# JACOBI ELLIPTIC ALGEBRAS OF $SO(3)$

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**Abstract.** A class of algebras that describe invariant pseudo-Riemannian connections on $SO(3)$ is shown to comprise Jacobi elliptic algebras arising from the Jacobi elliptic functions.

**1. Introduction.** We define the *Jacobi elliptic algebra* $J(k)$ of *modulus* $k \in \mathbf{R}$, (the field of real numbers), as the 3-dimensional real commutative algebra with multiplication $xy$ given by

$$e_i e_j = \tfrac{1}{2}\epsilon_{ijk}^2 \gamma_k e_k \qquad (i, j, k = 1, 2, 3), \tag{1}$$

with $\gamma_1 = -\gamma_2 = -1$ and $\gamma_3 = -k^2$, where $\{e_1, e_2, e_3\}$ is a basis of $J(k)$ and $\epsilon_{ijk}$ is the Levi-Civita symbol with $\epsilon_{123} = 1$. The term of $J(k)$ originated from the Jacobi elliptic functions of modulus $k$ which may be defined as the solutions of the autonomous system of quadratic differential equations

$$\frac{dx_1}{dt} - x_2 x_3 = 0, \qquad \frac{dx_2}{dt} + x_3 x_1 = 0, \qquad \frac{dx_3}{dt} - k^2 x_1 x_2 = 0 \tag{2}$$

with the initial values $x_1(0) = 0$ and $x_2(0) = x_3(0) = 1$. See [**1**, **7**]. If $x(t) = \sum_{i=1}^{3} x_i(t) e_i \in J(k)$ and $x'(t) = \frac{dx}{dt} = \sum_{i=1}^{3} x_i'(t) e_i$ then, since $J(k)$ is commutative, using the product in $J(k)$ we can rewrite (2) in the form

$$\frac{dx}{dt} + x(t)^2 = 0 \tag{3}$$

with the initial value $x(0) = e_2 + e_3$. Equations of the form (3) have appeared in several contexts dealing with quadratic dynamical or mechanical systems. (See, for example, [**3**, **5**, **8**] and the references therein.)

The Jacobi elliptic algebras $J(k)$ also comprise those algebras which determine all left-invariant pseudo-Riemannian connections on the Lie group $SO(3)$ corresponding to distinct moments of inertia. The primary concern of this note is to determine all left-invariant pseudo-Riemannian connections on $SO(3)$ by classifying its corresponding algebras, and we show that those algebras with distinct moments of inertia are isomorphic to Jacobi elliptic algebras of certain moduli.

**2. Preliminaries.** Let $G$ be a real Lie group (of dimension $n$) with Lie algebra $\mathbf{g}$. As is well known, (for example, see [**4**, **5**]), there is a one-to-one correspondence between the set of all left $G$-invariant connections $\nabla$ on $G$ and the set of all algebras $(\mathbf{g}, *)$ defined on

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$\mathbf{g}$, under the relation $(\nabla_{\tilde{x}}\tilde{y})_e = x*y$ for $x, y \in \mathbf{g}$, where $\tilde{x}$ denotes the unique left-invariant vector field on $G$ determined by $x$. Thus, the affine space $\mathscr{A}(G)$ of all such connections on $G$ is isomorphic to $\mathrm{Hom}_{\mathbf{R}}(\mathbf{g}\otimes\mathbf{g}, \mathbf{g})$. If $\nabla \in \mathscr{A}(G)$ and $(\mathbf{g}, *)$ is the algebra associated with $\nabla$, then $x*y$ decomposes as

$$x*y = \tfrac{1}{2}[x,y] + x\circ y \tag{4}$$

for a bilinear multiplication $x\circ y$ on $\mathbf{g}$, and $\nabla$ is torsion free if and only if $(\mathbf{g},\circ)$ is commutative. In this case, $x\circ y = \frac{1}{2}(x*y+y*x)$, $x*y-y*x=[x,y]$ for $x,y\in\mathbf{g}$, and $(\mathbf{g}, *)$ is said to be *compatible* with the Lie algebra $\mathbf{g}$. Thus, the torsion free connections in $\mathscr{A}(G)$ are determined by $\mathrm{Hom}_{\mathbf{R}}(S(\mathbf{g}\otimes\mathbf{g}),\mathbf{g})$ (of dimension $\frac{1}{2}n^2(n+1)$), where $S(\mathbf{g}\otimes\mathbf{g})$ is the $\mathbf{R}$-space of symmetric elements in $\mathbf{g}\otimes\mathbf{g}$.

