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QUASI-FLOWS II
ADDITIVE FUNCTIONALS AND TQ-SYSTEMS

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0. Introduction

We have given in [3] the definition of the quasi-flow and discussed the representation and the time-change of the quasi-flow. Further we have defined the TQ-system and studied some properties of them using the time-change and the representation of the quasi-flow. These properties are very useful to study the ergodicity, the entropy and increasing partitions of the automorphism. We are now going to extend the definition of the TQ-system and to study the similar problems as the above.

We shall begin with the extension of the additive functional and the multiplicative functional of the quasi-flow. The additive functional of flow has been introduced by G. Maruyama [5] in connection with the time-change of the flow. In that case, it has been required that the additive functional is continuous and increasing. But for the study of the (quasi-) flow it is convenient that we give a weaker definition of the additive functional. We call an $\mathfrak{R}_0\times\mathfrak{B}$-measurable function $\varphi(t,\omega)$ *an additive functional* of a quasi-flow $\{Z_t\}$ if it has the additivity

$$\varphi(t+s,\omega)=\varphi(s,\omega)+\varphi(t,Z_s\omega).$$

We call an $\mathfrak{R}_0\times\mathfrak{B}$-measurable function $\psi(t,\omega)>0$ *a multiplicative functional* if $\log\psi(t,\omega)$ is an additive functional.

Suppose that we are given a quasi-flow $\{Z_t\}$ and a flow $\{T_s\}$ and a multiplicative functional $\lambda(s,\omega)$ of the flow $\{T_s^{-1}\}$. If $[\{Z_t\},\ \lambda(s,\omega);\ T_s]$ is a TQ-system in the sense of [3] for each s, then we call the triple $[\{Z_t\},\ \lambda(s,\omega),\ \{T_s\}]$ a TQ-system.[1]

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[1] We have discussed in [3] a special class of the TQ-system such that $\lambda(s,\omega)$ is $\exp\left[\int_0^s \kappa(T_{-u}\omega)\,du\right]$.

The first purpose of this paper is *to represent the additive functional of the quasi-flow and to study its limit theorems.* The second purpose is *to give a representation of the TQ-system.* If these problems are solved, the similar results as in [3] are easily derived for the TQ-system $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$.

In section **1** we shall show that an additive functional $\varphi(t,(x,u))$ of the S-quasi-flow $\{\tilde{Z}_t\}$ built up by $(X,\mathfrak{A},\mu,f(x),p(x,u),S)$ can be expressed in the form

$$\varphi(t,(x,u)) = \varphi(\tilde{Z}_t(x,u)) - \varphi(x,u) + \varphi_n(x)$$

for $f_n(x) \leq u+t < f_{n+1}(x)$, where $\varphi(x, u)$ is a measurable function on $\tilde{\Omega} = \{(x,u);\ 0 \leq u < f(x),\ x \in X\}$ and $\varphi_n(x)$ is an additive functional of the quasi-automorphism S of the basic space X. This expression is useful not only to know the local structure of the additive functional but also to study its asymptotic behaviors as $t \to \infty$. In fact, the study of asymptotic behaviors of the additive functional can be reduced to that of φ_n, since any quasi-flow is expressed as an S-quasi-flow (c.f. §3 in [3]).

In section **2** we shall investigate the asymptotic behaviors of increasing additive functionals of the quasi-flow (see Theorem 2.1). In Theorem 2.2, we shall show that for any quasi-integrable additive functional of the flow $\{T_s\}$, the following equalities hold for almost every $\omega \in \Omega$,

$$\begin{aligned} E[\varphi(t,\cdot)|\nu_{\{T_s\}}](\omega) &= tE[\varphi(1,\cdot)|\nu_{\{T_s\}}](\omega) \\ &= E[\varphi(t,\cdot)|\nu_{T_t}](\omega) \\ &= \lim_{m\to\infty} \frac{1}{m}\varphi(mt,\omega) \\ &= m\text{-}\lim_{s\to\infty} \frac{1}{s}\varphi(st,\omega)\,. \end{aligned}$$

In section **3** we shall discuss several properties of the TQ-system $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ corresponding to those of the TQ-system (the case of automorphisms discussed in §5 and §6 in [3]) with the help of Theorem 2.2.

In section **4** we shall discuss the representation of the TQ-system which is anologous to the representation of the quasi-flow, especially with multi-dimensional parameter (see Theorem 4.1 and c.f. [4]). The representation is very useful to observe the local structure of the given TQ-system, and will be applied to the proof of the theorems in the last section. If the representation of the TQ-system $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ exists, then $\{Z_t\}$ may be called a transversal quasi-flow of the flow $\{T_s\}$ following the terminology of Ya. G. Sinai [7].

In section 5 the increasing partitions and the entropy of the flow will be discussed in the connection with TQ-systems.

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1. Additive functionals

We shall use the terminologies in [3]. Let $(\Omega, \mathfrak{B}, P)$ be a Lebesgue space, $P(\Omega)=1$. A non-singular bimeasurable point transformation S of Ω is called *a quasi-automorphism*. A one parameter group of quasi-automorphisms $\{Z_t\}$ is called *a quasi-flow* if the mapping $(t, \omega) \to Z_t\omega$ is measurable. Let S be a quasi-automorphism of a Lebesgue space $(\Omega, \mathfrak{B}, P)$. A finite function $\varphi_n(\omega)$ on $N \times \Omega$ is called *an additive functional of* S, if

$$\varphi_{n+m}(\omega) = \varphi_n(\omega) + \varphi_m(S^n\omega) \tag{1.1}$$

holds and $\varphi_n(\omega)$ is measurable for each n[2].

PROPOSITION 1.1. *Let $\varphi_n(\omega)$ be an additive functional of a quasi-automorphism S. Then it holds that*

$$\varphi_n(\omega) = \begin{cases} \sum_{k=0}^{n-1} \varphi_1(S^k\omega) & n \geq 1 \\ 0 & n = 0 \\ \sum_{k=1}^{-n} \varphi_1(S^{-k}\omega) & n \leq -1\,. \end{cases} \tag{1.2}$$

Conversely, for a measurable function $\varphi_1(\omega)$, the functional $\varphi_n(\omega)$ defined by the right terms in (1.2) *is an additive functional of* S.

Proof. Since $\varphi_0(\omega) = \varphi_0(\omega) + \varphi_0(\omega)$ holds, it follows that $\varphi_0(\omega) = 0$. For $n \geq 1$, we have that

$$\varphi_n(\omega) = \varphi_{n-1}(\omega) + \varphi_1(S^{n-1}\omega) = \sum_{k=0}^{n-1} \varphi_1(S^k\omega)$$

by the additivity (1.2). Since $0 = \varphi_0(\omega) = \varphi_1(\omega) + \varphi_{-1}(S\omega)$ holds, it follows that

2) $N = \{0, \pm 1, \pm 2, \cdots\}$.

$$\varphi_n(\omega) = \varphi_{n+1}(\omega) + \varphi_{-1}(S^{n+1}\omega) = \varphi_{n+1}(\omega) - \varphi_1(S^n\omega) = -\sum_{k=1}^{-n}\varphi_1(S^{-k}\omega)$$

for $n \leq -1$. Thus we have the assertion. The converse is obvious.

Given a $\mathfrak{B}$-measurable function $\varphi_1(\omega)$, the additive functional $\varphi_n(\omega)$ of S defined by (1, 2) is said *to be constructed from* φ_1. A positive function $\phi_n(\omega) > 0$ on $N \times \Omega$ is called *a multiplicative functional of* S if $\log\phi_n(\omega)$ is an additive functional of S. Any multiplicative functional $\phi_n(\omega)$ is represented in the form

$$\phi_n(\omega) = \begin{cases} \prod_{k=0}^{n-1} \phi_1(S^k\omega) & n \geq 1 \\ 1 & n = 1 \\ (\prod_{k=1}^{-n} \phi_1(S^{-k}\omega))^{-1} & n \leq -1. \end{cases} \tag{1.3}$$

Now, we extend the definition of the additive functional of the quasi-flow as the following manner.

DEFINITION 1.1. Let $\{Z_t\}$ be a quasi-flow. An $\mathfrak{R}_0 \times \mathfrak{B}$-measurable function $\varphi(t, \omega)$ is called *an additive functional* of $\{Z_t\}$ if for each $\omega \in \Omega$,

$$\varphi(t + s, \omega) = \varphi(s, \omega) + \varphi(t, Z_s\omega) \tag{1.4}$$

holds for every t and $s \in \boldsymbol{R}$.[3] A positive function $\phi(t, \omega) > 0$ is called *a multiplicative functional* of $\{Z_t\}$, if $\log\phi(t, \omega)$ is an additive functional of $\{Z_t\}$.

DEFINITION 1.2. An $\mathfrak{R} \times \mathfrak{B}$-measurable function $\varphi(t, \omega)$ is called *an almost additive functional* of $\{Z_t\}$, if for each pair of t and s, (1.4) holds for almost every $\omega \in \Omega$ (P). A positive function $\phi(t, \omega) > 0$ is *an almost multiplicative functional* of $\{Z_t\}$, if $\log\phi(t, \omega)$ is an almost additive functional of $\{Z_t\}$.

We now discuss the representation of the additive functional of the quasi-flow. We have shown in [3] that any quasi-flow $\{Z_t\}$ without fixed points is isomorphic to an S-quasi-flow. Here the S-quasi-flow built up by $(X, \mathfrak{A}, \mu, f(x), p(x, u), S)$ is defined as the following way. Let $(X, \mathfrak{A}, \mu)$ be a Lebesgue space, which is called a basic space. Let $f(x)$ be an $\mathfrak{A}$-measurable function on X, which is called a ceiling function and let S be a quasi-automorphism of X such that $\sum_{k>0} f(S^k x) = \sum_{k<0} f(S^k x) = \infty$ for $x \in X$. Set

[3] We denote by $\mathfrak{R}_0$ the topological Borel field of $\boldsymbol{R} = (-\infty, \infty)$. $\mathfrak{R} = \overline{\mathfrak{R}_0}$ is the completion of $\mathfrak{R}_0$.

$\tilde{\Omega}=\{(x,u);\ 0\leq u<f(x),\ x\in X\}$. Let $\tilde{\mathfrak{B}}$ be the restriction of $\overline{\mathfrak{A}\times\mathfrak{R}}$ to $\tilde{\Omega}$. Let $p(x,u)$ be a $\tilde{\mathfrak{B}}$-measurable function such that $p(x,u)>0$ and $\iint_{\tilde{\Omega}} p(x,u)\,du\,d\mu(x)=1$. Define a measure $\tilde{P}$ by $d\tilde{P}(x,u)=p(x,u)\,du\cdot d\mu(x)$. Let $f_n(x)$ be the additive functional of S constructed from $f_1(x)=f(x)$ by (1.2). Then we can define a quasi-flow $\{\tilde{Z}_t\}$ on the Lebesgue space $(\tilde{\Omega},\tilde{\mathfrak{B}},\tilde{P})$ by

$$\tilde{Z}_t(x,u)=(S^n x,\ u+t-f_n(x))\quad \text{for } f_n(x)\leq u+t<f_{n+1}(x). \tag{1.5}$$

The quasi-flow defined by (1.5) is called the *S-quasi-flow built up by* $(X,\ \mathfrak{A},\ \mu,\ f(x),\ p(x,u),S)$.

