(\Omega) := \{f\colon \mathbb{R}^d \to \mathbb{R} : \operatorname{supp} f \subseteq \Omega,\ f(0) = 1,\ f \text{ positive definite.}\}\tag{2}$$

Recall that positive definiteness of functions (and even measures and tempered distributions) can be defined or equivalently characterized by nonnegativity of the

Received by the editors February 9, 2003; revised March 30, 2005.

The first author was supported in part by European Commission IHP Network HARP (Harmonic Analysis and Related Problems) Contract Number: HPRN-CT-2001-00273-HARP. The second author was supported by the Hungarian National Foundation for Scientific Research, Grant T-049693 and T-049301, and by the Hungarian-French Scientific and Technological Governmental Cooperation, Project F-10/04.

AMS subject classification: Primary: 42B10; secondary: 26D15, 42A82, 42A05.

Keywords: Fourier transform, positive definite functions and measures, Turán's extremal problem, convex symmetric domains, positive trigonometric polynomials, dual extremal problems.

[1] Note that $0 \notin \Omega$ entails $f(0) = 0$, hence the function classes $\mathcal{F}^*(\Omega)$ and $\mathcal{F}(\Omega)$ defined in (1) and (2) are empty; therefore, it suffices to restrict attention to the case $0 \in \Omega$.

Fourier transform. In case (1) positive definiteness means $\widehat{f}(n) \geq 0$ ($\forall n \in \mathbb{Z}^d$), while in case (2) it means $\widehat{f}(x) \geq 0$ ($\forall x \in \mathbb{R}^d$).

In 1970 in a discussion with S. B. Stechkin [17], Paul Turán posed the following problem. Let $d = 1$ and $\Omega := (-h, h) \subset \mathbb{T}$: what is the largest possible value of the integral $\int_{\mathbb{T}} f$ over all $f \in \mathcal{F}^*((-h, h))$? The question was later investigated in higher dimensions and in $\mathbb{R}^d$ as well. As a natural condition for the above Turán extremal problem, convexity of the underlying domain Ω is usually supposed.

For an account of the problem see [1, 9] and the references therein. However, no authors seem to have noticed that Boas and Kac had already settled the analogous (and relatively easy) case of an interval $(-h, h) \subset \mathbb{R}$, see [3, Theorem 5].

The natural pointwise analogue of the above question of Turán for intervals in $\mathbb{T}$ or $\mathbb{R}$ was studied in [2]. For general domains in arbitrary dimension these problems can be formulated as follows.

Problem 1.1 (Boas–Kac-type pointwise extremal problem for the space) *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a positive definite function with* supp $f \subseteq \Omega$ *and $f(0) = 1$. Let also $z \in \Omega$. What is the largest possible value of $f(z)$? In other words, determine*

$$\mathcal{M}(\Omega, z) := \sup_{f \in \mathcal{F}(\Omega)} f(z). \tag{3}$$

Remark 1.2 Obviously, $\mathcal{M}(\Omega, z) \leq 1$, as

$$1 \pm f(z) = \int_{\mathbb{R}} (1 \pm \exp(2\pi i z t))\widehat{f}(t)\, dt = \int_{\mathbb{R}} (1 \pm \cos(2\pi z t))\widehat{f}(t)\, dt \geq 0.$$

One might miss a more precise specification of the function class $f\colon \mathbb{R}^d \to \mathbb{R}$ here and similarly in the problems listed below. The fact that considering L^1, C or C^∞ leads to the same answer, *i.e.,* the same extremal values, will be discussed at the beginning of §2.

Problem 1.3 (Turán-type pointwise extremal problem for the torus) *Let $\Omega \subseteq \mathbb{T}^d$ be any open set, and let $f\colon \mathbb{T}^d \to \mathbb{R}$ be a positive definite function with* supp $f \subseteq \Omega$ *and $f(0) = 1$. Let also $z \in \Omega$. What is the largest possible value of $f(z)$? In other words, determine*

$$\mathcal{M}^*(\Omega, z) := \sup_{f \in \mathcal{F}^*(\Omega)} f(z). \tag{4}$$

Remark 1.4 Let $\Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$ and $f\colon \Omega \to \mathbb{R}$. For the function f to be positive definite *on the torus* means a nonnegativity condition for the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} f(x)\, dx$$

only for a discrete set of values of ξ, namely $\xi \in \mathbb{Z}^d$, while positive definiteness of f as a function on $\mathbb{R}^d$ is equivalent to nonnegativity of the Fourier transform $\widehat{f}$ for all occurring values. From this it follows that we always have

$$\mathcal{M}^*(\Omega, z) \geq \mathcal{M}(\Omega, z). \tag{5}$$

The extremal value in Problem 1.1 was estimated together with its periodic analogue Problem 1.3 in [2] for dimension $d = 1$. However, Boas and Kac had already solved the $d = 1$ case of Problem 1.1, which seems to have been unnoticed in [2].

These problems are not only analogous, but also related to each other and, in fact, Problem 1.1 is only a special limiting case of the more complex Problem 1.3 (see Theorem 6.6). On the other hand, Boas and Kac have already observed that Problem 1.1 (dealt with for $\mathbb{R}$ in [3]) is connected to trigonometric polynomial extremal problems. In particular, from the solution to the interval case they deduced the value (20) below of the extremal problem due to Carathéodory [4] and Fejér [5]. They also established a connection (see [3, Theorem 6]) that corresponds to the one-dimensional case of the first part of our Theorem 2.1.

It is appropriate at this point to consider also the following type of trigonometric polynomial extremal problems. Let us define for any $H \subseteq \mathbb{N}_2 := \mathbb{N} \cap [2, \infty)$,

$$(6)\quad \Phi(H) := \Big\{ \varphi \colon \mathbb{T} \to \mathbb{R}_+ \mid \lambda \in \mathbb{R},\ \varphi \geq 0, \\ \varphi(t) \sim 1 + \lambda \cos 2\pi t + \sum_{k \in H} c_k \cos 2\pi k t \Big\}$$

and with a given $m \in \mathbb{N}_2$ and $H \subseteq \mathbb{N}_2$ also

$$(7)\quad \Phi_m(H) := \Big\{ \varphi \colon \mathbb{T} \to \mathbb{R} \mid \lambda \in \mathbb{R},\ \varphi\Big(\frac{j}{m}\Big) \geq 0 \ \ (j \in \mathbb{Z}), \\ \varphi(t) = 1 + \lambda \cos 2\pi t + \sum_{k \in H} c_k \cos 2\pi k t \Big\}.$$

Problem 1.5 (Carathéodory–Fejér type trigonometric polynomial problem) *Determine the extremal quantity*

$$(8)\qquad M(H) := \sup\{\lambda = 2\widehat{\varphi}(1) \mid \varphi \in \Phi(H)\}.$$

Remark 1.6 Observe that $M(H) \leq 2$, always, as

$$|\lambda/2| = |\widehat{\varphi}(1)| \leq \|\varphi\|_1 = \int \varphi = \widehat{\varphi}(0) = 1.$$

Problem 1.7 (Discretized Carathéodory-Fejér type extremal problem) *Determine*

$$(9)\qquad M_m(H) := \sup\{\lambda = 2\widehat{\varphi}(1) \mid \varphi \in \Phi_m(H)\}.$$

Remark 1.8 It should be remarked here that obviously we have $\Phi(H) \subseteq \Phi_m(H)$. So we always have $M_m(H) \geq M(H)$.

