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A CHARACTER-THEORY-FREE CHARACTERIZATION OF THE MATHIEU GROUP M_{12}

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Abstract

The known characterization of the Mathieu group M_{12} by the structure of the centralizer of a 2-central involution is based on the application of the theory of exceptional characters and uses in addition a block theoretic result which asserts that a simple group of order $|M_{12}|$ is isomorphic to M_{12}. The details of the proof of the latter result had never been published. We show here that M_{12} can be handled in a completely elementary and group theoretical way.

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The object of this paper is to present a character theory free proof of the following result.

THEOREM. *Let* G *be a finite, nonabelian simple group which possesses an involution such that its centralizer in* G *is isomorphic to the centralizer of a 2-central involution in* M_{12}. *Then* G *is isomorphic to* M_{12}, *the Mathieu group on* 12 *letters.*

The main point here is as in [7], that our proof will be completely free of applications of the theory of group characters. Predecessors of this theorem are [1, Theorem (6A)] and [12, Theorem]. The results of both papers had been easily combined in [4] to show that the above theorem holds. However, the proofs in [1] and in [12] rely upon a theorem of R. G. Stanton [8] which asserts that a simple group of order 95,040 or 244,823,040 is isomorphic to either M_{12} or M_{24}.

As a matter of fact, the Ph.D. thesis of R. G. Stanton, written under the supervision of Richard Brauer, had not been published, and there is only the short summary [8] available in which he describes his methods, which are heavily block theoretical and computational. Thus, it seems worthwhile to present an elementary and complete proof of the charcterization of M_{12} by a 2-central involution. We remark that in [6] we had shown that the characterization of M_{24} of [5] can be done without referring to the result of [8]; moreover, the second author has shown that for the characterizations of M_{22} and M_{23}, originally due to Z. Janko, one does not need recourse to the theory of exceptional characters.

1. The centralizer of a 2-central involution in M_{12}

Denote by H the centralizer of a 2-central involution z_1 of M_{12}. According to [4], H can be generated by elements $z_1, z_2, z_3, a, b, c, d$, where all elements except c are involutions and c is an element of order 3. The subgroup E generated by z_1, z_2, z_3 is elementary abelian of order 8 and normal in H, and $\langle a, b\rangle\langle c\rangle$ is isomorphic to A_4 such that $a^c = b$, $b^c = ab$. The involution d inverts c. The action on E for the elements a, b, c, d is described by the following matrices with entries from $GF(2)$ with respect to the basis $\{z_1, z_2, z_3\}$ of the "vector space" E over $GF(2)$:

$$a \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad b \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad ab \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$d \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad c \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We have $a^d = z_1 z_2 a$ and $b^d = z_1^\beta z_2 z_3 ab$, where $\beta \in \{0, 1\}$. Since d inverts c, we get from $a^{dcdc} = a$ that $\beta = 1$.

It is now a routine matter to calculate the conjugacy clases of H. The results are listed in Table I.

2. The fusion of involutions and the possible orders for G

In what follows, G denotes a finite simple group possessing an involution z such that $\mathbf{C}(z)$ is isomorphic to H. We put $\mathbf{C}(z) = H$ and use the notation developed for H so far. Thus, in particular, $z = z_1$, and since the center of a S_2-subgroup of H is cyclic, we see that z_1 is 2-central in G.