Suppose that a quasi-flow $\{Z_t\}$ on $(\Omega,\mathfrak{B},P)$ is isomorphic to an S-quasi-flow on $\tilde{\Omega}$ with an isomorphism H of Ω onto $\tilde{\Omega}$, that is, $\tilde{Z}_t=HZ_tH^{-1}$ holds for $t\in\boldsymbol{R}$. For an additive functional $\varphi(t,\omega)$ of $\{Z_t\}$, if we put

$$\tilde{\varphi}(t,(x,u))=\varphi(t,H^{-1}(x,u)), \tag{1.6}$$

then $\tilde{\varphi}(t,(x,u))$ is an additive functional of $\{\tilde{Z}_t\}$, obviously. Hence we discuss the additive functional of the S-quasi-flow.

THEOREM 1.1. *Let $\varphi(t,(x,u))$ be an additive functional of the S-quasi-flow $\{\tilde{Z}_t\}$ built up by $(X,\ \mathfrak{A},\ \mu,\ f(x),\ p(x,u),\ S)$. Then there exist a $\tilde{\mathfrak{B}}$-measurable function $\varphi(x,u)$ and an $\mathfrak{A}$-measurable function $\varphi_1(x)$ such that*

$$\varphi(t,(x,u))=\varphi(\tilde{Z}_t(x,u))-\varphi(x,u)+\varphi_n(x) \tag{1.7}$$

for $f_n(x)\leq u+t<f_{n+1}(x)$, where $\varphi_n(x)$ is the additive functional of S constructed from $\varphi_1(x)$.

Proof. From (1.5) and the additivity of $\varphi(t,(x,u))$, it follows that

$$\varphi(t+s,(x,u))=\varphi(s,(x,u))+\varphi(t,(x,u+s))\quad \text{for}\quad 0\leq u+s<f(x). \tag{1.8}$$

Therefore we have

$$\varphi(t,(x,s))=\varphi(t+s,(x,0))-\varphi(s,(x,0))\quad \text{for}\quad 0\leq s<f(x). \tag{1.9}$$

If $\qquad f(x)\leq u+s<f(x)+f(Sx)$, it follows that

$$\varphi(t+s,(x,u))=\varphi(s,(x,u))+\varphi(t,(Sx,u+s-f(x))). \tag{1.10}$$

In particular, we have

$$\varphi(t+f(x),(x,0))=\varphi(f(x),(x,0))+\varphi(t,(Sx,0)) \tag{1.11}$$

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by the setting $u=0$ and $s=f(x)$ in (1.10).

For $(x,u)\in\tilde{\Omega}$, we define $\varphi(x,u)$ by $\varphi(u,(x,0))$ and $\varphi_1(x)$ by $\varphi(f(x),(x,0))$. Then $\varphi(x,u)$ is $\tilde{\mathfrak{B}}$-measurable and $\varphi_1(x)$ is $\mathfrak{A}$-measurable. By (1.9) and (1.11), we have that

$$\begin{aligned}\varphi(t,(x,u)) &= \varphi(t+u,\ (x,0)) - \varphi(u,(x,0))\\ &= \varphi(f_n(x),\ (x,0)) + \varphi(u+t-f_n(x),\ \tilde{Z}_{f_n}(x)(x,0)) - \varphi(u,(x,0))\\ &= \varphi(f_{n-1}(x),\ (x,0)) + \varphi(f(S^n x),\ (S^{n-1}x,0))\\ &\quad + \varphi(u+t-f_n(x),\ (S^n x,0)) - \varphi(x,u)\\ &= \varphi_n(x) + \varphi(\tilde{Z}_t(x,u)) - \varphi(x,u)\end{aligned}$$

for $f_n(x)\leq u+t< f_{n+1}(x)$. Thus we have obtained (1.7).

We discuss the problem whether some version of the almost additive functional is an additive functional. Here we say that an additive functional $\varphi'(t,\omega)$ is *a version* of the almost additive functional $\varphi(t,\omega)$ if $P(\varphi'(t,\omega)\neq\varphi(t,\omega))=0$ for each $t\in\boldsymbol{R}$.

PROPOSITION 1.2. *Let $\{\tilde{Z}_t\}$ be an S-quasi-flow with the ceiling function $f(x)>\theta$ for some $\theta>0$, and let $\varphi(t,(x,u))$ be an almost additive functional of $\{\tilde{Z}_t\}$. Then there exists a version $\varphi'(t,(x,u))$ of $\varphi(t,(x,u))$ which is an additive functional of $\{\tilde{Z}_t\}$.*

Proof. Since $\varphi(t,(x,u))$ is $\mathfrak{R}\times\tilde{\mathfrak{B}}$-measurable, there exists an $\mathfrak{R}_0\times\tilde{\mathfrak{B}}^0$-measurable function $\varphi''(t,(x,u))$ such that

$$\varphi(t,(x,u)) = \varphi''(t,(x,u)) \qquad \text{a.e. } (dt\ d\tilde{P}), \tag{1.12}$$

where we denote by $\tilde{\mathfrak{B}}^0$ the restriction of the σ-field $\mathfrak{R}_0\times\mathfrak{A}$ to $\tilde{\Omega}$. From (1.10) and the almost additivity of $\varphi(t,(x,u))$ it follows that

$$\varphi''(t+s,\ (x,u)) = \varphi''(s,(x,u)) + \varphi''(t,(S^n x,\ u+s-f_n(x)) \tag{1.13}$$

holds for $f_n(x)\leq u+s<f_{n+1}(x)$ a.e. $(dt\ ds\ d\tilde{P})$. Hence there exists $u_0\in(0,\theta)$ such that (1.13) holds almost every (t,s,x) $(dt\ ds\ d\mu)$ for $u=u_0$. Setting $n=0$, $u=u_0$ and $s=v-u_0$, it follows that

$$\varphi''(t,(x,v)) = \varphi''(t+v-u_0,\ (x,u_0)) - \varphi''(v-u_0,\ (x,u_0)) \tag{1.14}$$

holds for almost every $(t,(x,v))\in\boldsymbol{R}\times\tilde{\Omega}$. By the same reasoning, there exists $v_0\in(0,u_0)$ such that

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$$(1.15)\qquad \varphi''(t+u-u_0,\ (x,u_0)) = \varphi''(t+u-u_0-f_n(x)+v_0,\ (S^n x, u_0-v_0)) + \varphi''(f_n(x)-v_0,\ (x,u_0))$$

holds for almost every $(t,(x,u))\in \boldsymbol{R}\times\tilde{\Omega}$ and that

$$(1.16)\qquad \varphi''(u-u_0,\ (x,u_0)) = \varphi''(u-u_0+v_0,\ (x,u_0-v_0)) + \varphi''(v_0,(x,u_0-v_0))$$

holds for almost $(x,u)\in\tilde{\Omega}$. Define a $\mathfrak{B}^0$-measurable function $\varphi''(x,u)$ and $\mathfrak{A}$-measurable functions $\varphi''_n(x)$, $n=0,\ \pm 1,\ \pm 2,\cdots$, by

$$\varphi''(x,u) = \varphi''(u-u_0+v_0,\ (x,u_0-v_0)),$$
$$\varphi''_n(x) = \varphi''(f_n(x)-v_0,\ (x,u_0)) + \varphi''(v_0,\ (x,u_0-v_0)).$$

Then by (1,14), (1.15) and (1.16), we have

$$(1.17)\qquad \varphi(t,(x,u)) = \varphi''(\tilde{Z}_t(x,u)) - \varphi''(x,u) + \varphi''_n(x)$$

for $f_n(x)\le t+u < f_{n+1}(x)$ a.e. $(dt\ d\tilde{P})$. Now we easily see that $\varphi''_n(x)$ is an almost additive functional of S. Hence

$$(1.18)\qquad \varphi'_n(x) = \varphi''_n(x) \qquad \text{a.e.} \quad (d\mu)$$

holds, where $\varphi'_n(x)$ is the additive functional of S constructed from $\varphi'_1(x)=\varphi''_1(x)$. Define an $\mathfrak{R}_0\times\mathfrak{B}^0$-measurable function $\varphi'(t,(x,u))$ by

$$(1.19)\qquad \varphi'(t,(x,u)) = \varphi''(\tilde{Z}_t(x,u)) - \varphi''(x,u) + \varphi'_n(x)$$

for $f_n(x)\le u+t < f_{n+1}(x)$. Then $\varphi'(t,(x,u))$ is an additive functional of $\{\tilde{Z}_t\}$. Further, from (1.17), (1.18) and (1.19), it follows that

$$(1.20)\qquad \tilde{P}(\varphi(t,(x,u)) \ne \varphi'(t,(x,u))) = 0 \qquad \text{for a.e.} \quad t\in\boldsymbol{R}.$$

Set $\Phi(t,(x,u)) = \varphi(t,(x,u)) - \varphi'(t,(x,u))$. Set $\tilde{\Omega}_t = \{(x,u);\ \Phi(t,(x,u))=0\}$ and set $\tilde{\Omega}_{t,s} = \{(x,u);\ \Phi(t+s,\ (x,u)) = \Phi(s,(x,u)) + \Phi(t,\tilde{Z}_s(x,u))\}$. Then $P(\tilde{\Omega}_t)=1$ for almost every $t\in\boldsymbol{R}$ and $P(\tilde{\Omega}_{t,s})=1$ for any pair of t and s. For any $r\in\boldsymbol{R}$, there exist t and s, $t+s=r$, such that $P(\tilde{\Omega}_t)=P(\tilde{\Omega}_s)=1$. For $(x,u)\in\tilde{\Omega}_{t,s}\cap\tilde{\Omega}_s\cap\tilde{Z}_{-s}\tilde{\Omega}_t$, $\Phi(r,(x,u)) = \Phi(s,(x,u)) + \Phi(t,\ \tilde{Z}_s(x,u)) = 0$. Hence we have $P(\varphi(r,(x,u)) \ne \varphi'(r,\ (x,u))) = 0$ for any $r\in\boldsymbol{R}$, that is, $\varphi'(t,(x,u))$ is a version of $\varphi(t,(x,u))$. Thus we have the assertion.

By this proposition and the representation theorem of quasi-flows (c.f. Theorem 3.1 and Remark 3.1 in [3]), we have the following theorem.

THEOREM 1.2. *Let $\varphi(t,\omega)$ be an almost additive functional of a quasi-flow $\{Z_t\}$. Then $\varphi(t,\omega)$ has a version which is an additive functional of $\{Z_t\}$.*

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2. Limit theorems related to additive functionals

Let $\{Z_t\}$ be a quasi-flow and $\varphi(t,\omega)$ be an additive functional of $\{Z_t\}$. It follows from the additivity of $\varphi(t,\omega)$ that $\varphi(t,\omega)$ is non-decreasing (resp. strictly increasing) in t for each $\omega\in\Omega$ if and only if

$$\varphi(t,\omega)\geqq 0 \quad \text{for every} \quad t>0,\ \omega\in\Omega$$

$$(\text{resp. } \varphi(t,\omega)>0 \quad \text{for every} \quad t>0,\ \omega\in\Omega),$$

We shall discuss a limit theorem for such additive functionals. First we notice a lemma concerning the conservativity of a quasi-flow. We say that a quasi-flow $\{\{Z_t\}$ is *conservative* if

$$\int_0^\infty g(Z_u\omega)du=\infty \qquad \text{a.e. } (P), \tag{2.1}$$

$$\int_0^\infty g(Z_{-u}\omega)du=\infty \qquad \text{a.e. } (P) \tag{2.2}$$

hold for any measurable function $g(\omega)>0$.