Assume that $\mathbf{g}$ possesses a pseudometric $\mu$, so that $\mu$ induces a pseudo-Riemannian structure on $G$ and there is a unique torsion free connection $\nabla\in\mathscr{A}(G)$, called a pseudo-Riemannian connection on $G$, such that the algebra $(\mathbf{g},*)$ of $\nabla$ satisfies the invariant condition

$$\mu(x*y,z)+\mu(y,x*z)=0 \tag{5}$$

for $x,y,z\in\mathbf{g}$. See [**4**, **5**]. Suppose next that there is an invariant Riemannian metric $(\,,)$ on $G$; that is

$$([x,y],z)+(y,[x,z])=0 \tag{6}$$

for $x,y,z\in\mathbf{g}$. Notice that if $G$ is compact and semisimple, then the Killing form on $\mathbf{g}$ induces such a metric.

If $\mu$ is a pseudometric on $\mathbf{g}$, then since $\mu$ is nondegenerate and symmetric, there is a unique symmetric operator $I\in GL(\mathbf{g})$ relative to $(\,,)$, called an *inertia operator* on $\mathbf{g}$, such that

$$\mu(x,y)=(Ix,y) \tag{7}$$

for all $x,y\in\mathbf{g}$. Conversely, for any inertia operator $I$ on $\mathbf{g}$, the bilinear form $\mu$ defined by (7) is a pseudometric on $\mathbf{g}$ since $\mu(x,y)=(Ix,y)=(x,Iy)=(Iy,x)=\mu(y,x)$ for $x,y\in\mathbf{g}$. In fact, the inertia operators determine all pseudometrics on $\mathbf{g}$ satisfying (5) and hence all left-invariant pseudo-Riemannian connections on $G$; (for a proof of this, see [**4**]). Here, we give a simpler and more direct proof of this for a finite-dimensional Lie algebra $\mathbf{g}$ over an arbitrary field $F$ of characteristic $\neq 2$.

LEMMA 1. *Let $\mathbf{g}$ be a finite-dimensional Lie algebra over a field $F$ of characteristic $\neq 2$, and let $(\mathbf{g},*)$ be an algebra over $F$ compatible with $\mathbf{g}$. Then, for any symmetric nondegenerate form $\mu$ on $\mathbf{g}$, the identity* (5) *is equivalent to the identity*

$$\mu(x\circ y,z)=\tfrac{1}{2}(\mu([z,x],y)+\mu([z,y],x)), \tag{8}$$

*for all $x,y,z\in\mathbf{g}$.*

*Proof.* Assume that (5) holds. It follows from (4) that

$$\begin{aligned}\tfrac{1}{2}(\mu([x,z],y)+\mu([y,z],x)) &= \mu(z\circ x,y)+\mu(x,z\circ y)\ -\mu(z*x,y)-\mu(z*y,x)\\ &= \mu(z\circ x,y)+\mu(x,z\circ y).\end{aligned}$$

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Cyclic permutations of $x \to y \to z$ in this yield

$$\tfrac{1}{2}(\mu([y,x],z)+\mu([z,x],y))=\mu(x\circ y,z)+\mu(y,x\circ z),$$
$$\tfrac{1}{2}(\mu([z,y],x)+\mu([x,y],z))=\mu(y\circ z,x)+\mu(z,y\circ x).$$

Since $\circ$ is a commutative product and $\mu$ is symmetric, subtracting the first relation from the addition of the last two implies (8).