In this note we present the exact solution of Problem 1.1 that is in line with what Boas–Kac [3] suggests. Actually, we have to acknowledge that Boas and Kac mentioned the possibility of extending one of their methods, the Poisson summation, to

https://doi.org/10.4153/CJM-2006-017-8 Published online by Cambridge University Press

higher dimensions, so some parts of what follows can be interpreted as implicitly present already in their work [3]. But here we also obtain some results for the more complex periodic version.

However, the main result of the present investigation is perhaps the understanding that the above point-value extremal problems are in fact equivalent to the above trigonometric polynomial extremal problems, thus transferring information on one problem to the equivalent other problem in several cases. Until now the equivalence formulated below remained unclear in spite of the fact that Boas and Kac found ways to deduce the solution of the trigonometric extremal problems in Problem 1.5 from their results on Problem 1.1. We also obtain a clear picture of the limiting relation between torus problems and space problems, and, parallel to this, between the finitely conditioned trigonometric polynomial extremal problems of Problem 1.7 and the positive definite trigonometric polynomial extremal problems of Problem 1.5.

2 Preliminaries: Formulation of the Equivalence Results

Note that in the above definitions (1), (2) or (6), (7) it is left a bit unclear what function classes are considered as $\mathbb{R}^d \to \mathbb{R}$, $\mathbb{T}^d \to \mathbb{R}$ or $\mathbb{T} \to \mathbb{R}$. However, this causes no ambiguity, since it is not hard to see that the extremal problems (3), (4), (8) or (9) yield the same extremal values when integrable functions (with continuity of f supposed only at z in case of (3) or (4)) are considered, and when compactly supported C^∞ functions are taken into account. Indeed, on $\mathbb{T}$ or $\mathbb{T}^d$ this follows after a convolution by the Fejér kernels. The same way we can restrict ourselves even to trigonometric polynomials in $\Phi(H)$ or $\Phi_m(H)$ as well.

Passing on to the case of the real space $\mathbb{R}^d$, first we show that it suffices to consider *bounded* open sets only. To this end let us consider the auxiliary positive definite function

$$\Delta_R(x) := \frac{1}{|B_{R/2}|}\chi_{B_{R/2}} * \chi_{B_{R/2}} \tag{10}$$

with $B_r := \{x \in \mathbb{R}^d \mid |x| \le r\}$, and take $f_N := f\Delta_N$ to obtain

$$\mathcal{M}(\Omega, z) = \lim_{N\to\infty} \mathcal{M}(\Omega_N, z) = \lim_{N\to\infty} \mathcal{M}(\operatorname{int}\Omega_N, z),$$

where $\Omega_N := \{x \in \Omega \mid |x| \le N\} = \Omega \cap B_N$, and thus $\Omega_N \subseteq \operatorname{int}\Omega_{N+1}$.

Next observe that for any bounded open Ω, the condition $\operatorname{supp} f \subseteq \Omega$ entails that $\operatorname{supp} f$ is compact and of a fixed positive distance η from the boundary of Ω. Thus convolution of f with the (convolution) square of some approximate identity k_δ with $\operatorname{supp} k_\delta \subseteq B_\delta$ leads to a function $f_\delta := f*k_\delta*k_\delta$ satisfying $\operatorname{supp} f_\delta \subseteq \operatorname{supp} f + B_{2\delta} \subseteq \Omega$ if $\delta < \frac{1}{2}\eta$. Hence, with a smooth k_δ we have $f_\delta \in \mathcal{F}(\Omega) \cap C^\infty(\Omega)$, while for arbitrary fixed $\epsilon > 0$ and with δ correspondingly small enough, $f_\delta(z) \ge f(z) - \epsilon$, in view of the continuity of f at z.

Now let us define for $z \in \Omega$ the derived set

$$H(\Omega, z) := \{k \in \mathbb{N}_2 \mid kz \in \Omega,\ -kz \in \Omega\}. \tag{11}$$

Our first goal is to show that in fact the Boas–Kac type Problem 1.1 is a one-dimensional problem. This is contained in the following result.

Theorem 2.1 *Let* $0 \in \Omega \subseteq \mathbb{R}^d$ *be any open set and* $z \in \Omega \cap (-\Omega)$. *With the above notations we have*

$$\mathcal{M}(\Omega, z) = \frac{1}{2} M(H(\Omega, z)).$$

Remark 2.2 Note that in case $z \in \Omega$, $z \notin -\Omega$, we trivially conclude that $\mathcal{M}(\Omega, z) = 0$ since for all $f \in \mathcal{F}(\Omega)$, $\operatorname{supp} f \subseteq \Omega \cap (-\Omega)$ follows from (16) below. Also $0 \in \Omega$ is necessary, for a positive definite function f must vanish everywhere if $0 \notin \operatorname{supp} f$.

To tackle the Turán-type Problem 1.3, one may consider $f \in L^1(\mathbb{T}^d)$ with continuity supposed at z, or even $f \in C^\infty(\mathbb{T}^d)$.

Here positive definiteness of f is equivalent to $\widehat{f}(n) \geq 0\, (\forall n \in \mathbb{Z}^d)$, and, similarly to (16), one gets $f(x) = f(-x)\, (\forall x \in \mathbb{T}^d)$. Thus $\operatorname{supp} f$ is symmetric, hence $\operatorname{supp} f \subseteq \Omega \cap (-\Omega)$.

Once again we see that (4) vanishes unless $z \in \Omega \cap (-\Omega)$ and that it suffices to restrict ourselves to sets symmetric about the origin. In other words, if $z \notin \Omega$ or if $z \notin (-\Omega)$, then $\mathcal{M}^*(\Omega, z) = 0$, while for $z = 0$ obviously $\mathcal{M}^*(\Omega, 0) = 1$. These are the trivial cases, and for the remaining cases we introduce a further notation. Put

$$\mathcal{Z} := \mathcal{Z}(z) := \{nz \pmod{\mathbb{T}^d} \mid n \in \mathbb{Z}\}. \tag{12}$$

The set $\mathcal{Z}$ is finite if and only if we have $z \in \mathbb{Q}^d$, that is, $z = (\frac{p_1}{q_1}, \dots, \frac{p_d}{q_d})$ with $p_j, q_j \in \mathbb{Z}, (p_j, q_j) = 1 (j = 1, \dots, d)$. In this case we have with $m = [q_1, \dots, q_d]$, the least common multiple of the denominators, that $mz = 0 (\operatorname{mod} \mathbb{T}^d)$, and for arbitrary $n, n' \in \mathbb{Z}\ nz = n'z (\operatorname{mod} \mathbb{T}^d)$ if and only if $n \equiv n' (\operatorname{mod} m)$.