TABLE I

x	x^2	$o(x)$	$\mathbf{C}_H(x)$	$\lvert ccl_H(x)\rvert$
1		1	H	1
z_1		2	H	1
z_2		2	$E\langle a, d\rangle$	6
a		2	$\langle z_1, z_2, a, b\rangle$	12
z_3a	z_1	4	$\langle z_3a, z_2, b, z_3d\rangle$	6
z_2z_3a	z_1	4	$\langle z_2z_3a, z_2, z_3b, z_3d\rangle$	6
c		3	$\langle z_1, c\rangle$	32
d		2	$\langle z_1, z_2, d\rangle$	24
z_3d	z_2	4	$\langle z_1, z_3a, z_3d\rangle$	12
z_1z_3d	z_2	4	$\parallel$	12
bd	z_2z_3a	8	$\langle bd\rangle$	24
z_3bd	z_1z_3a	8	$\langle z_3bd\rangle$	24
z_1c		6	$\langle z_1, c\rangle$	32
				192

(2.1) LEMMA. *A S_2-subgroup of $\mathbf{C}(a)$ has order 2^4.*

PROOF. We have $\mathbf{C}_H(a) = \langle a\rangle \times \langle z_1, z_2, b\rangle$. Thus $\mathbf{C}_H(a)' = \langle z_1\rangle$. The assertion follows.

(2.2) LEMMA. *If $z_1 \sim z_2$ in G, then $O(\mathbf{C}(a)) = \langle 1\rangle$.*

PROOF. Put $K = O(\mathbf{C}(a))$ and act with the four-group $\langle z_1, z_2\rangle$ on K. If $x \in \langle z_1, z_2\rangle^{\#}$, then $\mathbf{C}(a) \cap \mathbf{C}(x)$ is a 2-group by the structure of H, and so, x operates fixed-point-freely on K. Application of [9, 5.1.9] yields that $[K, z_1z_2] = \langle 1\rangle$. We conclude that $K = \langle 1\rangle$.

(2.3) LEMMA. *If $z_1 \sim z_2$ in G, then $\mathbf{C}(a) = \langle a\rangle \times U$, where U possesses a subgroup of index* 2.

PROOF. A result of W. Gaschütz yields the existence of a subgroup U such that $\mathbf{C}(a) = \langle a\rangle \times U$. Clearly, $P = \langle a, z_1, z_2, b\rangle \cap U$ is dihedral of order 8 with center equal to z_1 as $\mathbf{C}_H(a)' = \langle z_1\rangle$. We have that a S_2-subgroup of U is selfnormalizing in U. Application of a result of O. Grün [3, 7.4.2] yields that $P \cap U'$ is contained in $\langle z_1, z_2\rangle$; note that the elements of $\langle z_1, z_2\rangle^{\#}$ are the only G-conjugates of z_1 in $\langle a, z_1, z_2, b\rangle$. Now, from [3, 7.3.1] it follows that U has a subgroup of index 2; use the fact that $P/P\cap U' \cong U/U^*$ for some normal subgroup U^* of U. The lemma is proved.

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(2.4) LEMMA. *The normalizer of E in G is H.*

PROOF. By way of contradiction assume that $\mathbf{N}(E) \supset H$. Then $\mathbf{N}(E)/E \cong GL(3, 2)$; note that E is selfcentralizing. Thus, $z_1 \sim z_2$ and $a \sim d$ hold in $\mathbf{N}(E)$. It follows that G possesses precisely two classes of involutions. The representatives of these classes are z_1 and a.