LEMMA 2.1. *Let $\{Z_t\}$ be a quasi-flow. Then the following conditions are equivalent:*

(i) $\{Z_t\}$ *is conservative*;

(ii) *for any measurable function $g(\omega)>0$, (2.1) holds*;

(iii) *for any measurable function $g(\omega)>0$, (2.2) holds*;

(iv) *if $Z_sB\subset B$ for any $s>0$, then $P(B-Z_sB)=0$ for any $s>0$*;

(v) *if $Z_sB\supset B$ for any $s>0$, then $P(Z_sB-B)=0$ for any $s>0$.*

Proof. (ii) $\Longrightarrow$ (iv). We suppose that there exists a measurable set B such that $Z_sB\subset B$ for any $s>0$. Set $B_0=B-Z_1B$ and $B_k=Z_kB_0$. Then $B_k\cap B_n=\phi$ and $\bigcup_k B_k=\bigcup_s Z_sB-\bigcap_s Z_sB$. Define a measurable function $g(\omega)$ by

$$g(\omega)=\begin{cases}\dfrac{1}{k^2+1} & \omega\in B_k,\\[2ex] 1 & \omega\notin\bigcup_k B_k.\end{cases}$$

Then we can see that

$$\int_{-\infty}^\infty g(Z_u\omega)\ du=\sum_{k=-\infty}^\infty\frac{1}{k^2+1}\int_{-\infty}^\infty \chi_{B_k}(Z_u\omega)\ du=\sum_{k=-\infty}^\infty\frac{1}{k^2+1}<\infty$$

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for $\omega \in \cup B_k$. Hence if (ii) holds, $P(\bigcup_k B_k) = 0$, that is, $P(B - Z_s B) \leq P(\cup Z_s B - \bigcap_s Z_s B) = 0$.

(iv) $\Longrightarrow$ (v). Since $Z_s B \subset B$ holds if and only if $Z_s B^c \supset B^c$ and since $P(Z_s B \ominus B) = P(B^c \ominus Z_s B^c)$, the assertion is obvious.

(v) $\Longrightarrow$ (iii). Set $B = \left\{\omega;\ \int_0^\infty g(Z_{-u}\omega)\, du < \infty\right\}$ and $B_a = \left\{\omega;\ \int_0^\infty g(Z_{-u}\omega) dn < a\right\}$. Then $Z_s B_a = \left\{\omega;\ \int_0^\infty g(Z_{-u}\omega)\, du - \int_0^s g(Z_{-u}\omega)\, du < a\right\} \supset B_a$ for $s > 0$, because

$$\int_0^\infty g(Z_{-u-s}\omega)\, du = \int_0^\infty g(Z_{-u}\omega)\, du - \int_0^s g(Z_{-u}\omega)\, du. \tag{2.3}$$

Hence if (iv) holds, $P(Z_s B_a - B_a) = 0$ for $s > 0$. On the other hand, from (2.3), it follows that $\lim_{s\to\infty} \int_0^\infty g(Z_{-u-s}\omega)\, du = 0$ for $\omega \in B$. Hence we have that $\lim_{s\to\infty} P(B - Z_s B_a) = 0$. So we have $P(B - B_a) = 0$. Since $\lim_{a\to 0} P(B_a) = 0$, we have that $P(B) = 0$.

The proof of (iii) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) is similar. From the equivalence of (ii) and (iii), it follows that (i) is equivalent to the others.

THEOREM 2.1. *Let $\{Z_t\}$ be a conservative quasi-flow and $\varphi(t, \omega)$ be an additive functional of $\{Z_t\}$. Then, if $\varphi(t, \omega)$ is non-decreasing in t for any fixed ω, the following conditions are equivalent,*

(i) $\lim_{t\to\infty} \varphi(t, \omega) > 0$ a.e. (P);

(i)′ $\lim_{t\to-\infty} \varphi(t, \omega) < 0$ a.e. (P);

(ii) $\lim_{t\to\infty} \varphi(t, \omega) = \infty$ a.e. (P);

(ii)′ $\lim_{t\to-\infty} \varphi(t, \omega) = -\infty$ a.e. (P);

(iii) $\{\omega;\ \varphi(t, \omega) = 0\}$ *has no $\{Z_t\}$-invariant subset with positive measure for some* $t \neq 0$ (*or any* $t \neq 0$);

(iii)′ $E[\varphi(t, \cdot) | \nu_{\{Z_t\}}]\ (\omega) > 0$ a.e. (P) *for some* $t > 0$ (*or any* $t > 0$);

(iv)′ $\{\omega;\ \varphi(t, \omega) = 0\}$ *has no Z_t-invariant subset with positive measure for some* $t \neq 0$ (or *any* $t \neq 0$);

(iv)′ $E[\varphi(t, \cdot) | \nu_{Z_t}]\ (\omega) > 0$ a.e. (P) *for some* $t > 0$ (*or any* $t > 0$).

Proof. (i) $\Longrightarrow$ (iv). If there exists a measurable subset B of $\{\omega;\ \varphi(t, \omega) = 0\}$ with positive measure such that $Z_t B = B$ for some $t \neq 0$, then $Z_{kt} B = B$ for any $k \in \boldsymbol{N}$ and hence $\varphi(t, Z_{kt}\omega) = 0$ for any k and $\omega \in B$. It is easily seen that $\varphi(nt, \omega)$ is an additive functional of a quasi-automorphism Z_t for fixed t. Hence by the formula (1.2), we have $\varphi(nt, \omega) = 0$ for $\omega \in B$. Since $\varphi(t, \omega)$ is non-decreasing in t, $\varphi(s, \omega) = 0$ for any $s \in \boldsymbol{R}$, $\omega \in B$.

(iv) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (ii). Set $h(\omega) = \lim_{t \to \infty} \varphi(t, \omega)$ and set $B = \{\omega;\ h(\omega) < \infty\}$. Then

$$h(Z_s\omega) = \lim_{t \to \infty} \varphi(t, Z_s\omega) = \lim_{t \to \infty} \varphi(t+s, \omega) - \varphi(s, \omega) = h(\omega) - \varphi(s, \omega) \tag{2.4}$$

holds. Hence it holds that $Z_sB_a \subset B_a$ for any $s > 0$, where $B_a = \{\omega;\ 0 \leq h(\omega) \leq a\}$. By the concervativity of $\{Z_t\}$, we have $P(Z_sB_a \ominus B_a) = 0$ for any s. Since it follows from (2.4) that $\lim_{s \to \infty} h(Z_s\omega) = 0$ for $\omega \in B$, $\bigcup_s Z_sB_a = B$ holds. Hence we have $P(B) = P(\bigcup_s Z_sB_a) = P(B_a) = P(\bigcap_s Z_sB_a)$. The set $\bigcap_s Z_sB_a$ is a $\{Z_t\}$-invariant measurable set which is contained in B_a. Set $B_* = \bigcap_{a>0} \bigcap_s Z_sB_a$, then $B_* \subset \bigcap_{a>0} B_a = \{\omega;\ h(\omega) = 0\}$ holds and B_* is a $\{Z_t\}$-invariant measurable set with $P(B_*) = P(\bigcap_s Z_sB_a) = P(B)$. If we assume the condition (iii), $P(B) = P(B_*) = 0$ holds. Thus we have the assertion.

(ii) $\Longrightarrow$ (i) is obvious.

The equivalence of (iii) and (iii)′ (or (iv) and (iv)′) is easily seen. The equivalence of (i) and (i)′ (or (ii) and (ii)′) is proved by virtue of (iii) and the fact that $\varphi(-t, \omega)$ is an additive functional of the quasi-flow $\{Z_{-t}\}$ which is conservative if $\{Z_t\}$ is so.

In the following, we shall discuss a limit theorem for the additive functional of the flow $\{T_t\}$. Let $\varphi(t, \omega)$ be an additive functional of a flow $\{T_t\}$. Suppose that $\varphi(t, \omega)$ is integrable for any $t \in \boldsymbol{R}$. Set

$$\Omega_t = \left\{\omega;\ \text{there exists the limit of } \frac{1}{2K}\int_{-K}^{K} \varphi(t, T_u\omega)\, du \text{ as } K \to \infty\right\}.$$

By Birkhoff's ergodic theorem, the $\{T_t\}$-invariant function

$$\varphi^*(t, \omega) = \begin{cases} \lim_{K \to \infty} \dfrac{1}{2K}\displaystyle\int_{-K}^{K} \varphi(t, Z_u\omega)\, du & w \in \Omega_t \\ 0 & w \notin \Omega_t \end{cases}$$

is a version of the conditional expectation $E[\varphi(t, \cdot)|\nu_{\{T_t\}}](\omega)$ and Ω_t is a $\{T_t\}$-invariant set with $P(\Omega_t) = 1$ for each $t \in \boldsymbol{R}$. Since

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(2.5) $$\frac{1}{2K}\int_{-K}^{K}\varphi(t+s,\ T_u\omega)\ du = \frac{1}{2K}\int_{-K}^{K}\varphi(s, T_u\omega)\ du + \frac{1}{2K}\int_{-K}^{K}\varphi(t, T_{u+s}\omega)\ du$$

holds by (1.4), $\Omega_{t+s}\subset\Omega_s\cap T_{-s}\Omega_t = \Omega_s\cap\Omega_t$ and $\Omega_t\subset\Omega_{t+s}\cap\Omega_s$ for any t and $s\in\boldsymbol{R}$. Hence Ω_t is independent of $t\in\boldsymbol{R}$, that is, $\Omega_t = \Omega_0$ for any $t\in\boldsymbol{R}$. From (2.5), it follows that

(2.6) $$\varphi^*(t+s,\omega) = \varphi^*(t,\omega) + \varphi^*(s,\omega) \text{ for } t, s\in\boldsymbol{R},\ \omega\in\Omega.$$

Since $\varphi^*(t,\omega)$ is $\mathfrak{R}_0$-measurable for each fixed $\omega\in\Omega$, we have

(2.7) $$\varphi^*(t,\omega) = t\varphi^*(1,\omega) \text{ for } t\in\boldsymbol{R} \text{ and } \omega\in\Omega.$$

On the other hand, we have

(2.8) $$E[\varphi(s,\cdot)|\nu_{T_t}]\ (\omega) = \lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\varphi(s, T_{kt}\omega) \quad \text{a.e. } (P)$$

by Birkhoff's ergodic theorem. In particular, we have that

(2.9) $$E[\varphi(t,\cdot)|\nu_{T_t}]\ (\omega) = \lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\varphi(t, T_{kt}\omega) = \lim_{m\to\infty}\varphi(mt,\omega) \quad \text{a.e. } (P),$$

by (1.2) and (1.4). From (1.4) and (2.8), it follows that

(2.10) $$\begin{aligned} E[\varphi(t,\cdot)|\nu_{T_t}]\ (T_s\omega) &= \lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\varphi(t, T_{kt+s}\omega) \\ &= \lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}[\varphi(t, T_{kt}\omega) + \varphi(s, T_{kt+t}\omega) - \varphi(s, T_{kt}\omega)] \\ &= E[\varphi(t,\cdot)|\nu_{T_t}]\ (\omega) \qquad \text{a.e. } (P) \end{aligned}$$

for each t and $s\in\boldsymbol{R}$. The last equality shows that $E[\varphi(t,\cdot)|\nu_{T_t}]\ (\omega)$ is $\{T_t\}$-invariant (mod 0). By the relation $\nu_{T_t}\geq\nu_{\{T_t\}}$, we have

(2.11) $$E[\varphi(t,\cdot)|\nu_{T_t}]\ (\omega) = E[\varphi(t,\cdot)|\nu_{\{T_t\}}]\ (\omega) \quad \text{a.e. } (P)$$

for each $t\in\boldsymbol{R}$.