Let us keep the definition (11) with an interpretation $(\operatorname{mod} \mathbb{T}^d)$ for infinite $\#\mathcal{Z}$. On the other hand, in case $\#\mathcal{Z} = m$ we put

$$H_m(\Omega, z) := \{k \in [2, m/2] \mid kz \in \Omega,\ -kz \in \Omega\} = H(\Omega, z) \cap [2, m/2]. \tag{13}$$

Moreover, for any set $H \subset \mathbb{Z}$ we define

$$H(m) := \{k \in [2, m/2] \mid \exists h \in H \text{ such that } \pm k \equiv h \pmod{m}\}.$$

Remark 2.3 Note the following relations for an arbitrary $H \subseteq \mathbb{N}_2$. First, if there exists any index $k \in H$ with $k \equiv 1 (\operatorname{mod} m)$, then we obtain $M_m(H) = \infty$, because $1 + a \cos 2\pi t - a \cos 2k\pi t$ is nonnegative at j/m for all $j = 1, \dots, m$ and for any $a \in \mathbb{R}$. Similarly, for $k \equiv \ell \pmod{m}$ $\cos 2k\pi t - \cos 2\ell\pi t$ vanishes at all points of the form j/m, hence the frequencies can be changed mod m to reduce φ to a trigonometric polynomial of degree at most m. Moreover, since this can be used even for negative indices, and as $\cos(-k2\pi t) = \cos k2\pi t$, we can reduce the support of $\widehat{\varphi}$ to $[0, m/2]$. That is, either $M_m(H) = \infty$ (in case there is a $k \in H$ with $k \equiv 0, \pm 1 (\operatorname{mod} m)$), or $M_m(H) = M_m(H(m))$.

Now we can formulate

Theorem 2.4 *Let $0 \in \Omega \subseteq \mathbb{T}^d$ be any open set and $z \in \Omega \cap (-\Omega)$. Then the extremal quantity (4) depends only on the set $\mathcal{Z}$. In case $\#\mathcal{Z}$ is infinite, we have*

$$\mathcal{M}^*(\Omega, z) = \frac{1}{2} M(H(\Omega, z)). \tag{14}$$

In case $\#\mathcal{Z} = m$ is finite, we have

$$\mathcal{M}^*(\Omega, z) = \frac{1}{2} M_m(H_m(\Omega, z)). \tag{15}$$

3 Proof of Theorem 2.1

First note that it suffices to consider symmetric sets $\Omega' = \Omega \cap (-\Omega)$ only. Indeed, if Ω is arbitrary, and $f \in \mathcal{F}(\Omega)$, $f \in C_0^\infty(\mathbb{R}^d)$, then by $\widehat{f} \geq 0$ Fourier inversion yields

$$f(x) = \overline{f(x)} = \overline{\int \widehat{f}(y) e^{2\pi i \langle x, y\rangle}\, dy} = \int \widehat{f}(y) e^{-2\pi i \langle x, y\rangle}\, dy = f(-x). \tag{16}$$

Thus for all $f \in \mathcal{F}(\Omega)$, $\operatorname{supp} f$ is necessarily symmetric. On the other hand, $H(\Omega, z)$ is symmetrized by definition (11) with respect to Ω. Hence we can restrict ourselves to symmetric sets. Without loss of generality we can assume that Ω is also bounded.

Now given a bounded symmetric open set Ω the proof consists of proving the two inequalities below.

(A) $\mathcal{M}(\Omega, z) \leq M(H(\Omega, z))/2$

Let f have $f(0) = 1$, be positive definite and have support in Ω. Define also the positive definite Radon measure

$$\mu_z := \sum_{k \in \mathbb{Z}} \delta_{kz}.$$

The function f being continuous, the measure

$$\nu_z = f \cdot \mu_z = \sum_{k \in \mathbb{Z}} f(kz) \delta_{kz} \tag{17}$$

is well defined and positive definite as well.

Notice now, because of the boundedness of Ω, that the sum in (17) is actually a finite one. More precisely, if we have *e.g.*, $\Omega \subseteq B_n$, then we find

$$\nu_z := \sum_{k=-(n-1)}^{n-1} f(kz) \delta_{kz} = \delta_0 + f(z)(\delta_z + \delta_{-z}) + \sum_{k \in H(\Omega, z)} f(kz)(\delta_{kz} + \delta_{-kz}),$$

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and that

$$0 \le \widehat{\nu_z}(x) = 1 + 2f(z)\cos 2\pi\langle z, x\rangle + \sum_{k\in H(\Omega,z)} 2f(kz)\cos 2\pi k\langle z, x\rangle, \quad (x \in \mathbb{R}^d).$$

Setting $t = \langle z, x\rangle$ and observing that the trigonometric polynomial

$$1 + 2f(z)\cos 2\pi t + \sum_{k\in H(\Omega,z)} 2f(kz)\cos 2\pi kt$$

is nonnegative, we obtain $2f(z) \le M(H(\Omega, z))$.

(B) $\mathcal{M}(\Omega, z) \ge M(H(\Omega, z))/2$

For a function $\varphi\colon \mathbb{T} \to \mathbb{R}$ let us call the *(restricted) spectrum* of φ the set $S := S(\varphi) := \operatorname{supp}\widehat{\varphi} \cap \mathbb{N}_2 \subseteq \mathbb{N}_2$. Also, we will use the term *full spectrum* and the notation $S' := S'(\varphi)$ for the set $S' := \{-1, 0, 1\} \cup S \cup (-S)$, whether the exponential Fourier coefficients at $-1, 0$ or 1 happen to vanish or not.

Take any trigonometric polynomial $\varphi \in \Phi(H)$ with spectrum $S \subseteq H := H(\Omega, z)$. Recall that taking the supremum in (8) over the function class (6) yields the same result as considering such trigonometric polynomials only. Consider the measure

$$\alpha_z := \delta_0 + (\lambda/2)(\delta_z + \delta_{-z}) + \sum_{k\in S}(c_k/2)(\delta_{kz} + \delta_{-kz}),$$

whose Fourier transform is essentially equal to the polynomial $\varphi(t)$ in (6). Hence α_z is a positive definite measure.

Take now the "triangle function" Δ_ϵ defined as in (10), but here with a subscript ϵ small enough to guarantee that

(1) the sets $kz + B_\epsilon$, $k \in S'$, are disjoint, *i.e.,* $\epsilon < \frac{|z|}{2}$;
(2) these sets are all contained in Ω, *i.e.,* $\epsilon < \operatorname{dist}\{\partial\Omega, S'z\}$.