From (2.3) we know that $\mathbf{C}(a) = \langle a \rangle \times U$, where U possesses a subgroup U_1 of index 2, and from (2.2) we get that $O(\mathbf{C}(a)) = \langle 1 \rangle$. If a S_2-subgroup of U_1 is cyclic then U is a 2-group by transfer results, and in this case $\mathbf{C}(a) = \mathbf{C}_H(a)$. Assume that a S_2-subgroup X of U_1 is a four-group. If $\mathbf{C}_H(a) \subset \mathbf{C}(a)$, then all involutions of U_1 are conjugate in U_1 as otherwise U_1 would possess a normal 2-complement which is $\langle 1 \rangle$ by (2.2). It follows that $(\mathbf{N}(X) \cap U_1)/(\mathbf{C}(X) \cap U_1)$ has order 3. Clearly, the commutator subgroup of $\mathbf{C}(a)$ lies in U_1. Since $\mathbf{C}_H(a)' = \langle z_1 \rangle$, we get $z_1 \in U_1$ and this implies that $\langle z_1, z_2 \rangle$ is a S_2-subgroup of U_1. Thus, $\mathbf{C}(X) \cap U_1 = X$ and $|\mathbf{N}(X) \cap U_1| = 2^2 \cdot 3$. Note that $\langle z_1, z_2 \rangle = \mathbf{Z}(E\langle a, d \rangle)$. Since $\mathbf{N}(E)/E \cong GL(3, 2)$, we see that $\mathbf{N}(\langle z_1, z_2 \rangle)$ lies in $\mathbf{N}(E)$ and has order $2^6 \cdot 3$. Thus $\mathbf{N}(\langle z_1, z_2 \rangle) \cap U_1 \subseteq \mathbf{N}(E)$. But then a would centralize an element of order 3 in $\mathbf{N}(E)$ which contradicts the structure of $GL(3, 2)$. We have shown that $\mathbf{C}(a) = \mathbf{C}_H(a)$.

We know that G has precisely two classes of involutions and that if x is an involution of $\mathbf{N}(E)$ then $\mathbf{C}(x) \subseteq \mathbf{N}(E)$. Application of [10, Lemma 5.35] yields $\mathbf{N}(E) = G$ which contradicts the simplicity of G. The assertion is proved.

(2.5) LEMMA. *The involution z_1 is conjugate in G to an element of $H \backslash \langle z_1 \rangle$.*

PROOF. Assume by way of contradiction that G is 2-normal. Put $T = E\langle a, b, d \rangle$. Since $\mathbf{Z}(T) = \langle z_1 \rangle$ and $T \in \mathrm{Syl}_2(G)$, we get from O. Grün's theorem [3, 7.5.2] that $T \cap G' = T \cap H'$. It would follow that G had a normal subgroup of index 2. Therefore G is not 2-normal. This implies the existence of an element g in G such that $z_1 \in T \cap T^g$, but $\langle z_1 \rangle \neq \mathbf{Z}(T^g)$. The center of T^g is $\langle z_1^g \rangle$, and so, $z_1 \neq z_1^g$. Since $z_1 \in T^g$, we have $[z_1, z_1^g] = 1$. It follows $z_1^g \in H \backslash \langle z_1 \rangle$.

(2.6) LEMMA. *In G we have $z_1 \sim z_2 \sim d$ and a is not conjugate to z_1.*

PROOF. We know that $\mathbf{N}(E) = H$ and that z_1 is not conjugate to a in G; remember that a S_2-subgroup of $\mathbf{C}(a)$ has order 2^4.

By way of contradiction we suppose that $d \sim a$ holds in G. From (2.5) we get $z_1 \sim z_2$. In $\langle a\rangle \times \langle z_1, z_2, b\rangle$ there are precisely two elementary abelian subgroups of order 8. Thus, as $d \sim a$, there are the following two possibilities:

(i) $$\langle a, z_1, z_2\rangle \sim \langle d, z_1, z_2\rangle;$$

(ii) $$\langle a, z_1, b\rangle \sim \langle d, z_1, z_2\rangle.$$

The possibility (ii) does not occur as in the group on the left there is only one G-conjugate of z_1 whereas in the group on the right there are three G-conjugates of z_1. Thus, we are in case (i). We get that the conjugation (i) is performed by an element of $\mathbf{N}(\langle z_1, z_2\rangle)$. Denote the latter group by N. Our assumptions imply that $|N| = 2^6 \cdot 3$, since $\langle z_1, z_2\rangle = \mathbf{Z}(E\langle a, d\rangle)$. Clearly, $O_2(N) = E\langle a, d\rangle = \mathbf{C}(\langle z_1, z_2\rangle)$. By a result of Baer-Suzuki [3, 3.8.2] there is an element k of order 3 in N which is inverted by b. We may assume that $k: z_1 \to z_2 \to z_1z_2$. The group $\langle k, b\rangle \cong \Sigma_3$ acts on $E\langle a, d\rangle/\langle z_1, z_2\rangle$. Clearly, $k \notin \mathbf{N}(E)$ as $\mathbf{N}(E) = H$. We have $o(z_3) = o(a) = o(d) = o(z_3ad) = 2$ and $o(z_3a) = o(z_3d) = o(ad) = 4$. Since k acts fixed-point-freely on $\langle z_1, z_2\rangle$, we see that k does not leave invariant any of the following cosets: $\langle z_1, z_2\rangle z_3a$, $\langle z_1, z_2\rangle z_3d$, $\langle z_1, z_2\rangle ad$. It follows that

$$k: \langle z_1, z_2\rangle z_3a \to \langle z_1, z_2\rangle z_3d \to \langle z_1, z_2\rangle ad.$$

Furthermore, $k: \langle z_1, z_2\rangle z_3 \to \langle z_1, z_2\rangle a$ and $k: \langle z_1, z_2\rangle z_3 \to \langle z_1, z_2\rangle d$ are both impossible as $z_1 \sim z_2 \sim z_3$, $a \sim d$, and a is not conjugate to z_1. But also $k: \langle z_1, z_2\rangle z_3 \to \langle z_1, z_2\rangle z_3ad$ cannot hold as $Ead \sim Ed$ in H and all involutions of Ed are conjugate. We have obtained a contradiction which shows that d cannot be conjugate to a in G.

Thompson's transfer theorem [10, 5.38] gives $d \sim z_2$ or $d \sim z_1$. Suppose that z_1 is not conjugate to z_2 in G. Application of (2.5) yields that $z_1 \sim d$ in G. Clearly, z_2 is not conjugate to a in G by (2.1). Thus, in G there are precisely three classes of involutions; representatives of these classes are z_1, z_2, a. There is an element x in $\mathbf{C}(d)$ normalizing $\langle z_1, z_2, d\rangle$ such that $x^2 \in \langle z_1, z_2, d\rangle$ and $x \notin \langle z_1, z_2, d\rangle$. Note that all elements of $\langle z_1, z_2\rangle d$ are conjugate, that $z_1 \sim d$, and that $z_2 \sim z_1z_2 \nsim z_1$. It follows that x centralizes z_1; this, however, is not possible. The contradiction shows that $z_1 \sim z_2 \sim d \nsim a$. The lemma is proved.

(2.7) LEMMA. $\mathbf{C}(a) = \langle a\rangle \times L$, *where* $L \cong \Sigma_4$ *or* $L \cong \Sigma_5$. *The group* $\langle z_1, z_2\rangle\langle \gamma\rangle$ *is isomorphic to* A_4 *and lies in* L'; *here,* γ *is an element of order* 3 *of* $N = \mathbf{N}(\langle z_1, z_2\rangle)$.

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PROOF. We have $\mathbf{C}_H(z_2) = E\langle a, d\rangle$ and $\mathbf{Z}(E(\langle a, d\rangle)) = \langle z_1, z_2\rangle$. As $z_1 \sim z_2 \sim z_1 z_2$, we get that $N = \mathbf{N}(\langle z_1, z_2'\rangle)$ has order $2^6 \cdot 3$. Since $b \notin O_2(N)$, we get from the result of Baer-Suzuki that there is an element k of order 3 in N which is inverted by b. We know that $\mathbf{N}(E) = H$, and this implies that E is not normal in N. Hence, $[k, z_3] \notin \langle z_1, z_2\rangle$. Obviously, $\langle z_1, z_2\rangle z_3$ cannot be mapped onto $\langle z_1, z_2\rangle a$ as all involutions of E are conjugate and a is not conjugate to z_1. Via b, the cosets Ed and Ez_3ad are conjugate; so, all involutions of Ed and Ead are conjugate to z_1 in G. It follows that $\langle z_1, z_2\rangle a$ is kept fixed under the action of k in N. Therefore, k centralizes a conjugate of a in $\langle z_1, z_2\rangle a$. Clearly, $\langle z_1\rangle$ is a S_2-subgroup of $\mathbf{C}(c)$. If $\langle k\rangle$ was conjugate to $\langle c\rangle$ in G, then we would get $a \sim z_1$, which is impossible. We have shown that in G there are at least two classes of elements of order 3 and that 3 divides $|\mathbf{C}(a)|$; note that $c \sim c^{-1}$ in H.