We shall discuss the relations (2.7), (2.9) and (2.11) under weaker conditions and prove the following theorem.

THEOREM 2.2. *Let $\{T_t\}$ be a flow and $\varphi(t,\omega)$ be an additive functional of $\{T_t\}$. If $\varphi(t,\omega)$ is quasi-integrable for each $t\in\boldsymbol{R}$, then the following equalities hold*[4]:

[4] We say that a measurable function $g(\omega)$ is quasi-integrable, if either the positive or the negative part of $g(\omega)$ is integrable.

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$$(2.12)\qquad E[\varphi(t,\cdot)|\nu_{\{T_t\}}](\omega) = t\ E[\varphi(1,\cdot)|\nu_{\{T_t\}}](\omega)$$
$$= E[\varphi(t,\cdot)|\nu_T](\omega)$$
$$= \lim_{m\to\infty}\frac{1}{m}\varphi(mt,\omega)$$
$$= m\text{-}\lim_{s\to\infty}\frac{1}{s}\varphi(st,\omega)\quad \text{a.e. } (P)^{5)}$$

Proof. First we shall prove the assertion in the following case: Let $\{T_t\}$ be the S-flow built up by $(X, \mathfrak{A}, \mu, f(x), 1, S)$ with an automorphism S, and suppose that there exists a constant $\theta>0$, such that $f(x)>\theta$ for any $x\in X$. Let $\varphi(t,(x,u))$ be an additive functional of $\{T_t\}$ which is quasi-integrable for each $t\in\boldsymbol{R}$. Now, let $\varphi(x,u)$ and $\varphi_n(x)$ be the functions defined in Theorem 1.2. Then $\varphi(T_t(x,u))/t$ and $\varphi(x,u)/t$ converge to 0 in measure as $t\to\infty$. Further $f_n(x)/n$ converges to $E_\mu[f(\cdot)|\nu_S](x)$ as $n\to\infty$ for almost every $x\in X$ $(d\mu)$[6]. Therefore by (1.7) and (2.9), there exists $\varphi_1^*(x)=m\text{-}\lim\varphi_n(x)/n$ admitting infinite values and we have

$$(2.13)\quad \varphi_1^*(x) = m\text{-}\lim_{n\to\infty}\frac{f_n(x)}{n}\left[m\text{-}\lim_{m\to\infty}\frac{\varphi(mt,(x,u))}{mt} - m\text{-}\lim_{m\to\infty}\frac{\varphi(T_{mt}(x,u))-\varphi(x,u)}{mt}\right]$$
$$= \frac{1}{t}E_\mu[f(\cdot)|\nu_S](x)\,E[\varphi(t,\cdot)|\nu_{T_t}](x,u)\quad \text{a.e. } (\tilde{P}).$$

for $0<t<\theta$. Observing (1.7) again, we have the equality (2.13) for any $t\neq 0$. The expression shows in particular that $E[\varphi(t,\cdot)|\nu_{T_t}] = t\ E[\varphi(1,\cdot)|\nu_{T_1}]$. Now since the $\mathfrak{A}$-measurable function $\varphi_1^*(x)$ is S-invariant, $\frac{1}{t}E[\varphi(t,\cdot)|\nu_{\{T_t\}}](x,u) = \varphi_1^*(x)/E_\mu[f(\cdot)|\nu_S](x)$ is $\{T_t\}$-invariant. The relation $\nu_{T_t}\geq\nu_{\{T_t\}}$ implies that

$$(2.14)\qquad E[\varphi(t,\cdot)|\nu_{\{T_t\}}] = E[\varphi(t,\cdot)|\nu_{T_t}] = t\ E[\varphi(1,\cdot)|\nu_{T_1}]\quad \text{a.e. } (\tilde{P}).$$

From (1.7) and (2.12), it follows that

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5) The terminology $m\text{-}\lim_{s\to\infty} g_s(\omega)=g(\omega)$ means that $g_s(\omega)$ converges to $g(\omega)$ in measure as $s\to\infty$, that is, $\lim_{s\to\infty}\left[P(|g_s-g|>\varepsilon,\ |g|<\infty)+P(g_s>\frac{1}{\varepsilon},\ g=\infty)+P(g_s<\frac{-1}{\varepsilon},\ g=-\infty)\right]=0$ for any $\varepsilon>0$.

6) We denote the conditional expectation of $g(x)$ with respect to a measurable partition ζ by $E_\mu[g(\cdot)|\zeta](x)$.

$$\begin{aligned} m\text{-}\lim_{t\to\infty}\frac{1}{t}\,\varphi(t,(x,u)) &= m\text{-}\lim_{t\to\infty}\frac{1}{t}\,[\varphi(T_t(x,u))-\varphi(x,u)] + m\text{-}\lim_{n\to\infty}\frac{\varphi_n(x)}{n}\lim_{n\to\infty}\frac{n}{f_n(x)} \\ &= \frac{\varphi_1^*(x)}{E_\mu[f(\cdot)\,|\,\nu_S]\,(x)} \\ &= E[\varphi(1,\cdot)\,|\,\nu_{T_1}]\,(x,u) \quad \text{a.e. } (P). \end{aligned}$$

Hence we have the assertion in our case. Generally, if ω is a fixed poit of $\{T_t\}$, then $\varphi(t+s,\omega)=\varphi(s,\omega)+\varphi(t,T_s\omega)=\varphi(s,\omega)+\varphi(t,\omega)$ hold for $t,s\in\boldsymbol{R}$. Since $\varphi(t,\omega)$ is $\mathfrak{R}_0$-measurable for fixed ω, $\varphi(t,\omega)=t\varphi(1,\omega)$. Hence by the same reasoning of the proof of Theorem 1.2, we have the assertion.

Keeping Theorem 2.1 and 2.2 in mind, we shall give the following definitions.

Definition 2.1. Let $\varphi(t,\omega)$ be an additive functional of a quasi-flow $\{Z_t\}$. We say that $\varphi(t,\omega)$ *belongs to Class* (I), if $\varphi(t,\omega)\geqq 0$ for $t>0$ and $E[\varphi(1,\cdot)\,|\,\nu_{\{Z_t\}}]>0$ a.e. (P) hold. We say that $\varphi(t,\omega)$ *belongs to Class* (D) if $-\varphi(t,\omega)$ belongs to Class (I).

Definition 2.2. An additive functional $\varphi(t,\omega)$ of a flow $\{T_t\}$ *belongs to Class* (A) (resp. *Class* (AI) or *Class* (AD)) if $E[\varphi(1,\cdot)\,|\,\nu_{\{T_t\}}]\neq 0$ a.e. (P) (resp. $E[\varphi(1,\cdot)\,|\,\nu_{\{T_t\}}]>0$ a.e. or <0 a.e.) holds.

Now we show two remarks,

Remark 2.1. By Theorem 2.1, an additive functional $\varphi(t,\omega)$ of a conservative quasi-flow $\{Z_t\}$ belongs to Class (I) if and only if $\varphi(t,\omega)$ is non-decreasing in t and

$$\lim_{t\to\infty}\varphi(t,\omega) = -\lim_{t\to-\infty}\varphi(t,\omega)=\infty \tag{2.15}$$

hold for almost every $\omega\in\Omega$ (P).

Remark 2.2. By the above remark, (2.15) holds for almost every regular point (in the sense of Totoki [8]) if $\{Z_t\}$ is conservative, especially if $\{Z_t\}$ is a flow. This fact means that we may omit the condition (2.15) in the definition of additive functionals in [8], if we discuss only the properties which are preserved under isomorphism (mod 0).

3. *TQ*-systems

The definition of *TQ*-system has been given by the author in [3], but we need to extend it for the purpose of studying the classical dynamics.

Let $\{Z_t\}$ be a quasi-flow and $\{T_s\}$ be a flow. Let $\lambda(s, \omega)$ be a multiplicative functional of $\{T_s^{-1}\}$ such that $\lambda(s, \omega)$ is integrable along the trajectory of $\{Z_t\}$ for any fixed s and ω, that is, $\lambda(s, Z_u\omega)$ is locally integrable in u for any s and ω. Moreover we suppose that

$$\int_0^\infty \lambda(s, Z_u\omega)\, du = \int_0^\infty \lambda(s, Z_{-u}\omega)\, du = \infty \tag{3.1}$$

for any s and ω. Let us define an $\mathfrak{R}_0 \times \mathfrak{R}_0 \times \mathfrak{B}$-measurable function $\varphi(t, s, \omega)$ by

$$\varphi(t, s, \omega) = \int_0^t \lambda(s, Z_u\omega)\, du \tag{3.2}$$

and let $\tau(t, s, \omega)$ be the inverse function of $\varphi(t, s, \omega)$ for fixed s and ω. Let $\{\hat{Z}_t^{(s)}\}$ be the time-changed quasi-flow of $\{Z_t\}$ by $\lambda(s, \omega)$, that is,

$$\hat{Z}_t^{(s)}\omega = Z_{\tau(t,s,\omega)}\omega.$$

Definition 3.1. The triple $[\{Z_t\}, \lambda(s, \omega), \{T_s\}]$ of a quasi-flow $\{Z_t\}$, a flow $\{T_s\}$ and a multiplicative functional $\lambda(s, \omega)$ of $\{T_s^{-1}\}$ is called *a TQ-system* if

$$T_s Z_t T_s^{-1} = \hat{Z}_t^{(s)} \tag{3.3}$$

holds. We say that the *TQ*-system has *Property* (A) (resp. (AC) or (AD)), if the additive functional $\log \lambda(s, \omega)$ of $\{T_s^{-1}\}$ belongs to Class (A) (resp. (AI) or (AD)).

By Theorem 2.2, it holds that

$$E[\log\lambda(s, \cdot)\,|\,\nu_{\{T_t\}}] = E[\log\lambda(s, \cdot)\,|\,\nu_{T_s}] = s\; E[\log\lambda(1, \cdot)\,|\,\nu_{T_1}]$$

for almost every $\omega \in \Omega$ (P). So we have the following proposition.