Conversely, if (8) holds for all $x,y,z \in \mathbf{g}$, then $\mu(x\circ y,y)=\tfrac{1}{2}\mu([y,x],y)$ for $x,y\in\mathbf{g}$, which is equivalent to $\mu(x*y,y)=0$ for $x,y\in\mathbf{g}$, by (4). Relation (5) now follows from a linearization of this. □

If, in addition, $\mathbf{g}$ has a symmetric nondegenerate invariant form $(\,,)$, then an algebra $(\mathbf{g},*)$ compatible with $\mathbf{g}$ satisfying (5) is uniquely determined by $\mu$ and hence by an inertia operator.

THEOREM 2. *Let $\mathbf{g}$ be a finite-dimensional Lie algebra over a field $F$ of characteristic $\neq 2$ with a symmetric nondegenerate invariant form $(\,,)$. Then, for any symmetric nondegenerate bilinear form $\mu$ on $\mathbf{g}$, there is a unique algebra $(\mathbf{g},*)$ compatible with $\mathbf{g}$ satisfying* (5), *and $x*y$ is given by*

$$x*y=\tfrac{1}{2}[x,y]+\tfrac{1}{2}I^{-1}([x,Iy]-[Ix,y]) \tag{9}$$

*for $x,y\in\mathbf{g}$, where $I$ is the inertia operator given by $\mu$ and* (7). *Conversely, for any symmetric $I\in GL(\mathbf{g})$ relative to $(\,,)$, the algebra $(\mathbf{g},*)$ given by* (9) *satisfies* (5) *with $\mu$ defined by* (7).

*Proof.* Assume that $(\mathbf{g},*)$ satisfies (5), and let $I$ be the unique symmetric operator in $GL(\mathbf{g})$ determined by (7). For (9), it suffices to verify that the product $x\circ y$ is given by

$$x\circ y=\tfrac{1}{2}I^{-1}([x,Iy]-[Ix,y]). \tag{10}$$

By Lemma 1 and (7), we have

$$\begin{aligned}(I(x\circ y),z)&=\tfrac{1}{2}(([z,x],Iy)+(Ix,[z,y]))\\&=\tfrac{1}{2}(([x,Iy],z)+([y,Ix],z)),\end{aligned}$$

using the invariance of $(\,,)$, which implies (10), since $(\,,)$ is nondegenerate.

For the converse, if $x,y,z\in\mathbf{g}$ and $(\mathbf{g},*)$ is given by (9) then, since $I$ is symmetric relative to $(\,,)$, we have

$$\begin{aligned}\mu(x*y,z)+\mu(y,x*z)&=\tfrac{1}{2}(I[x,y]+[x,Iy]-[Ix,y],z)\\&\quad+\tfrac{1}{2}(y,I[x,z]+[x,Iz]-[Ix,z])\\&=\tfrac{1}{2}([x,y],Iz)+\tfrac{1}{2}(y,[x,Iz])+\tfrac{1}{2}([x,Iy],z)\\&\quad+\tfrac{1}{2}(Iy,[x,z])-\tfrac{1}{2}([Ix,y],z)-\tfrac{1}{2}(y,[Ix,z])\\&=0,\end{aligned}$$

using the invariance of $(\,,)$ in $\mathbf{g}$. This gives (5), as desired. □

We return to a Lie group $G$ with an invariant Riemannian metric $(\,,)$, and let $\mathcal{I}(G)$ be the set of all inertia operators. Then, $\mathcal{I}(G)$ is a closed submanifold of the Lie group $GL(\mathbf{g})$ of dimension $\tfrac{1}{2}n(n+1)$. If $I\in\mathcal{I}(G)$, then denote by $(\mathbf{g},*,I)$ and $(\mathbf{g},\circ,\mathrm{I})$ the algebras given by (9) and (10), respectively. The foregoing remarks and Theorem 2 show that all left-invariant pseudo-Riemannian connections on $G$ are given by the class of algebras $\{(\mathbf{g}, *, I) \mid I \in \mathcal{I}(G)\}$ or $\{(\mathbf{g}, \circ, \mathrm{I}) \mid \mathrm{I} \in \mathcal{I}(G)\}$. For each $I \in \mathcal{I}(G)$, there is an orthonormal basis $x_1, \ldots, x_n$ (principal axes) of $\mathbf{g}$ consisting of the eigenvectors of $I$ with real eigenvalues $I_1, \ldots, I_n$ (the moments of inertia). Thus, by (7), (9) and (10), $(\mathbf{g}, *, I)$ and $(\mathbf{g}, \circ, I)$ are given by