Finally define

$$f := \alpha_z * \Delta_\epsilon,$$

which is a positive definite function supported in Ω with value 1 at the origin and with $f(z) = \lambda/2$. This proves that $\mathcal{M}(\Omega, z) \ge M(H(\Omega, z))/2$, as desired. ∎

4 Applications of Theorem 2.1

The first application concerns the original convex case of the pointwise Boas–Kac type problem formulated in Problem 1.1. A symmetric, bounded convex domain with nonempty interior, *i.e.,* a *convex body*, defines a norm. So for a vector x let $\|x\|$ denote the norm of x defined by Ω, that is

$$\|x\| := \inf\{\lambda > 0 \ \frac{1}{\lambda}x \in \Omega\}.$$

In other words, Ω is the unit ball of the norm $\|\cdot\|$.

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Corollary 4.1 (Boas–Kac [3]) *Let $\Omega \subseteq \mathbb{R}^d$ be a convex open domain, symmetric about* 0*. Suppose that*

$$\frac{1}{n+1} \le \|z\| < \frac{1}{n}, \tag{18}$$

for some $n \ge 1$. Then

$$\mathcal{M}(\Omega, z) = \cos\frac{\pi}{n+2}.$$

Proof of Corollary 4.1 First observe that for the symmetric, convex, bounded, open set Ω the norm of z satisfies (18) if and only if $H(\Omega, z) = [2, n]$. Thus by Theorem 2.1 the problem reduces to the extremal problem

$$M_n := \sup\{\lambda \mid \exists \varphi(t) \ge 0, \varphi(t) = 1 + \lambda\cos 2\pi t + \sum_{k=2}^{n} c_k \cos 2\pi kt\}. \tag{19}$$

This problem was settled by Fejér, see *e.g.*, [5] or [6, pp. 869–870]. To finish the proof, we quote from these or from [11, Problem VI. 52, p. 79] the formula

$$M_n = 2\cos\frac{\pi}{n+2}. \tag{20}$$

■

Note that [2, Theorem 2] gave the estimate $\frac{n}{n+1} \le \mathcal{M}(\Omega, z) \le \frac{1}{2}(1 + \cos(\frac{\pi}{n+1}))$ for the one-dimensional case. The above exact solution and some calculation shows that both of these estimates are sharp for $n = 1$, but none of them is for $n > 1$. However, this is covered (at least for $d = 1$) by [3, Theorem 2].

Now the $n \to \infty$ limiting case easily leads to

Corollary 4.2 (Boas–Kac [3]) *Suppose that the open set $\Omega \subseteq \mathbb{R}^d$ contains all integer multiples of the point $z \in \mathbb{R}^d$. Then $\mathcal{M}(\Omega, z) = 1$.*

Moreover, we also derive easily the d-dimensional extension of [3, Theorem 3].

Corollary 4.3 (Boas–Kac) *Suppose that for some $n \in \mathbb{N}$ the open set $\Omega \subset \mathbb{R}^d$ contains no integer multiples kz of the point $z \in \mathbb{R}^d$ with $k > n$. Then we have again $\mathcal{M}(\Omega, z) \le M_n = 2\cos\frac{\pi}{n+2}$.*

Apart from the convex case there are several cases of (3) when through the trigonometric extremal problem (8), either the precise value, or at least some estimate can be found.

Theorem 4.4 *Let Ω be a symmetric open set and $z \in \Omega$. Then the value of the extremal quantity* (3) *satisfies the following relations.*

(i) *If $H(\Omega, z) = \{n\}$, then $\mathcal{M}(\Omega, z) = \frac{1}{2\cos\frac{\pi}{2n}}$.*
(ii) *If $H(\Omega, z) = \mathbb{N}_2 \setminus \{n\}$, then $\mathcal{M}(\Omega, z) = \cos\frac{\pi}{2n}$.*
(iii) *If $H(\Omega, z) = (n, \infty) \cap \mathbb{N}_2$, then $\mathcal{M}(\Omega, z) = \frac{1}{2\cos\frac{\pi}{n+2}}$.*

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(iv) *If* $H(\Omega, z) = 2\mathbb{N} + 1$, *then* $\mathcal{M}(\Omega, z) = \frac{2}{\pi}$.
(v) *If* $H(\Omega, z) = 2\mathbb{N}$, *then* $\mathcal{M}(\Omega, z) = \frac{\pi}{4}$.

Remark 4.5 The extremal quantities $\mathcal{M}$ and M are monotonic in the sets Ω and H, respectively, hence the above relations imply the corresponding inequalities when we know only that *e.g.*, $nz \in \Omega$, etc. We skip the formulation.

Proof of Theorem 4.4 In view of Theorem 2.1, the calculation of $\mathcal{M}(\Omega, z)$ hinges on finding the value of $M(H(\Omega, z))$. The solutions of the corresponding trigonometric polynomial extremal problems, relevant to the above list (i)–(v), can be looked up from the literature as follows.

(i) An easy calculation, see [12].
(ii) See [12, Proposition 1].
(iii) See [13].
(iv) See the end of [16].
(v) See [15, pp. 492–493].

When $\mathcal{M}(\Omega, z)$ is known for a certain $H(\Omega, z)$, then further cases can be obtained via the following duality result.

Lemma 4.6 (see [12]) *Let* $H \subseteq \mathbb{N}_2$ *be arbitrary. Then we have*

$$M(H)M(\mathbb{N}_2 \setminus H) = 2.$$

In fact, this gives (ii) once (i) is known; (iii) and Corollary 4.3 and also (iv) and (v) are similarly related, although they were obtained differently in the works mentioned above.

To formulate the corresponding relation in Problem 1.1 we can record

Corollary 4.7 *For any open set* $\Omega \subseteq \mathbb{R}^d$ *and* $z \in \Omega$ *we have*

$$\mathcal{M}(\Omega, z)\mathcal{M}(\Omega^*, z) = \frac{1}{2},$$

where Ω^* *is any open, symmetric set containing* 0, z *and* $(\mathbb{N}_2 \setminus H(\Omega, z))z$, *but disjoint from* $H(\Omega, z)z$.

Ending this section, let us recall that investigation of Turán-type problems started with keeping an eye on number theoretic applications and connected problems. The interesting papers of Gorbachev and Manoshina [7, 8] mention [10].

Problem 4.8 *Determine*

$$\Delta(n) := \sup\{M(H)/2 \mid H \subseteq \mathbb{N}_2, |H| = n\}.$$

We only know (*cf.* [12])

$$1 - \frac{5}{(n+1)^2} \le \Delta(n) \le 1 - \frac{0.5}{(n+1)^2}.$$

The question is relevant to the Beurling theory of generalized primes, see [14].

5 Proof of Theorem 2.4

As above, without loss of generality we can restrict ourselves to sets Ω symmetric about the origin. Similarly to the proof of Theorem 2.1, we are to prove two inequalities for both cases.