Proposition 3.1. *Let* $[\{Z_t\}, \lambda(s, \omega), \{T_s\}]$ *be a TQ-system. Then the triple* $[\{Z_t\}, \lambda(s, \omega); T_s]$ *is a TQ-system for each* s *and the following-conditions are equivalent:*

(i) *the TQ-system* $[\{Z_t\}, \lambda(s, \omega), \{T_s\}]$ *has property* (A) (*resp.* (AC) *or* (AD);

(ii) *the TQ-system* $[\{Z_t\}, \lambda(s, \omega); T_s]$ *has property* (A) (*resp.* (AC) *or* (AD)) *for some* $s > 0$;

(iii) *the TQ-system* $[\{Z_t\}, \lambda(s, \omega); T_s]$ *has property* (A) (*resp.* (AC) *or* (AD)) *for any* $s > 0$.

The following Theorems 3.1~3.4 are counterparts of Theorems 5.1, 5.2, 6.1 and 6.2 in [3], respectively. The proofs are immediate from the corresponding theorems and Proposition 3.1.

TTHEOREM 3.1. *Let* $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ *be a TQ-system with property (A). Then the following conditions are equivalent:*

(i) $\{Z_t\}$ *is a flow;*

(ii) $\lambda(s,\omega)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable for any* s;

(iii) $\lambda(s,\omega)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable for some* s $(\neq 0)$.

THEOREM 3.2. *Let* $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ *be a TQ-system with property (A). If* $\{Z_t\}$ *is a flow, the entropy of the flow* $\{Z_t\}$ *is* 0 *or* ∞.

THEOREM 3.3. *Let* $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ *be a TQ-system. If the additive functional* $\log\lambda(s,\omega)$ *of* $\{T_s^{-1}\}$ *belongs to Class (I) (or Class (D)), it holds that*

$$\nu_{\{T_t\}} \leq \nu_{\{Z_t\}}. \tag{3.4}$$

We shall prove only Theorem 3.3. By Proposition 3.1, $[\{Z_t\}, \lambda(s,\omega), T_s]$ is a TQ-system belonging to Class (AC) (resp. (AD)) for fixed $s>0$. Hence

$$\nu_{T_s} \leq \nu_{\{Z_t\}}$$

holds by Theorem 6.1 in [3]. Since it is obvious that $\nu_{\{T_t\}} \leq \nu_{T_s}$, we have the assertion.

THEOREM 3.4. *Let* $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ *be a TQ-system with property (A). If* $\lambda(s,\omega)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable for some* s $(\neq 0)$, (3.4) *holds.*

4. Representation of TQ-systems

In order to study the local structure of the TQ-system, it is necessary to consider the representation of the TQ-system. We say that a TQ-system $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ on Ω is *isomorphic* to a TQ-system $[\{Z'_t\}, \lambda'(s,\omega), \{T'_s\}]$ on Ω', if there exists an isomorphism H of Ω to Ω' such that

$$Z_t = H^{-1}Z'_tH, \quad T_s = H^{-1}T'_sH \quad \text{and} \quad \lambda(s,\omega) = \lambda'(s,H\omega).$$

We shall obtain a special type of TQ-system which is isomorphic to a given TQ-system with suitable conditions. A similar representation of the flow (or the quasi-flow with multi-dimensional parameter) has been obtained

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under the condition that it has no fixed points by [1] (resp. under the condition (ND) by [4]). We now assume the following two conditions.

Condition (ND): For any $\{Z_t\}$ and $\{T_s\}$-invariant set $\Omega_0 \subset \Omega$, $P(\Omega_0) > 0$, there exists a measurable subset V of Ω_0, $P(V) > 0$, and exist positive constants a_1, a_2 such that

$$V \cap Z_t T_s V = \phi \tag{4.1}$$

for any t and s of $a_1 < \max\{|t|, |s|\} < a_2$.

Condition (C): For any $t \in \boldsymbol{R}$ and $\omega \in \Omega$, it holds that

$$\lim_{s \to 0} \int_0^t |\lambda(s, Z_u \omega) - 1| du = 0. \tag{4.2}$$

First of all, we shall show the following simple lemma.

Lemma 4.1. *Let* $[\{Z_t\}, \lambda(s, \omega), \{T_s\}]$ *be a TQ-system. Then it holds that*

$$\varphi(t, s, \omega) = \tau(t, -s, T_s^{-1}\omega) \tag{4.3}$$

and

$$\varphi(\varphi(t, s_1, \omega), s_2, T_{s_1}^{-1}\omega) = \varphi(t, s_1 + s_2, \omega). \tag{4.4}$$

Moreover, $\varphi(t, s, \omega)$ *and* $\tau(t, s, \omega)$ *are continuous in* (t, s).

Proof. By the multiplicativity of $\lambda(s, \omega)$ and by (3,2) and (3.3),

$$\begin{aligned} t = \varphi(u, -s, T_s^{-1}\omega) &= \int_0^u \lambda(-s, Z_r T_s^{-1}\omega)\, dr \\ &= \int_0^u \frac{dr}{\lambda(s, T_s Z_r T_{-s}\omega)} \\ &= \int_0^u \frac{dr}{\lambda(s, Z_{\tau(r,s,\omega)}\omega)} \\ &= \int_0^{\tau(u,s,\omega)} dv = \tau(u, s, \omega) \end{aligned}$$

for $u = \tau(t, -s, T_s^{-1}\omega)$. Hence $u = \varphi(t, s, \omega)$. The relation (4.4) is obvious by

$$\begin{aligned} \int_0^t \lambda(s_1 + s_2, Z_u\omega)\, du &= \int_0^t \lambda(s_1, Z_u\omega)\lambda(s_2, T_{s_1}^{-1} Z_u \omega)\, du \\ &= \int_0^t \lambda(s_1, Z_u\omega)\lambda(s_2, Z_{\tau(u, -s_1, T-s_1\omega)} T_{s_1}^{-1}\omega)\, du \\ &= \int_0^{\varphi(t, s_1, \omega)} \lambda(s_2, Z_v T_{s_1}^{-1}\omega)\, dv \\ &= \varphi(\varphi(t, s_1, \omega), s_2, T_{s_1}^{-1}\omega). \end{aligned}$$

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Since

$$\varphi(t, s_1 + s_2, \omega) - \varphi(t, s_1, \omega) = \int_0^{\varphi(t, s_1, \omega)} (\lambda(s_2, Z_u T_{s_1}^{-1}\omega) - 1)\, du,$$

$\varphi(t, s, \omega)$ is continouus in t and equicontinuous in s for t in any bounded interval. Hence the inverse function $\tau(t, s, \omega)$ has the same property. These facts mean that $\varphi(t, s, \omega)$ and $\tau(t, s, \omega)$ are continuous in (t, s).

By Lemma 4.1, there exist positive numbers $\delta_j < \delta_0 = (a_2 - a_1)/3$, $j = 1, 2, 3, 4$, and exists a measurable subset V_1 of V, $P(V_1) > 0$, such that the following inequalities hold for $\omega \in V_1$;

$$|\varphi(t_1, s, \omega) - \varphi(t_2, s, \omega)| < \delta_0 \quad \text{for} \quad |t_1 - t_2| < \delta_1,\ |t_1|,\ |t_2|,\ |s| < 2a_2$$

$$|\tau(t, s_1, \omega) - \tau(t, s_2, \omega)| < \delta_1 \ \text{for} \ |s_1 - s_2| < \delta,\ |t|,\ |s_1|,\ |s_2| < 2a_2$$

$$|\varphi(t, s, \omega| < \frac{\delta_0}{2} \quad \text{for} \ |t| < \delta_3 \ \text{and} \ |s| < 2a_2,$$

$$|\tau(t, s, \omega)| < \delta_3 \quad \text{for} \ |t|,\ |s| < \delta_3.$$

We can easily see

$$\begin{aligned}\tau &\equiv \tau(u + t + \tau(-u, s, T_{s+v}\omega), -v, T_s\omega) \\ &= \varphi(\tau(u + t + \varphi(\tau(-u, -v, \omega), -s, \omega), -s - v, \omega), -s, \omega)\end{aligned}$$

by Lemma 4.1. Hence

$$|\tau - t| < \delta_0 \tag{4.5}$$

holds for any $\omega \in V_1$, $|t|$, $|s| < a_2$ and $|u|$, $|v| < \delta_4$.

Let us define a function $\psi(a, b; \omega)$ by

$$\psi(a, b;\ \omega) = \frac{1}{ab}\int_0^a\!\!\int_0^b \chi_{V_1}(T_v^{-1} Z_u \omega)\ du\ dv,$$

where $\chi_{V_1}(\omega)$ is the indicator function of V_1. Then by Wiener's ergodic theorem (see Lemma 3.2 in [3]), there exist a and $b \in (0, \delta_4)$, such that $P(\{\omega;\ \psi(a, b;\ \omega) > 3/4\}) > 0$.

Let us fix such a and b and set

$$V' = \left\{\omega;\ \psi(a, b;\ \omega) > \frac{3}{4}\right\}.$$

We shall show that

$$Z_t T_s V' \cap V' = \phi \ \text{for} \ a_1 + \delta_0 < \max(|t|,\ |s|) < a_2 - \delta_0. \tag{4.6}$$

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Suppose on the contrary that the set $Z_tT_sV'\cap V'$ contains ω for suitable t, s. Then from the definition of ψ and V', it is easily seen that $T_v^{-1}Z_u\omega$ and $T_v^{-1}Z_{u+t}T_s\omega$ belong to V_1 for some $u\in(0,a)$ and $v\in(0,b)$. Set $\omega'=T_v^{-1}Z_u\omega$, then $T_v^{-1}Z_{u+t}T_s\omega=T_v^{-1}Z_{u+t}T_sZ_{-u}T_v\omega'=Z_\tau T_s\omega'$, where $\tau=\tau(u+t+\tau(-u,s,T_{s+v}\omega),-v,T_s\omega)$. Hence $Z_\tau T_s w'\in V\cap Z_\tau T_sV$ and $a_1<\max(|\tau|,|s|)<a_2$ hold by (4.5). This fact contradicts Condition (*ND*).