$$\begin{aligned} x_i * x_j &= \tfrac{1}{2}(1 + (I_j - I_i)I^{-1})[x_i, x_j], \\ x_i \circ x_j &= \tfrac{1}{2}(I_j - I_i)I^{-1}[x_i, x_j], \\ \mu(x_i, x_j) &= \delta_{ij} I_i, \quad \text{for } i, j = 1, \ldots, n. \end{aligned} \tag{11}$$

In the remainder of this paper, we focus on the rotation group $SO(3)$. Using (11) it is possible to determine the structure of $(\mathbf{g}, *, I)$ or $(\mathbf{g}, \circ, I)$, for all $I \in \mathcal{I}(SO(3))$.

**3. Jacobi elliptic algebras.** Let $G = SO(3)$ and $\mathbf{g} = so(3) = \mathbf{R}^3$. If $(x, y) = -\frac{1}{2}\kappa(x, y)$ for the Killing form $\kappa$ on $\mathbf{g}$, then $(\,,\,)$ gives an invariant Riemannian metric on $G$. If $I \in \mathcal{I}(G)$ has eigenvalues $I_1$, $I_2$, $I_3$, then let

$$a_1 = \frac{I_3 - I_2}{I_1}, \qquad a_2 = \frac{I_1 - I_3}{I_2}, \qquad a_3 = \frac{I_2 - I_1}{I_3}. \tag{12}$$

LEMMA 3. *For each $I \in \mathcal{I}(G)$, there is a basis $\{y_1, y_2, y_3\}$ of $\mathbf{g}$ such that $(\mathbf{g}, *, I)$ and $(\mathbf{g}, \circ, I)$ are given by*

$$\begin{aligned} y_i * y_j &= \tfrac{1}{2}\epsilon_{ijk}(1 + \epsilon_{ijk} a_k) y_k, \\ y_i \circ y_j &= \tfrac{1}{2}\epsilon_{ijk}^2 a_k y_k, \\ \mu(y_i, y_j) &= \alpha^{-2}\delta_{ij} I_i \qquad (i, j, k = 1, 2, 3), \end{aligned} \tag{13}$$

*for some $\alpha \neq 0$ in $\mathbf{R}$, where $a_1$, $a_2$, $a_3$ are given by* (12) *with eigenvalues $I_1$, $I_2$, $I_3$ of $I$.*

*Proof.* Let $\{x_1, x_2, x_3\}$ be an orthonormal basis of $\mathbf{g}$ consisting of the eigenvectors of $I$. From the invariance of $(\,,\,)$, we have $([x_i, x_j], x_i) = ([x_i, x_j], x_j) = 0$ for $i, j = 1, 2, 3$, which imply $[x_i, x_j] = \epsilon_{ijk}\alpha_k x_k$ for $i, j, k = 1, 2, 3$ and for some nonzero $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$. If $i$, $j$, $k$ are distinct, then from $([x_i, x_j], x_k) + (x_j, [x_i, x_k]) = 0$ for $i, j, k = 1, 2, 3$, it follows that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \neq 0$. Letting $y_i = \alpha^{-1}x_i$ $(i = 1, 2, 3)$, we obtain the desired relations (13) from (11). □

We now prove our principal result in this section, which determines $(\mathbf{g}, \circ, I)$ and hence $(\mathbf{g}, *, I)$ for all $I \in \mathcal{I}(G)$.