Case $\#\mathcal{Z} = \infty$: $\mathcal{M}^*(\Omega, z) \le M(H(\Omega, z))/2$**.**

Let $f \in \mathcal{F}^*(\Omega) \cap C^\infty(\mathbb{T}^d)$. We consider the measure

$$\sigma_z^{(N)} := \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N}\Big) \delta_{kz}.$$

This measure is positive definite since for all $n \in \mathbb{Z}^d$ we have

$$\widehat{\sigma_z^{(N)}}(n) = \int_{\mathbb{T}^d} e^{-2\pi i \langle n, x\rangle} d\sigma_z^{(N)}(x) = \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N}\Big) e^{2\pi i \langle n, kz\rangle} =: K^{(N)}(2\pi \langle n, z\rangle),$$

where $K^{(N)}$ is the usual Fejér kernel, which is nonnegative. Let us denote $H(N) := H(\Omega, z) \cap [2, N]$.

The function f being continuous and even, the measure

$$\rho_z := f \cdot \sigma_z^{(N)} = f(0)\delta_0 + \sum_{k \in \{1\} \cup H(N)} \Big(1 - \frac{k}{N}\Big) f(kz)(\delta_{kz} + \delta_{-kz}) \tag{21}$$

is well defined and, by $\widehat{\rho_z} = \widehat{f} * \widehat{\sigma_z^{(N)}}$, is positive definite as well. In view of $f(0) = 1$ we now find for arbitrary $n \in \mathbb{Z}^d$ that

$$0 \le \widehat{\rho_z}(n) = 1 + \Big(2 - \frac{2}{N}\Big) f(z) \cos 2\pi \langle z, n\rangle + \sum_{k \in H(N)} \Big(2 - \frac{2k}{N}\Big) f(kz) \cos 2\pi k \langle z, n\rangle.$$

Setting $t := \langle z, n\rangle$ yields

$$0 \le \varphi_N(t) := 1 + 2\Big(1 - \frac{1}{N}\Big) f(z) \cos 2\pi t + \sum_{k \in H(N)} 2\Big(1 - \frac{k}{N}\Big) f(kz) \cos 2\pi k t.$$

Since $\#\mathcal{Z} = \infty$, here for the various values of $n \in \mathbb{Z}^d$ the derived variable t will be dense in $\mathbb{T}$.

Hence we can conclude that in the infinite case $\varphi_N(t) \in \Phi(H(\Omega, z))$. This gives $2\big(1 - \frac{1}{N}\big) f(z) \le M(H(\Omega, z))$ for all $N \in \mathbb{N}$. Whence the stated inequality.

Case $\#\mathcal{Z} = m < \infty$: $\mathcal{M}^*(\Omega, z) \le M_m(H_m(\Omega, z))/2$**.**

Let again $f \in \mathcal{F}^*(\Omega) \cap C^\infty(\mathbb{T}^d)$. Now we consider the measure

$$\sigma_{z,m} := \frac{1}{2} \sum_{k=-[\frac{m-1}{2}]}^{[\frac{m-1}{2}]} \delta_{kz} + \frac{1}{2} \sum_{k=-[\frac{m}{2}]}^{[\frac{m}{2}]} \delta_{kz}.$$

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For all $n \in \mathbb{Z}^d$ we have

$$\widehat{\sigma_{z,m}}(n) = \int_{\mathbb{T}^d} e^{-2\pi i\langle n,x\rangle} d\sigma_{z,m}(x) = 1 + \sum_{k=1}^{[\frac{m-1}{2}]} \cos 2\pi k\langle n,z\rangle + \sum_{k=1}^{[\frac{m}{2}]} \cos 2\pi k\langle n,z\rangle.$$

Since $\#\mathcal{Z} = m < \infty$, where $m = [q_1,\dots,q_d]$ with $z = (\frac{p_1}{q_1},\dots,\frac{p_d}{q_d})$, $(p_j,q_j) = 1 (j = 1,\dots,d)$, for the various values of $n \in \mathbb{Z}^d$ the derived variable $t := \langle n,z\rangle$ will cover exactly the values of $j/m (\operatorname{mod} \mathbb{T})$. For these values, however, direct calculation shows that the above sum is either exactly m (in case $n \equiv 0 (\operatorname{mod} m)$, *i.e.,* $t \in \mathbb{Z}$), or vanishes. Thus, again, the measure $\sigma_{z,m}$ will be positive definite.

The function f being continuous and symmetric, the measure

$$\rho_{z,m} := f\cdot\sigma_{z,m} = f(0)\delta_0 + \sum_{k=1}^{[\frac{m-1}{2}]} f(kz)(\delta_{kz}+\delta_{-kz}) + \sum_{k=1}^{[\frac{m}{2}]} f(kz)(\delta_{kz}+\delta_{-kz}) \tag{22}$$

is well defined and, by $\widehat{\rho_{z,m}} = \widehat{f} * \widehat{\sigma_{z,m}}$, is positive definite as well. In view of $f(0) = 1$ we now find for all $n \in \mathbb{Z}^d$

$$0 \le \widehat{\rho_z}(n) = 1 + 2f(z)\cos 2\pi t + \sum_{k=2}^{[\frac{m-1}{2}]} f(kz)\cos 2\pi kt + \sum_{k=2}^{[\frac{m}{2}]} f(kz)\cos 2\pi kt, \tag{23}$$

where $t = \langle z,n\rangle$ as above. So let us now write

$$\varphi_{z,m}(t) := 1 + 2f(z)\cos 2\pi t + \sum_{k=2}^{[\frac{m-1}{2}]} f(kz)\cos 2\pi kt + \sum_{k=2}^{[\frac{m}{2}]} f(kz)\cos 2\pi kt.$$

It follows that

$$\varphi_{z,m}(t) = 1 + 2f(z)\cos 2\pi t + \sum_{k\in H_m(\Omega,z)} c_k^* \cos 2\pi kt,$$

for some $c_k^* \in \mathbb{R}$. Similarly as above, (23) implies $\varphi_{z,m}(j/m) \ge 0\,(j = 0,\dots,m-1)$. That is, we conclude $\varphi_{z,m} \in \Phi_m(H_m(\Omega,z))$ and thus $2f(z) \le M_m(H_m(\Omega,z))$. Hence the statement.

Case $\#\mathcal{Z} = \infty$ **:** $\mathcal{M}^*(\Omega,z) \ge M(H(\Omega,z))/2$**.**

Let φ be any trigonometric polynomial from the class (6). Then φ has (restricted) spectral set S and full spectrum $S' := \{-1,0,1\} \cup \pm S$ with $S \subseteq H := H(\Omega,z)$ necessarily finite. Note that the supremum in the definition (8) of $M(H(\Omega,z))$ can be restricted to the trigonometric polynomials of (6).

Consider the measure

$$\alpha_z = \delta_0 + (\lambda/2)(\delta_z+\delta_{-z}) + \sum_{k\in S}(c_k/2)(\delta_{kz}+\delta_{-kz}),$$

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whose Fourier transform $\widehat{\alpha_z}(n) = \varphi(\langle z, n\rangle)\,(n \in \mathbb{Z}^d)$ is essentially the polynomial $\varphi(t)$ itself. Hence α_z is a positive definite measure.