Lemma 4.2. *$\psi(a,b;\ Z_tT_s^{-1}\omega)$ is continuous in (t,s).*

Proof. Since

$$\begin{aligned}\psi(a,b;\ Z_tT_s^{-1}\omega)&=\frac{1}{ab}\int_0^b\chi_{V_1}(T_v^{-1}Z_uZ_tT_s^{-1}\omega)\,du\,dv\\&=\frac{1}{ab}\int_0^b\int_0^a\chi_{V_1}(T_{v+s}^{-1}Z_{\tau(u+t,s,\omega)}\omega)\,du\,dv\\&=\frac{1}{ab}\int_s^{b+s}\int_{\tau(t,s,\omega)}^{\tau(t+a,s,\omega)}\chi_{V_1}(T_v^{-1}Z_u\omega)\lambda(s,Z_u\omega)\,du\,dv\end{aligned}$$

hold, we have that

$$\begin{aligned}&|\psi(a,b;\ Z_{t_1}T_s^{-1}\omega)-\psi(a,b;\ Z_{t_2}T_s^{-1}\omega)|\\&\leq\frac{1}{ab}\left|\int_s^{b+s}\left(\int_{\tau(t_2+a,s,\omega)}^{\tau(t_1+a,s,\omega)}+\int_{\tau(t_2,s,\omega)}^{\tau(t_1,s,\omega)}\right)\lambda(s,z_u\omega)\,du\,dv\right|\\&=\frac{1}{ab}\int_s^{b+s}2|t_1-t_2|dv=\frac{2}{a}\,|t_1-t_2|\end{aligned}$$

and we have similarly that

$$\begin{aligned}&|\psi(a,b;\ Z_sT_s^{-1}\omega)-\psi(a,b;\ Z_t\omega)|\\&\leq\frac{1}{ab}\left|\left(\int_s^{b+s}-\int_0^b\right)\int_{\tau(t,s,\omega)}^{\tau(t+a,s,\omega)}\chi_{V_1}(T_v^{-1}Z_u\omega)\lambda(s,Z_u\omega)\,du\,dv\right|\\&\quad+\frac{1}{ab}\left|\int_0^b\left(\int_{\tau(t,s,\omega)}^{\tau(t+a,s,\omega)}-\int_a^{t+a}\right)\chi_{V_1}(T_v^{-1}Z_u\omega)\lambda(s,Z_u\omega)\,du\,dv\right|\\&\quad+\frac{1}{ab}\left|\int_0^b\int_t^{t+a}\chi_{V_1}(T_v^{-1}Z_u\omega)\,(\lambda(s,Z_u\omega)-1)\,du\,dv\right|\\&\leq\frac{2|s|}{ab}\int_{\tau(t,s,\omega)}^{\tau(t+a,s,\omega)}\lambda(s,Z_u\omega)\,du\\&\quad+\frac{b}{ab}\,|t+a-t-\varphi(t+a,s,\omega)+\varphi(t,s,\omega)|\\&\quad+\frac{b}{ab}\int_t^{t+a}|\lambda(s,Z_uw)-1|\,du\\&\leq\frac{2|s|}{b}+\frac{2}{a}\int_t^{t+a}|\lambda(s,Z_uw)-1|\,du.\end{aligned}$$

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Hence we have the assertion by Condition (C).

LEMMA 4.3. *Let D be the subset of $\boldsymbol{R}^2\times\Omega$ defined by*

$$D=\left\{(t,s,\omega);\ \begin{array}{l}\omega\in V' \text{ and } (t,s) \text{ belongs to the connected}\\ \text{component of } (0,0) \text{ in the set } \{(u,v);\\ \phi(a,b;\ Z_uT_{-v}\omega)>3/4\}\end{array}\right\} \tag{4.7}$$

Then D is $\mathfrak{R}_0\times\mathfrak{R}_0\times\mathfrak{B}$-measurable and it holds that

$$D(Z_tT_s^{-1}\omega)=\{(u-\varphi(t,v-s,T_s^{-1}\omega),\ v-s);\ (u,v)\in D(\omega)\} \tag{4.8}$$

for $(t,s)\in D(\omega)$, where we denote by $D(\omega)$ the ω-section of D: $D(\omega)=\{(u,v);(u,v,\omega)\in D\}$. Moreover, $D(\omega)$ is a bounded open domain of $\boldsymbol{R}^2$.

Proof. Set

$$\varDelta(q,k_1,k_2)=\left[\left(k_1-\frac{1}{2}\right)2^{-q},\ \left(k_1+\frac{1}{2}\right)2^{-q}\right]\times\left[\left(k_2-\frac{1}{2}\right)2^{-q},\ \left(k_2+\frac{1}{2}\right)2^{-q}\right]$$

for integers k_1, k_2 and q, and set

$$L(q)=\{\varDelta(q,k_1,k_2);\ k_1 \text{ and } k_2 \text{ are integers}\},$$

We say that two elements $\varDelta(q,k_1,k_2)$ and $\varDelta(q',k_1',k_2')$ are linked, in symbols $\varDelta\sim\varDelta'$, if either $k_1=k_1'$, $|k_2-k_2'|\leq 1$ or $k_2=k_2'$, $|k_1-k_1'|\leq 1$. Set

$$L(q,n)=\left\{\bigcup_{k=0}^{n}\varDelta_k;\ \begin{array}{l}\varDelta_0=\varDelta(q,0,0)\sim\varDelta_1,\ \varDelta_1\sim\varDelta_2,\cdots,\\ \varDelta_{n-1}\sim\varDelta_n,\ \varDelta_j\in L(q)\end{array}\right\}.$$

Then we have

$$D=\bigcup_{m,n,}\ \bigcup_{\varDelta\in L(q,n)}\left\{\varDelta\times\bigcap_{\substack{t,s\in\varDelta\\ t,s:\ \text{rational}}}\left\{\omega;\ \phi(a,b;\ Z_tT_s^{-1}\omega)\geq\frac{3}{4}+\frac{1}{m}\right\}\right\}$$

by Lemma 4.2. Hence D is $\mathfrak{R}_0\times\mathfrak{R}_0\times\mathfrak{B}$-measurable.

Notice that if $Z_uT_{-v}=Z_{u'}T_{-v'}Z_tT_{-s}\omega$, then

$$\begin{aligned}u'&=u-\varphi(t,v-s,T_s^{-1}\omega),\\ v'&=v-s\end{aligned} \tag{4.9}$$

hold. Since the mapping: $(u,v)\to(u-\varphi(t,v-s,\ T_s^{-1}\omega),\ v-s)$ is one-to-one and continuous by Lemma 4.1, the subset

$$\{(u-\varphi(t,v-s,\ T_s^{-1}\omega),\ v-s);\ (u,v)\in D(\omega)\}$$

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of $\boldsymbol{R}^2$ is connected. Hence (4.8) was proved. The boundedness of $D(\omega)$ follows from (4.6), immediately.

In what follows, we define two measurable functions $t'(\omega)$ and $s'(\omega)$. For $\varDelta\in L(q,n)$, define $s_{m,\varDelta}(\omega)$ by

$$s_{m,\varDelta}(\omega)=\begin{cases}\sup\{s;(-t,-s)\in\varDelta \text{ for some } t\} \text{ if } \varDelta\subset\left\{(t,s);\psi(a,b;Z_tT_s^{-1}\omega)\geq\frac{3}{4}+\frac{1}{m}\right\}\\ -\infty \qquad\qquad\qquad\qquad \text{otherwise}\end{cases}$$

for $\omega\in V'$. Since $\psi(a,b;Z_tT_{-s}\omega)$ is continuous in (t,s) by Lemma 4.2,

$$\bigcap_{t,s}\left\{\omega;\psi(a,b;Z_tT_s^{-1}\omega)\geq\frac{3}{4}+\frac{1}{m}\right\}=\bigcap_{t,s\text{: rational}}\left\{\omega;\psi(a,b,Z_tT_s^{-1}\omega)\geq\frac{3}{4}+\frac{1}{m}\right\}$$

holds by Lemma 4.2, $s_{m,\varDelta}(\omega)$ is measurable. Hence the function defined by

$$s'(\omega)=\sup_m\sup_{q,n}\sup_{\varDelta\in L(q,n)}s_{m,\varDelta}(\omega) \tag{4.10}$$

is measurable. With the same reasoning, the functions defined by

$$t_{m,\varDelta}(s,\omega)=\begin{cases}\sup\{(t;(-t,-s)\in\varDelta\} \text{ if } \varDelta\subset\left\{(t,s);\psi(a,b,Z_tT_s^{-1}w)\geq\frac{3}{4}+\frac{1}{m}\right\}\\ -\infty \qquad\qquad\qquad \text{othersiwe}\end{cases}$$

and

$$t'(s,\omega)=\sup_m\sup_{q,n}\sup_{\varDelta\in L(q,n)}t_{m,\varDelta}(s,\omega) \tag{4.11}$$

are measurable. Further

$$t'(\omega)=\overline{\lim_{s\to s'(\omega)-0}}\,t'(s,\omega) \tag{4.12}$$

is measurable. In fact, for any $k>0$, there exists $s_k\in(s'(\omega)-1/k,\ s'(\omega)]$ such that $t'(\omega)-1/k<t'(s_k,\omega)<t'(\omega)+1/k$. There exist m, q,n and $\varDelta\in L(q,n)$ such that $t'(s_k,\omega)-1/k<t_{m,\varDelta}(s_k,\omega)\leq t'(s_k,\omega)$ and exist a rational number s_k', integrers m', q', n' and $\varDelta'\in L(q'n')$ such that $1/k+t_{m',\varDelta'}(s_k',w)>t_{m,\varDelta}(s_k,\omega)$. Comparing the above inequalitiles, we have

$$t'(\omega)-\frac{3}{k}<t_{m',\varDelta'}(s_k',\omega)\leq t(s_k',\omega). \tag{4.13}$$

Hence we have

$$t'(\omega)=\overline{\lim_{\substack{s\to s'(\omega)-0\\ s;\text{ rational}}}}\,t'(s,\omega). \tag{4.14}$$

We can easily see that $t'(\omega)$ is measurable by the measurability of $t'(s,\omega)$ and (4.14). Hence the following lemma is easily seen.

LEMMA 4.4. *The functions $s'(\omega)$ and $t'(\omega)$ on V' defined by* (4.10) *and* (4.12), *respectively, are measurable and satisfy*

$$s'(Z_t T_s^{-1}\omega) = s'(\omega) + s \tag{4.15}$$

and

$$t'(Z_t T_s^{-1}\omega) = t'(\omega) + \varphi(t, -s'(\omega) - s, T_s^{-1}\omega) \tag{4.16}$$

for $(t, s) \in D(\omega)$.

LEMMA 4.5. *Set*

$$\tilde{D} = \{(u - \varphi(-t'(\omega), v + s'(\omega), T_{s'(\omega)}\omega), v + s'(\omega), \omega); (u, v, \omega) \in D\}. \tag{4.17}$$

Then $\tilde{D}$ is $\mathfrak{R}_0 \times \mathfrak{R}_0 \times \mathfrak{B}$-measurable and satisfies

$$\tilde{D}(\omega) = \tilde{D}(Z_t T_s^{-1}\omega) \quad \textit{for} \quad (t, s) \in D(\omega). \tag{4.18}$$

Proof. By the additivity of $\varphi(t, s, \omega)$ and Lemma 4.1, we have

$$\begin{aligned} &\varphi(-t'(\omega) - \varphi(t, s'(\omega) - s, T_s^{-1}\omega), v + s'(\omega) + s, T_{s'(\omega)+s} Z_t T_s^{-1}\omega) \\ &\qquad = \varphi(-t'(\omega), v + s'(\omega) + s, T_{s'(\omega)}\omega) - \varphi(t, v, T_s^{-1}\omega). \end{aligned} \tag{4.19}$$

By (4.8), (4.17) and (4.19), we have (4.18). The measurability is easily seen by Lemma 4.3 and 4.4.