THEOREM 4. *Let $I \in \mathcal{I}(G)$ and let $I_1$, $I_2$, $I_3$ be the eigenvalues of $I$.*

(i) *If $I_1 = I_2 = I_3$, then $(\mathbf{g}, \circ, I)$ is a zero algebra; that is $\mathbf{g} \circ \mathbf{g} = 0$.*

(ii) *If two of $I_1$, $I_2$, $I_3$ are equal, say $I_1 = I_2 \neq I_3$, then $(\mathbf{g}, \circ, I)$ is given by the multiplication*

$$\begin{aligned} y_i^2 = 0 \quad (i = 1, 2, 3), &\qquad y_1 \circ y_2 = y_2 \circ y_1 = 0, \\ y_1 \circ y_3 = y_3 \circ y_1 = -\tfrac{1}{2} a y_2, &\qquad y_2 \circ y_3 = y_3 \circ y_2 = \tfrac{1}{2} a y_1, \end{aligned} \tag{14}$$

*where $a = a_1 = -a_2$ is given by* (12) *and $\{y_1, y_2, y_3\}$ is a basis of $\mathbf{g}$.*

(iii) *If $I_1$, $I_2$, $I_3$ are distinct, then $a_1$, $a_2$, $a_3$ given by* (12) *have different signs and* $(\mathbf{g}, \circ, I)$ *is isomorphic to the Jacobi elliptic algebra $J(k)$ of a certain modulus $k$.*

*Proof.* (i) Since $a_1 = a_3 = 0$ by (12), from Lemma 3 we have $\mathbf{g} \circ \mathbf{g} = 0$.

(ii) If $I_1 = I_2 \neq I_3$, then $a = a_1 = -a_2 \neq 0$, $a_3 = 0$ by (12) and hence, by Lemma 3, the basis $\{y_1, y_2, y_3\}$ in (13) gives the desired multiplication for $(\mathbf{g}, \circ, I)$.

(iii) We establish an explicit isomorphism between $(\mathbf{g}, \circ, I)$ and $J(k)$, according to the signature of $(a_1, a_2, a_3)$. We first show that the $a_k$ have different signs. If the $I_k$ have the same sign, then since $\sum_{k=1}^{3} a_k I_k = 0$, the $a_k$ must have different signs. For the remaining cases, we observe that if $\sigma$ is a transposition on $\{1, 2, 3\}$, then it is easily seen that

$$\frac{I_{\sigma(j)} - I_{\sigma(i)}}{I_{\sigma(k)}} = -a_{\sigma(k)} \tag{15}$$

for $(ijk) = (231), (312), (123)$. For example, if $\sigma = (12)$, then

$$\frac{I_{\sigma(3)} - I_{\sigma(2)}}{I_{\sigma(1)}} = \frac{I_3 - I_1}{I_2} = -a_2 = -a_{\sigma(1)},$$
$$\frac{I_{\sigma(1)} - I_{\sigma(3)}}{I_{\sigma(2)}} = \frac{I_2 - I_3}{I_1} = -a_1 = -a_{\sigma(2)},$$
$$\frac{I_{\sigma(2)} - I_{\sigma(1)}}{I_{\sigma(3)}} = \frac{I_1 - I_2}{I_3} = -a_3 = -a_{\sigma(3)}.$$

Therefore, it suffices to treat the two cases: $I_1 < I_2 < 0 < I_3$ and $I_1 < 0 < I_2 < I_3$. But, these yield the signatures $(-, +, +)$ and $(-, -, +)$ for $(a_1, a_2, a_3)$. In view of (15), the remaining signatures $(+, +, -)$, $(+, -, -)$, $(-, +, -)$, $(+, -, +)$ are obtained by applying transpositions on $\{1, 2, 3\}$ to the two cases above or to the cases: $0 < I_1 < I_2 < I_3$ and $I_1 < I_2 < I_3 < 0$.