Now take the "triangle function" Δ_ϵ, defined in (10), with a parameter ϵ small enough to guarantee that

(1) the sets $kz + B_\epsilon$, $(k \in S')$, are disjoint;
(2) these sets are all contained in Ω, *i.e.,* $\epsilon < \operatorname{dist}\{\partial\Omega, S'z\}$.

Since we consider only a finite subset S of H, and $S' = \{-1, 0, 1\} \cup \pm S$, these conditions are met with some positive ϵ as no two different multiples of z are equal in $\mathbb{T}^d$. Finally define

$$f := \alpha_z * \Delta_\epsilon,$$

which is a positive definite function supported in Ω with value 1 at the origin and with $f(z) = \lambda/2$. This proves that $\mathcal{M}^*(\Omega, z) \geq \lambda/2$, hence taking supremum over all polynomials $\varphi \in \Phi(H)$ concludes the proof. ■

Case $\#\mathcal{Z} = m < \infty$: $\mathcal{M}^*(\Omega, z) \geq M_m(H_m(\Omega, z))/2$**.**

We denote here $H := H_m(\Omega, z)$. Now take any φ in (7).

Consider the measure

$$\alpha_z = \delta_0 + (\lambda/2)(\delta_z + \delta_{-z}) + \sum_{k<\frac{m}{2}, k\in H} (c_k/2)(\delta_{kz} + \delta_{-kz}) + c_{m/2}\delta_{mz/2},$$

with the last term appearing only if m is even and $m/2$ belongs to the spectral set (13). Observe that for the true spectrum of this measure we have

$$S^* := \operatorname{supp}\widehat{\alpha_z} := S^*(\alpha_z) \subseteq \{-1, 0, 1\} \cup \pm H \setminus \{-m/2\} = S' \setminus \{-m/2\}, \tag{24}$$

where the last term $(\setminus\{-m/2\})$ appears only if m is even. Thus it is easy to see that the multiples $kz\,(k \in S^*)$ are different even in $\mathbb{T}^d$.

Now let us prove that α_z is positive definite. Taking $n \in \mathbb{Z}^d$ arbitrarily, consider the Fourier transform

$$\widehat{\alpha_z}(n) = 1 + \lambda\cos 2\pi\langle z, n\rangle + \sum_{k<\frac{m}{2}, k\in H} c_k \cos 2\pi k\langle z, n\rangle + c_{m/2}e^{-im\pi\langle z,n\rangle}.$$

Here, by the condition $\langle z, n\rangle = j/m$ for some integer j, we have in the last term $e^{-m\pi\langle z,n\rangle} = (-1)^j = \cos\pi j = \cos m\pi\langle z, n\rangle$ and we get $\widehat{\alpha_z}(n) = \varphi(\langle z, n\rangle) = \varphi(j/n)$. It follows that $\widehat{\alpha_z}(n) \geq 0$ by definition (7).

Take now the "triangle function" Δ_ϵ defined in (10) with a parameter ϵ small enough to ensure

(1) the sets $kz + B_\epsilon$, $(k \in S^*)$, are disjoint;
(2) these sets are all contained in Ω, *i.e.,* $\epsilon < \operatorname{dist}\{\partial\Omega, S^*z\}$.

These conditions are met with some positive ϵ since no two different multiples $kz\,(k \in S^*)$ are equal in $\mathbb{T}^d$, and by definitions (7) and (24) we necessarily have $S^*z \subseteq \Omega$.

Finally define

$$f = \alpha_z * \Delta_\epsilon,$$

which is a positive definite function supported in Ω with value 1 at the origin and with $f(z) = \lambda/2$. This proves that $\mathcal{M}^*(\Omega, z) \geq \lambda/2$, hence taking supremum over all polynomials $\varphi \in \Phi_m(H)$ concludes the proof. ∎

6 Applications of Theorem 2.4 and Further Connections

Arestov, Berdysheva and Berens [2] mention the one dimensional symmetric interval special case of the following fact.

Proposition 6.1 *Suppose $\Omega \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ is an open set. Then*

$$\mathcal{M}(\Omega, z) \leq \mathcal{M}^*(\Omega, z).$$

Proof The original proof of [2] uses the natural periodization of functions $f \in \mathcal{F}(\Omega)$. Taking $g(x) := \sum_{n \in \mathbb{Z}^d} f(x - n)$ maps $\mathcal{F}(\Omega)$ injectively to $\mathcal{F}^*(\Omega)$, which proves the proposition. However, we have also an alternative argument here, as Theorems 2.1 and 2.4 translate the extremal problems in question to extremal problems for trigonometric polynomials. In case $\#\mathcal{Z} = \infty$ the $\mathbb{R}^d$ and $\mathbb{T}^d$ interpretations of (11) give $H_{\mathbb{R}^d}(\Omega, z) \subset H_{\mathbb{R}^d}(\Omega + \mathbb{Z}^d, z) = H_{\mathbb{T}^d}(\Omega, z)$. For $\#\mathcal{Z} = m < \infty$ $H_{\mathbb{R}^d}(\Omega, z) \subseteq [2, m-2]$. Indeed, $-z \in \Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$, and as $0 \neq mz$ but $mz \equiv 0 \pmod{\mathbb{T}^d}$, we obtain that $(m-1)z \notin \Omega$ in $\mathbb{R}^d$, and similarly for $k \geq m$ $kz \notin [-\frac{1}{2}, \frac{1}{2})^d$ excludes the possibility of $k \in H_{\mathbb{R}^d}(\Omega, z)$. Thus it is easy to see that

$$M_m(H_m(\Omega, z)) = M_m(H(\Omega, z) \cap [2, m-2]) = M_m(H_{\mathbb{R}^d}(\Omega, z)). \tag{25}$$

Now it is obvious that $\Phi_m(H) \supseteq \Phi(H)$ and thus $M_m(H) \geq M(H)$ for arbitrary $H \subseteq \mathbb{N}_2$, and we get the assertion even for the finite case.

Corollary 6.2 *Let $\Omega \subseteq (-\frac{1}{2}, \frac{1}{2})^d$ be a convex, symmetric domain. Then we have*

$$\mathcal{M}^*(\Omega, z) \geq w(\|z\|), \quad \text{where} \quad w(t) := \cos \frac{\pi}{\lceil 1/t \rceil + 1}.$$

Proof Corollary 4.1 gives $\mathcal{M}(\Omega, z) \geq w(\|z\|)$. Thus combining Proposition 6.1 and Corollary 4.1 proves the assertion. ∎

Remark 6.3 The above estimate is a sharpening of (14) in [2, Theorem 3].

The following assertion is obvious both directly and by Theorem 2.1.