We now define a partition ξ' of V' by

$$C_{\xi'}(\omega) = \{Z_t T_s^{-1}\omega; (t, s) \in D(\omega)\},$$

which is well defined because of Lemma 4.3. The measurability of ξ' is shown by the following lemma. Let $\Gamma = \{\Gamma_n\}$ be a basis of the Lebesgue space Ω. We denote by Γ_δ the countable family of sets $\Gamma_\delta = \{\Lambda\}$ such that

$$\Lambda = \Gamma_{n_1} \cap \Gamma_{n_2} \cap \cdots \cap \Gamma_{n_i} \cap \Gamma_{m_1}^c \cap \Gamma_{m_2}^c \cap \cdots \cap \Gamma_{m_j}^c.$$

LEMMA 4.6. *The system of measurable functions on V' defined by*

$$\psi_\Lambda(\omega) = \frac{\int_{\tilde{D}(\omega)} \chi_\Lambda(T_s^{-1} Z_{t-t'(\omega)} T_{s'(\omega)}\omega)\, dt\, ds}{\int_{\tilde{D}(\omega)} dt\, ds}, \quad \Lambda \in \Gamma_\delta, \tag{4.20}$$

induces the partition ξ'.

Proof. The measurability of the given functions is obvious by Lemma 4.4 and 4.5. By Lemma 4.5, $\psi_\Lambda(\omega)$ is invariant on each $C_{\xi'}$. Since $\Gamma=\{\Gamma_n\}$ is completly separable, $\psi_\Lambda(\omega)$ is a set function of $\Lambda\in\Gamma_\delta$ and hence it defines a Lebesgue measure on $\tilde{D}(\omega)$ for almost every $\omega\in V'$. Hence, if $\psi_\Lambda(\omega)=\psi_\Lambda(\omega^*)$ holds for any $\Lambda\in\Gamma_\delta$, then an isomorphism θ of $\tilde{D}(\omega)$ onto $\tilde{D}(\omega^*)$ is induced, naturally. Let $(u^*,v^*)=\theta(u,v)$, then $T_{-v}Z_{u-t'(\omega)}T_{s'(\omega)}(\omega)\in\Lambda$ if and only if $T_{-v^*}Z_{u^*-t'(\omega^*)}\omega^*T_{s'(\omega^*)}\omega^*\in\Lambda$ for almost every (u,v). By the separability of $\Gamma=\{\Gamma_n\}$, $C_{\xi'}(\omega)$ and $C_{\xi'}(\omega^*)$ contain the same point, i.e. $C_{\xi'}(\omega)=C_{\xi'}(\omega^*)$.

Set

$$\tilde{V}'=\{(x,u,v);\ (u,v)\in\tilde{D}(\omega),\ \ x=C_{\xi'}(\omega)\in V'_{/\xi'}\}.$$

Let H' be the mapping from V' onto $\tilde{V}'$ defined by

$$H'\omega=(C_{\xi'}(\omega),\ t'(\omega),\ s'(\omega)).$$

Define a function φ' by $\varphi'(t,s;x,u,v)=\varphi(t,s,H'^{-1}(x,u,v))$ for $(x,u,v)\in V'$. Then it holds that

$$H'Z_tH'^{-1}(x,u,v)=(x,\ u+\varphi'(t,\ -v;\ x,u,v),v) \tag{4.21}$$

$$H'T_s^{-1}H'^{-1}(x,u,v)=(x,u,v+s) \tag{4.22}$$

if $(x,\ u+\varphi'(t,\ -v;x,u,v),v)$ and $(x,u,v+s)$ belong to V'.

Let us define a measure $\tilde{P}'$ on $\tilde{V}'$ by $\tilde{P}'(\tilde{B}')=P(H'^{-1}B')$, $\tilde{\mathfrak{B}}'=\{H'B;\ B\subset V',\ B\in\mathfrak{B}\}$. Let $p(du\ dv|C_{\tilde{\xi}'})$ be the canonical system of measures with respect to the measurable partition $\tilde{\xi}'=H'\xi'$ of $\tilde{V}'$. Then $\tilde{\mathfrak{B}}'$ is the restriction of the product σ-field $\mathfrak{B}_{/\xi'}\times\mathfrak{R}\times\mathfrak{R}$ to $\tilde{V}'$ by the measurability of the set D and of the functions $t'(\omega)$ and $s'(\omega)$.

Since $\{Z_t\}$ and $\{T_s\}$ are non-singular transformation groups, it follows from (4.21) and (4.22) that the canonical system of measures $p(du\ dv|C_{\tilde{\xi}'})$ is equivalent to the ordinary Lebesgue measure $du\ dv$, that is, the meausre P' is expressed in the form

$$\tilde{P}'(B')=\iint_{D(x)}\chi_{B'}(x,u,v)\,p(x,u,v)\ du\ dv\ dP_{\xi'}(x)$$

with some positive measurable function $p(x,u,v)$, where $D(x)=D(\omega)$, $\omega\in C_{\xi'}=x$. (c.f. Lemma 3.1 in [3]).

Since $\{T_t\}$ is measure preserving $p'(x,u,v)$ dose not depend on v, that is,

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(4.23) $$d\tilde{P}'(x,u,v) = p(x,u)\ du\ dv\ dP_{\xi'}(x).$$

Thus we have the following lemma.

LEMMA 4.6. *The mapping H' is an isomorphism from V' onto $(\tilde{V}', \tilde{\mathfrak{B}}', \tilde{P}')$ which satisfies* (4.21) *and* (4.22).

Since $D(x)$ is an open domain in $\boldsymbol{R}^2$, there exist real numbers d_i, $i = 1, 2, 3, 4$, and a measurable ξ'-set $A \subset V'$, $\tilde{P}'(A) > 0$, such that

$$[d_1, d_2) \times [d_3, d_4) \subset \tilde{D}(\omega) \quad \text{for} \quad \omega \in A.$$

Define two functions $t_0(\omega)$ and $s_0(\omega)$ by

(4.24) $$t_0(\omega) = \varphi(t'(\omega) - d_1, d_3, Z_{d_1 - t'(\omega)} T_{s'(\omega)}\omega), \quad s_0(\omega) = s'(\omega) - d_3.$$

Then there exists a positive constant d_5 and exists a measurable ξ'-set $A_0 \subset A$ with positive measure such that

$$d_5 \leq \varphi(d_2 - d_1, d_3, Z_{d_1 - t'(\omega)} T_{s'(\omega)}\omega) \quad \text{for} \quad \omega \in A.$$

Set $$V_0 = \{\omega;\ 0 \leq t_0(\omega) < d_5,\ 0 \leq s_0(\omega) < d_4 - d_3,\ \omega \in A_0\}.$$

Now we have the following main lemma by Lemma 4.6.

LEMMA 4.7. *Let $[\{Z_t\}, \lambda(s, \omega), \{T_s\}]$ be a TQ-system which satisfies Condition (ND) and (C). Then for any $\{Z_t\}$ and $\{T_s\}$-invariant set Ω_0 with $P(\Omega_0) > 0$, there exists a measurable subset V_0, $P(V_0) > 0$, and exists a measurable partition ξ_0 of V_0 such that V_0 is isomorphic to a Lebesgue space $(\tilde{V}_0, \tilde{\mathfrak{B}}_0, \tilde{P}_0)$ defined as follows. Set $X = V_{0/\xi_0}$ and set $\tilde{V}_0 = \{(x, u, v);\ 0 \leq u < a_0,\ 0 \leq v < b_0,\ x \in X\}$. Let $\tilde{\mathfrak{B}}_0$ be the restriction of the σ-field $\mathfrak{B}_{/\xi_0} \times \mathfrak{R}$ to $\tilde{V}_0$ The measure $\tilde{P}_0$ on $\tilde{V}_0$ is defined by*

$$d\tilde{P}_0(x, u, v) = p(x, u)\ du\ dv\ d\mu(x), \quad d\mu(x) = dP_{\xi_0}(x)$$

with some positive $\tilde{\mathfrak{B}}_0$-measurable function $p(x, u)$ which are independent of v. Moreover, there exists an isomorphism H_0 of V_0 onto $\tilde{V}_0$ which satisfies

(4.25) $$H_0 Z_t H_0^{-1}(x, u, v) = (x, u + \tilde{\varphi}(t, -v, (x, u, v)), v)$$

(4.26) $$H_0 T_s^{-1} H_0^{-1}(x, u, v) = (x, u, v + s)$$

for u, $u + \tilde{\varphi}(t, -v, (x, u, v)) \in [0, a_0)$ and v, $v + s \in [0, b_0)$, where

$$\tilde{\varphi}(t, s, (x, u, v)) = \varphi(t, s, H_0^{-1}(x, u, v)).$$

Let $(X, \mathfrak{A}, \mu)$ be a Lebesgue space and $f^{(1)}(x)$ and $f^{(2)}(x)$ be measurable functions on X. Set $\tilde{\Omega} = \{(x, u, v);\ 0 \leq u < f^{(1)}(x),\ 0 \leq v < f^{(2)}(x),\ x \in X\}$,

$\tilde{X}^{(1)} = \{(x,v);\ 0 \leq v < f^{(2)}(x),\ x \in X\}$ and $\tilde{X}^{(2)} = \{(x,u);\ 0 \leq u < f^{(1)}(x),\ x \in X\}$. We shall identify the space $\{((x, v), u);\ 0 \leq u < f^{(1)}(x),\ (x, u) \in X^{(1)}\}$ and $\{((x,u),v);\ 0 \leq v < f^{(2)}(x),\ (x,u) \in X^{(2)}\}$ with $\tilde{\Omega}$. Let $\mathfrak{B}$ (resp. $\mathfrak{A}^{(1)}$ or $\mathfrak{A}^{(2)}$) be the restriction of $\overline{\mathfrak{A} \times \mathfrak{R} \times \mathfrak{R}}$ to $\tilde{\Omega}$ (resp. $\overline{\mathfrak{A} \times \mathfrak{R}}$ to $X^{(1)}$ or $\overline{\mathfrak{A} \times \mathfrak{R}}$ to $X^{(2)}$). Let $p(x,u,v) = p(x,u) > 0$ be a measurable function on $\tilde{\Omega}$ which is independent of v. Suppose that

$$\int_X \int_0^{f^{(2)}(x)} \int_0^{f^{(1)}(x)} p(x,u)\ du\ dv\ d\mu(x) = 1.$$

Set $d\tilde{P}(x, u, v) = p(x,u)\ du\ dv\ d\mu(x)$, $d\mu^{(1)}(x,v) = dv\ d\mu(x)$ and $d\mu^{(2)}(x,u) = p(x,u)\ du\ d\mu(x)$. Thus we have three Lebesgue space $(\tilde{\Omega}, \mathfrak{B}, \tilde{P})$, $(X^{(1)}, \mathfrak{A}^{(1)}, \mu^{(1)})$ and $(X^{(2)}, \mathfrak{A}^{(2)}, \mu^{(2)})$. Define mappings $\pi_X^j(x,u)$ and $\pi_R^j(x,u)$, $j = 1,2$, by

$$\pi_X^j(x,u) = x \text{ and } \pi_R^j(x,u) = u \text{ for } (x,u) \in X^{(j)},\ j = 1,2.$$