Denote by γ an element of order 3 of N which centralizes a; there is such an element as all elements of $\langle z_1, z_2\rangle a$ are conjugate by the action of $\langle z_3, d\rangle$. Let $H_2 = \mathbf{C}(a)$. We have that $H_2 \supseteq \langle a, z_1, z_2, \gamma, b\rangle$ and that a S_2-subgroup of H_2 is of type $Z_2 \times D_8$. Since $\gamma \in N$, we get $\langle z_1, z_2\rangle\langle\gamma\rangle \cong A_4$. We know from (2.1) and (2.3) that $H_2 = \langle a\rangle \times L$ and that a S_2-subgroup of L is dihedral of order 8. Moreover, L possesses a subgroup L_1 of index 2. Obviously, $\langle z_1, z_2\rangle\langle\gamma\rangle \subseteq L_1$. Note that if $x \in \langle z_1, z_2\rangle^{\#}$ then $\mathbf{C}(x) \cap L_1 = \langle z_1, z_2\rangle$.

Let us assume that $\langle z_1, z_2\rangle\langle\gamma\rangle \subset L_1$. As $O(L_1)$ is characteristic in L_1 and hence normal in H_2, we get $O(L_1) = \langle 1\rangle$. Since $\mathbf{C}(z_1) \cap L_1 = \langle z_1, z_2\rangle$, there is no normal 2-subgroup of L_1 different from $\langle 1\rangle$. Denote by K a minimal normal subgroup of L_1. Then K has even order and is a simple group. It follows from [9, p. 129] that $K \cong A_5$. Thus, as K is normal in L_1, we get $L_1 = K$. It follows $L \cong \Sigma_5$. Clearly, if $L_1 \cong A_4$ then $L \cong \Sigma_4$. The lemma is proved.

In what follows we shall make use of J. G. Thompson's order formula [9, 5.1.7]. Thus, if x is an involution of G, we denote by $a(x)$ the number of pairs (u, v) such that $u \sim z_1$, $v \sim a$, and $x \in \langle uv\rangle$.

(2.8) LEMMA. *The integer* $a(z_1)$ *is equal to* 240.

PROOF. The roots of z_1 lie in the H-classes with the representatives z_1, z_3a,
z_2z_3a, bd, z_3bd, and z_1c.

Assume that $uv = z_1$. Then $u = z_1v$ and v is conjugate to a in H. But $z_1a \sim a$ in H and this shows that $uv = z_1$ is not possible.

Assume that $o(uv) = 4$. Then $uvuv = z_1$, and so $u^v = z_1 u$ which implies that $v \in \mathbf{N}_H(\langle z_1, u\rangle)\backslash\mathbf{C}(\langle z_1, u\rangle)$. First, we handle the case $u = z_2$. We have $\mathbf{C}(\langle z_1, z_2\rangle) = E\langle a, d\rangle$, and therefore, v is an involution of the coset $E\langle a, d\rangle b$. The relevant cosets with respect to E containing v are Eb, Eab. In $Ea \cup Eab$ there are 8 involutions conjugate to a. Thus, if $u = z_2$, there are 8 possibilities for v. But z_2 has precisely 6 conjugates in H, and therefore we get that there are $8 \cdot 6 = 48$ pairs (u, v) such that $o(uv) = 4$ and $u \sim z_2$ in H. Now assume that $u = d$. Thus $v \in \langle z_1, z_2, d\rangle z_3 a$. This coset contains precisely four involutions; these involutions form $\langle z_1, z_2\rangle d z_3 a$; but all these involutions are conjugate to z_1 in G and we cannot find such a v. Thus, if $o(uv) = 4$, the number of pairs (u, v) such that $z_1 \in \langle u, v\rangle$ is equal to 48.