Let $\{y_1, y_2, y_3\}$ be the basis of $\mathbf{g}$ given by (13). If $(a_1, a_2, a_3)$ has signature $(\mp, \pm, \mp)$, then let $k = \sqrt{a_1 a_3}$ and put

$$f_1 = \sqrt{-a_1 a_2^{-1}}\, y_1, \qquad f_2 = y_2, \qquad f_3 = \sqrt{-(a_1 a_2)^{-1}}\, y_3.$$

Then, the linear map $\lambda : J(k) \to (\mathbf{g}, \circ, I)$ with $\lambda(e_1) = \pm f_1$ and $\lambda(e_i) = f_i$ $(i = 2, 3)$ gives an algebra isomorphism, where "$\pm$" denotes the sign of $a_2$. In fact, for $\lambda(e_1) = -f_1$,

$$f_1 \circ f_2 = -\sqrt{-a_1 a_2^{-1}}\, y_1 \circ y_2 = -\tfrac{1}{2} a_3 \sqrt{-a_1 a_2^{-1}}\, y_3 = -\tfrac{1}{2} a_1 a_3 f_3 = -\tfrac{1}{2} k^2 f_3,$$
$$f_1 \circ f_3 = -\sqrt{-a_1 a_2^{-1}} \sqrt{-(a_1 a_2)^{-1}}\, y_1 \circ y_3 = -\tfrac{1}{2} a_2 \sqrt{a_2^{-2}}\, y_2 = \tfrac{1}{2} f_2,$$
$$f_2 \circ f_3 = \sqrt{-(a_1 a_2)^{-1}}\, y_2 \circ y_3 = \tfrac{1}{2} a_1 \sqrt{-(a_1 a_2)^{-1}}\, y_1 = -\tfrac{1}{2} f_1,$$

using (13) and $a_2 < 0$. Thus, $\lambda$ is an isomorphism.

For the signatures $(\mp, \mp, \pm)$, we take modulus $k = \sqrt{a_1 a_2}$ and let

$$f_1 = \sqrt{-a_1 a_3^{-1}}\, y_1, \qquad f_2 = y_3, \qquad f_3 = \sqrt{-(a_1 a_3)^{-1}}\, y_2.$$

It easily follows that the map: $J(k) \to (\mathbf{g}, \circ, I)$: $e_1 \to \pm f_1$, $e_i \to f_i$ $(i = 2, 3)$ induces an isomorphism, where "±" varies with the sign of $a_3$. Similarly, for $(\pm, \mp, \mp)$, we let $k = \sqrt{a_2 a_3}$ and

$$f_1 = \sqrt{-a_1^{-1} a_2}\, y_2, \qquad f_2 = y_1, \qquad f_3 = \sqrt{-(a_1 a_2)^{-1}}\, y_3.$$

Then, $\{\pm f_1, f_2, f_3\}$ has the same multiplication as the basis $\{e_1, e_2, e_3\}$ for $J(k)$. (See (1).) □

We notice that $(\mathbf{g}, \circ, I)$ for $I_2 = I_3 \neq I_1$ or $I_1 = I_3 \neq I_2$ in Theorem 4(ii) is isomorphic to the algebra $(\mathbf{g}, \circ, I)$ given by (14). If $I_2 = I_3 \neq I_1$, then $b = a_2 = -a_3$, $a_1 = 0$ and the map $(\mathbf{g}, \circ, I_1 = I_2) \to (\mathbf{g}, \circ, I_2 = I_3)$: $y_1 \to y_2$, $y_2 \to y_3$, $y_3 \to ab^{-1} y_1$ gives an isomorphism. Equation (3) for the algebra $(\mathbf{g}, \circ, I)$ given by (13) with $I_1 > I_2 > I_3 > 0$ or $I_1 < I_2 < I_3 < 0$ gives Euler's equations for the motion of a free rotating rigid body [**2, 6**]. In both cases, the signature of $(a_1, a_2, a_3)$ is $(-, +, -)$ and hence $(\mathbf{g}, \circ, I)$ is isomorphic to $J(k)$ of modulus $k = \sqrt{a_1 a_3}$.

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