Let $S_{(1)}$ be a quasi-automorphism of $X^{(1)}$ such that there exists $\delta^{(1)} = \delta^{(1)}(x,v) > 0$ and

$$\pi_X^1(x,v') = \pi_X^1(x,v) \text{ and } \pi_R^1(x,v') - \pi_R^1(x,v) = v' - v$$

hold for any $v' \in [v, v + \delta^{(1)})$. Let $\{\tilde{Z}'_t\}$ be the S-quasi-flow built up by $(X^{(1)}, \mathfrak{A}^{(1)}, \mu^{(1)}, f^{(1)}(x), p(x,u), S_{(1)})$. Let $S_{(2)}$ be an automorphism of $X^{(2)}$ such that there exists $\delta^{(2)} = \delta^{(2)}(x,u) > 0$ and

$$\pi_X^2(x,u') = \pi_X^2(x,u) \text{ and } \pi_R^2(x,u') - \pi_R^2(x,u) = \int_u^{u'} \lambda_1(x,u)\ du$$

hold for any $u' \in [u, u + \delta^{(2)})$ with some positive $\mathfrak{A}^{(2)}$-measurable function $\lambda_1(x,u)$. Let $\{\tilde{T}_s^{-1}\}$ be the S-flow built up by $(X^{(2)}, \mathfrak{A}^{(2)}, \mu^{(2)}, f^{(2)}(x), 1, S_{(2)})$. Define a multiplicative functional $\lambda'(s,(x,u,v))$ of $\{T_s^{-1}\}$ by

$$\lambda'(s,(x,u,v)) = \lambda_n(x,u) \text{ for } f_n^{(2)}(x,u) \leq v + s < f_{n+1}^{(2)}(x,u),$$

where $\lambda_n(x,u)$ and $f_n(x,u)$ are multiplicative and additive functional of $S_{(2)}$ constructed from $\lambda_1(x,u)$ and $f_1^{(2)}(x,u) = f^{(2)}(x)$, respectively. Then it holds that

$$\tilde{T}_s \tilde{Z}'_t \tilde{T}_s^{-1}(x,u,v) = \tilde{Z}'_{\tau'(t,s,(x,u,v))}(x,u,v) \tag{4.17}$$

if either $|t|$ or $|s|$ is sufficiently small where $\tau'(t,s,(x,u,v))$ is the inverse function of $\varphi'(t,s,(x,u,v)) = \int_0^t \lambda'(s, \tilde{Z}'_t(x,u,v))\ dt$ for each $(x,u,v) \in \tilde{\Omega}$ and $s \in R$.

Suppose that (4.17) holds for any $t, s \in R$ and $(x,u,v) \in \tilde{\Omega}$. Then we have a TQ-system $[\{\tilde{Z}'_t\}, \lambda'(s,(x,u,v)), \{\tilde{T}_s\}]$ on $\tilde{\Omega}$. Let $\lambda(x,u,v)$ be a positive measurable function and let $\{\tilde{Z}_t\}$ be the time-changed quasi-flow of $\{\tilde{Z}'_t\}$ by

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$\lambda(x,u,v)$, that is,

$$\tilde{Z}_t(x,u,v) = \tilde{Z}'_{\sigma(t,(x,u,v))}(x,u,v) \qquad (x,u,v)\in\tilde{\Omega},$$

where $\sigma(t,(x,u,v))$ is the inverse function of $\psi(t,(x,u,v)) = \int_0^t \lambda(\tilde{Z}'_t(x,u,v))\,dt$ for fixed $(x,u,v)\in\tilde{\Omega}$. Then it follows from Proposition 5.1 in [3] that $[\{\tilde{Z}_t\}, \lambda(s,(x,u,v)), \{\tilde{T}_s\}]$ is a TQ-system, where $\lambda(s,(x,u,v))$ is the multiplicative functional of $\{\tilde{T}_s^{-1}\}$ defined by

$$\lambda(s,(x,u,v)) = \frac{\lambda(\tilde{T}_s^{-1}(x,u,v))}{\lambda(x,u,v)}\,\lambda'(s,(x,u,v)).$$

We say that the TQ-system $[\{\tilde{Z}_t\}, \lambda(s,(x,u,v)), \{\tilde{T}_s\}]$ is *an S-TQ-system.* By virtue of Lemma 4.7, we have the following theorem.

THEOREM 4.1. *Let us assume that the TQ-system* $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ *satisfies Condition* (*C*). *Then there exists an S-TQ-system isomorphic to it if and only if the original TQ-system satisfies Condition* (*ND*).

REMARK. If $\lambda(s,\omega) = e^{\kappa s}$ with constant $\kappa \neq 0$, then $\{Z_t\}$ is a flow by Theorem 3.1. Hence we can choose $p(x,u) = 1$, $\lambda(x,u,v) = e^{\kappa v}$ and $\lambda_1(x,u) = e^{\kappa f^{2}(x)}$. Then we have

$$Z_t T_s^{-1}(x,u,v) = (x, u + te^{-\kappa(v+s)}, v+s)$$

for $0 \leq u$, $u + te^{-\kappa(v+s)} < f^{(1)}(x)$, $0 \leq v$, $v + s < f^{(2)}(x)$.

5. *TQ*-systems and increasing partitions of Flows

We now discuss increasing partitions of flows with the help of the representation of TQ-systems. The similar results are given in the case of automorphisms [3].

THEOREM 5.1. *Let* $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ *be a TQ-system which satisfies Condition* (*ND*) *and* (*C*). *If the system has property* (*AC*) *and* $\lambda(s,\omega)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable for some* $s\,(\neq 0)$, *then there exists a partition* ζ *of* Ω *such that almost every element of* Ω *is a segment of a trajectory of* $\{Z_t\}$ *and it holds that*

(i) $T_s\zeta \geq \zeta$ (mod 0) *for* $s > 0$,

(ii) $\bigvee_s T_s\zeta = \varepsilon$ (mod 0),

(iii) $\bigwedge_s T_s\zeta = \nu_{\{Z_t\}}$ (mod 0),

(iv) $H(T_s\zeta\,|\,\zeta) = E[\log\lambda(s,\omega)] = s\;E[\log\lambda(1,\omega)]$, $s > 0$.

Proof. By the same reasoning as the proof of Theorem 7.1 in [3], it suffices to prove that, for any $\{Z_t\}$ and $\{T_s\}$-invariant set Ω_0 with positive measure, there exists a $\{Z_t\}$ and $\{T_s\}$-invariant subset $\Omega_1 \subset \Omega_0$ with positive measure and exists a measurable partition ζ_1 of Ω_1 which satisfies the following four conditions: (i)′ $T_s\zeta_1 \geq \zeta_1$ (mod 0) for $s > 0$; (ii)′ $\bigvee_s T_s\zeta_1$ is the partition of Ω_1 to the individual points which is denoted by ε^1; (iii)′ $\bigwedge_s T_s\zeta_1$ is the restriction of $\nu_{\{Z_t\}}$ onto Ω_1, we denote simply it by $\nu^1_{\{Z_t\}}$; (iv)′ the following equality holds

$$\int_{\Omega_1} \log P(C_{T_s\zeta_1}(\omega) | C_{\zeta_1}(\omega))\, dP = \int_{\Omega_1} \log \lambda(s, \omega)\ dP. \tag{5.1}$$

We shall fix a $\{Z_t\}$ and $\{T_s\}$-invariant measurable set Ω_0 with positive measure. Let V_0, $\tilde{V}_0, p(x, u), \mu(x)$ and H_0 be the ones given by Lemma 4.7. Since $\{Z_t\}$ is a flow by Theorem 3.1, we may assume that $p(x, u) = 1$. Let $\tilde{\eta}$ be the measurable partition of $\tilde{V}_0$ defined by

$$C_{\tilde{\eta}}(x, u, v) = \{(x, u', v);\ 0 \leq u' < a_0\}$$

and η be the partition of V_0 given by $\eta = H_0^{-1}\tilde{\eta}$. Let $\Omega_1 = \bigcup_k T_{kr}V_0$ with $r = b_0$, and define a partition ζ_1 of Ω_1 by

$$\zeta_1 = \bigvee_{k \leq 0} T_{kr}\eta_1, \tag{5.2}$$

where η_1 is the partion of Ω_1 which is eaual to η on V_0 and is degenerated on $\Omega_1 - V_0$.

Then Ω_1 satisfies the following conditions (i)″ ~ (iv)″ by Theorem 7.1 in [3]: (i)″ $T_r\zeta_1 \geq \zeta_1$ (mod 0); (ii)″ $\bigvee_k T_{kr}\zeta_1 = \varepsilon^1$ (mod 0); (iii)″ $\bigwedge_k T_{kr}\zeta_1 = \nu^1_{\{Z_t\}}$ (mod 0); (iv)″ $H(T_r\zeta_1 | \zeta_1) = \int_{\Omega_1} \log \lambda(r, \omega)\, dP$.

By the construction of ζ_1, it is easily seen that

$$T_s\eta_1 \vee T_{s-r}\eta_1 \geq \eta_1 \quad 0 \leq s \leq r \tag{5.3}$$

and that

$$T_s\eta_1 \geq \nu^1_{\{Z_t\}}. \tag{5.4}$$

By (5.3) and (ii)″, we have

$$(5.5)\qquad \begin{aligned} T_s\zeta_1 &= \bigvee_{k\leq 0} T_s T_{kr}\eta_1 \\ &= \bigvee_{k\leq 0} T_{kr}(T_s\eta_1 \vee T_{s-r}\eta_1) \\ &\geq \bigvee_{k\leq 0} T_{kr}\eta_1 = \zeta_1 \end{aligned}$$

for $r \geqq s \geqq 0$. From (ii)″, (iii)″ and (5.4), it follows that

$$(5.6)\qquad \bigvee_s T_s\zeta_1 \geq \bigvee_k T_{kr}\zeta_1 = \varepsilon^1$$

and

$$(5.7)\qquad \nu^1_{\{Z_t\}} \leq \bigwedge_s T_s\zeta_1 \leq \bigwedge_k T_{kr}\zeta_1 = \nu^1_{\{Z_t\}}.$$

By (iv)″ and Theorem 2.2, we have that

$$\begin{aligned} H(T_s\zeta|\zeta_1) &= \frac{s}{r} H(T_r\zeta_1|\zeta_1) \\ &= \frac{s}{r}\int_{\Omega_1} \log\lambda(r,\omega)\, dP \\ &= s\int_{\Omega_1} \log\lambda(1,\omega)\, dP. \end{aligned}$$

Thus our proof is concluded.

THEOREM 5.2. *Let* $[\{Z_t\}, \lambda(s,\omega), \{T_s\}]$ *be a TQ-system which satisfies Condition (ND) and (C). If the additive functional* $\log\lambda(s,\omega)$ *of* $\{T_s^{-1}\}$ *belongs to Class* (I), *then there exists a measurable partition* ζ *of* Ω *which satisfies the conditions* (i)∼(iv) *in Theorem* 5.1.

The proof of this theorem is performed similarly as the proof of Theorem 5.1 (see Theorem 7.2 in [3]).

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