Proposition 6.4 *For all open sets $\Omega \subseteq \mathbb{R}^d$ and $z \in \mathbb{R}^d$, $\alpha > 0$ we have*

$$\mathcal{M}(\alpha\Omega, \alpha z) = \mathcal{M}(\Omega, z).$$

Proposition 6.5 *For $\Omega \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ open, $z \in \mathbb{T}^d$ and $N \in \mathbb{N}$ we have*

$$\mathcal{M}^*\left(\frac{1}{N}\Omega, \frac{1}{N}z\right) \le \mathcal{M}^*(\Omega, z).$$

Proof One can work out the generalization of the proof of [2, Lemma 5], which is the one-dimensional interval special case of this assertion. Instead, we note that $k\frac{1}{N}z \in \frac{1}{N}\Omega \pmod{\mathbb{T}^d}$ entails $kz \in \Omega \pmod{\mathbb{T}^d}$, and by Theorem 2.4 the $\#\mathcal{Z} = \infty$ case follows.

On the other hand for finite $\#\mathcal{Z}(z) = m < \infty$ we have $\#\mathcal{Z}\left(\frac{1}{N}z\right) = Nm$ and $\Phi_m(H) \supseteq \Phi_{mN}(H)$. Thus combining (15) and (25) yields

$$\begin{aligned} 2\mathcal{M}^*(\Omega, z) &= M_m(H_m(\Omega, z)) = M_m(H_{\mathbb{R}^d}(\Omega, z)) \\ &= M_m\left(H_{\mathbb{R}^d}\left(\frac{1}{N}\Omega, \frac{1}{N}z\right)\right) \ge M_{mN}\left(H_{\mathbb{R}^d}\left(\frac{1}{N}\Omega, \frac{1}{N}z\right)\right) \\ &= M_{mN}\left(H_{mN}\left(\frac{1}{N}\Omega, \frac{1}{N}z\right)\right) = 2\mathcal{M}^*\left(\frac{1}{N}\Omega, \frac{1}{N}z\right). \qquad \blacksquare \end{aligned}$$

The next assertion is the generalization of [2, Theorem 4].

Theorem 6.6 *For any bounded open set $\Omega \subset \mathbb{R}^d$ and $z \in \mathbb{R}^d$ we have*

$$\lim_{\alpha \to +0} \mathcal{M}^*(\alpha\Omega, \alpha z) = \mathcal{M}(\Omega, z).$$

Remark 6.7 Here the condition of boundedness ensures that for α small enough we have $\alpha\Omega \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ and the expression under the limit on the left hand side is defined by (4).

Proof Again, extending the original arguments of [7, 8] or [2] leads to a proof. There the idea is to multiply $f \in \mathcal{F}^*(\alpha\Omega)$ by a fixed positive kernel, say $\Delta_{\frac{1}{4}}$, and exploit that for α small $\Delta_{\frac{1}{4}}|_{\alpha\Omega}$ is approximately 1.

Alternatively, we can argue as follows. Let Ω be bounded by R and let $\alpha < \frac{1}{2R}$; then $\alpha\Omega \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right)^d$. Moreover, using $\mathbb{R}^d$ interpretation of the arising sets we always have

$$H_{\mathbb{R}^d}(\Omega, z) = H_{\mathbb{R}^d}(\alpha\Omega, \alpha z) \subset \left[2, \frac{R}{|z|}\right], \tag{26}$$

while $m(\alpha) := \#\mathcal{Z}(\alpha z) \ge \frac{1}{\alpha|z|} \to \infty \quad (\alpha \to 0)$. Note that here for irrational α we can have $m(\alpha) = +\infty$, but defining the index function $m(\alpha)$ in this extended sense does not question the asserted limit relation.

In what follows we unify terminology by writing $H_\infty(\Theta, w) = H(\Theta, w)$ while keeping the notation $H_n(\Theta, w) = H(\Theta, w) \cap [2, n/2]$ for finite n. For the finite case we have $H_{\mathbb{R}^d}(\alpha\Omega, \alpha z) = H_{\mathbb{R}^d}(\Omega, z) \subseteq [2, \frac{m(\alpha)}{2}]$, and in view of (11) and (26), $H := H_{m(\alpha)}(\alpha\Omega, \alpha z) = H_{\mathbb{T}^d}(\alpha\Omega, \alpha z) \cap [2, \frac{m(\alpha)}{2}] = H_{\mathbb{R}^d}(\alpha\Omega, \alpha z) = H_{\mathbb{R}^d}(\Omega, z)$, too.

Now if $m(\alpha) = \infty$, then we are to consider the normalized, nonnegative trigonometric polynomials $\varphi \in \Phi_\infty(H) := \Phi(H)$ defined by (6), while for finite $m(\alpha) < \infty$, the function set to be considered is $\Phi_m(H)$ defined by (7).

Now let $\alpha_n \to 0$, and φ_n be an extremal polynomial in $\Phi_{m(\alpha_n)}(H)$. In view of the nonnegativity conditions for these sets we get $|c_k| \le 2 \quad (k \in H)$, applying finite Fourier Transform in case $m(\alpha_n) < \infty$. Hence with $K := \lceil 2R/|z| \rceil$ we find $\varphi_n \in \mathcal{F}_K := \{\varphi(t) = 1 + 2\sum_{k=1}^{K} a_k \cos 2\pi kt \mid |a_k| \le 1,\ k = 1, \dots, K\}$, which is a compact subset of $C(\mathbb{T})$. Thus without loss of generality we can suppose that $\varphi_n \to \phi \in \mathcal{F}_K$ uniformly as $n \to \infty$. Since $m(\alpha_n) \to \infty$, we must have $\phi \ge 0$. Moreover, if we write $\phi(t) = 1 + 2\sum_{k=1}^{K} a_k \cos 2\pi kt$ and $\varphi_n(t) = 1 + 2\sum_{k=1}^{K} a_k^{(n)} \cos 2\pi kt$, then $\lim_{n\to\infty} a_k^{(n)} = a_k$, so $\phi \in \Phi(H)$ and

$$\lim_{n\to\infty} \mathcal{M}^*(\alpha_n \Omega, \alpha_n z) = \lim_{n\to\infty} a_1^{(n)} = a_1 \le \mathcal{M}(\Omega, z).$$

On the other hand Propositions 6.1 and 6.4 give the converse inequality. ∎

7 Calculations of Extremal Values for Some Special Cases

Now we formulate a periodic case analogue of the Boas–Kac result Corollary 4.2.

Proposition 7.1 *Suppose that the open set $\Omega \subseteq \mathbb{T}^d$ contains all integer multiples of the point $z \in \mathbb{T}^d$,* i.e., *$\mathcal{Z} \subset \Omega$ with $\mathcal{Z}$ defined in* (12). *Then $\mathcal{M}^*(\Omega, z) = 1$.*

Proof In case $\#\mathcal{Z} = \infty$, Theorem 2.4 gives $\mathcal{M}^*(\Omega, z) = M(H(\Omega, z))/2 = M(\mathbb{N}_2)/2 = 1$ immediately. Now let $\#\mathcal{Z} = m < \infty$. Then Theorem 2.4 yields $\mathcal{M}^*(\Omega, z) = M_m(H_m(\Omega, z))/2 = M_m([2, m/2])/2$. To see that this quantity achieves 1, it suffices to consider the cosine polynomial

$$\varphi_m(t) := 1 + \sum_{k=1}^{[\frac{m-1}{2}]} \cos 2\pi kt + \sum_{k=1}^{[\frac{m}{2}]} \cos 2\pi kt.$$

Direct calculation proves again $\varphi_m(j/m) \ge 0\ (j \in \mathbb{N})$, thus $\varphi_m \in \Phi_m([2, m/2])$ and now we find $M_m([2, m/2])/2 = 1$. ∎

With the following applications in mind we first prove

Lemma 7.2 *Let $m \in 2\mathbb{N}$ be even. Then we have $M_m([2, m/2)) = 1 + \cos \frac{2\pi}{m}$.*

Proof Let $m = 2n$ and

$$\varphi(t) = 1 + \sum_{k=1}^{n-1} c_k \cos 2\pi kt \in \Phi_m([2, n)).$$

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Using the finite Fourier Transform coefficient formula and $\varphi(j/m) \geq 0\,(j \in \mathbb{N})$ we obtain

$$
\begin{aligned}
c_1 &= \frac{2}{m}\sum_{j=0}^{m-1}\varphi\Big(\frac{j}{m}\Big)\cos\frac{2\pi j}{m} \\
&= \frac{1}{n}\sum_{l=0}^{n-1}\varphi\Big(\frac{l}{n}\Big)\cos\frac{2\pi l}{n} + \frac{1}{n}\sum_{l=0}^{n-1}\varphi\Big(\frac{2l+1}{m}\Big)\cos\Big(\frac{2\pi l}{n}+\frac{\pi}{n}\Big) \\
&\leq \frac{1}{n}\sum_{l=0}^{n-1}\varphi\Big(\frac{l}{n}\Big) + \frac{1}{n}\sum_{l=0}^{n-1}\varphi\Big(\frac{l}{n}+\frac{1}{m}\Big)\cos\Big(\frac{\pi}{n}\Big) = 1+\cos\Big(\frac{\pi}{n}\Big).
\end{aligned}
$$

On the other hand, take the cosine polynomial

$$\phi_m(t) := 1 + \sum_{k=1}^{n-1}\Big(1+\cos\frac{\pi k}{n}\Big)\cos 2\pi k t.$$

Direct calculation gives

$$\phi_m\big(\frac{j}{m}\big) = \begin{cases} m & j \equiv 0 \pmod m, \\ m/2 & j \equiv \pm 1 \pmod m, \\ 0 & \text{otherwise}, \end{cases}$$

whence $\phi_m(\frac{j}{m}) \geq 0\,(j \in \mathbb{N})$ and $\phi_m \in \Phi_m([2,n))$. ∎

Corollary 7.3 (Arestov–Berdysheva–Berens [2]) *For dimension one we have*

(i) *For $(p,q) = 1$, q even we have $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2}),\frac{p}{q}) = \frac{1}{2}(1+\cos\frac{2\pi}{q})$.*
(ii) *For $(p,q) = 1$, q odd we have $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2}),\frac{p}{q}) = 1$.*
(iii) *For $z \notin \mathbb{Q}$ we have $\mathcal{M}^*((-\frac{1}{2},\frac{1}{2}),z) = 1$.*

Proof In case (i) $\#\mathcal{Z} = q = 2r$, and $H(\Omega,z) = \mathbb{N}_2 \setminus r\mathbb{N}$, $H_q(\Omega,z) = [2,r-1]$. Hence in view of Theorem 2.4 it suffices to show that $M_q([2,r)) = 1+\cos(2\pi/q)$, which follows from Lemma 7.2. For the cases (ii) and (iii) we clearly have $\mathcal{Z} \subseteq \Omega$, hence Proposition 7.1 applies. ∎

Similarly to the above result of [2], we can also answer the pointwise Turán extremal problem for $\Omega = (-\frac{1}{2},\frac{1}{2})^d$.

Theorem 7.4 *Let* $\Omega = (-\frac{1}{2}, \frac{1}{2})^d \in \mathbb{T}^d$*. Then we have*

(i) $\mathcal{M}^*((-\frac{1}{2}, \frac{1}{2})^d, z) = 1$ *if* $z \notin \mathbb{Q}^d$.

Moreover, if $z \in \mathbb{Q}^d$, $z = (\frac{p_1}{q_1}, \dots, \frac{p_d}{q_d})$ *with* $(p_j, q_j) = 1$, $q_j = 2^{s_j} t_j \quad (s_j \in \mathbb{N})$, $t_j \in 2\mathbb{N}+1 \quad (j = 1, \dots, d)$ *and* $m := [q_1, \dots, q_d] = 2^s t \quad t \in 2\mathbb{N}+1$, *then we have either*

(ii) $1 \le s = s_1 = \dots = s_d$, *and then* $\mathcal{M}^*((-\frac{1}{2}, \frac{1}{2})^d, z) = \frac{1}{2}(1 + \cos\frac{2\pi}{m})$, *or*
(iii) $s = 0$ *or* $\exists j$, $1 \le j \le d$ *with* $s_j < s$ *and then* $\mathcal{M}^*((-\frac{1}{2}, \frac{1}{2})^d, z) = 1$.

Proof Case (i) is covered by Proposition 7.1 above. If $z \in \mathbb{Q}^d$, then the set defined in (12) is finite and we have $\#\mathcal{Z} = m = [q_1, \dots, q_d]$. Let us determine the set $H(\Omega, z)$ first. For $k \in \mathbb{N}$ we have $kz \notin \Omega$ iff $kp_j/q_j \equiv 1/2 \pmod 1$ $(j = 1, \dots, d)$, *i.e.,* $2kp_j/q_j \equiv 1 \pmod 2$ $(j = 1, \dots, d)$. It follows that $q_j | 2k$ $(j = 1, \dots, d)$, and we can not have a solution $k \in \mathbb{N}$ if $\exists j$ so that q_j is odd, since then $2k/q_j$ must be even. Hence we can consider the case when all $s_j \ge 1$ and, by $(p_j, q_j) = 1$, all p_j is odd. Then using $p_j \in 2\mathbb{Z}+1$ the condition becomes $2k/q_j \equiv 1 \pmod 2$ $(j = 1, \dots, d)$. Hence $m = [q_1, \dots, q_d] | 2k$ and $s = s_j$ $(j = 1, \dots, d)$ since otherwise for any $s_j < s$ we get $2k/q_j = nm/q_j = n2^{s-s_j} t/t_j \equiv 0 \pmod 2$. In all, $kz \notin \Omega$ occurs only in case (ii), while case (iii) will again be covered by Proposition 7.1. In case (ii), when $kz \notin \Omega$ happens, it occurs precisely for multiples of $m/2 \in \mathbb{N}$. That is, case (ii) now reduces to the determination of $\mathcal{M}^*(\Omega, z) = M_m([2, m/2))/2 = (1 + \cos 2\pi/m)/2$ in view of Theorem 2.4 and Lemma 7.2. ■

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