Assume that $o(uv) = 6$. There is only one class of elements of order 6 in H; a representative for the class is $z_1 c$. We have $\mathbf{C}_H(z_1 c) = \langle z_1\rangle \times \langle c\rangle$. Thus $\mathbf{N}_H(\langle z_1 c\rangle) = \langle z_1\rangle \times \langle c, d\rangle$. Let $uv = z_1 c$. Then $v \sim a$ in H and v inverts c. This is not possible as all involutions of $\langle z_1, d\rangle$ are conjugate to z_1 in G. We have shown that $o(uv) = 6$ is not possible.

Finally, we handle the case $o(uv) = 8$. In H there are exactly two classes of elements of order 8; they are represented by bd and $z_3 bd$. The elements of $\langle bd\rangle$ are $1, bd, z_2z_3a, z_2z_3abd, z_1, z_1bd, z_1z_2z_3a, z_1z_2z_3abd$. The involutions inverting bd lie all in $\langle bd\rangle d$ and are the following elements: $d, b, z_2z_3ad, z_2z_3ab, z_1d, z_1b, z_1z_2z_3ad, z_1z_2z_3ab$. Let $uv = bd$. Then $ubd = v$. We see that there are precisely four pairs (u, v) such that $uv = bd$. Thus there are precisely $4 \cdot 24 = 96$ pairs (u, v) such that uv lies in $ccl_H(bd)$. Let $uv = z_3bd$. The elements of $\langle z_3bd\rangle$ are $1, z_3bd, z_1z_3a, abd, z_1, z_1z_3bd, z_3a, z_1abd$. The involutions inverting z_3bd are $d, z_3b, z_1z_3ad, ab, z_1d, z_1z_3b, z_3ad, z_1ab$. We have $uz_3bd = v$, and it is easy to compute that there are precisely four pairs (u, v) such that $uv = z_3bd$. Hence, there are precisely $4 \cdot 24 = 96$ pairs (u, v) such that $uv \in ccl_H(z_3bd)$. It follows that $a(z_1) = 48 + 96 + 96 = 240$.

(2.9) Lemma. *If $L \cong \Sigma_4$ then $a(a) = 3$.*

Proof. From (2.7) we get $\mathbf{C}(a) = \langle a, z_1, z_2, \gamma, b\rangle$ and $\langle z_1, z_2\rangle\langle\gamma\rangle \cong A_4$. The conjugacy classes of $\mathbf{C}(a)$ are listed in Table II.

Put $H_2 = \mathbf{C}(a)$. Roots of the involution a are in the H_2-classes with the representatives a and $a\gamma$.

All involutions of H_2 conjugate to z_1 in G are conjugate to z_1 and H_2. It follows that there are precisely three pairs (u, v) such that $uv = a$.

Assume that $uv = a\gamma$. Clearly, u and v both invert γ, and $\mathbf{N}(\langle\gamma\rangle)\cap H_2 = (\langle a\rangle \times \langle\gamma\rangle)\langle b'\rangle$, where b' is an involution conjugate to a and G and inverting

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TABLE II

x	$o(x)$	$\mathbf{C}_{\mathbf{C}(a)}(x)$	$\lvert ccl_{\mathbf{C}(a)}(x)\rvert$
1	1	$\langle a, z_1, z_2, \gamma, b\rangle$	1
a	2	$\Vert$	1
z_1	2	$\langle a, z_1, z_2, b\rangle$	3
az_1	2	$\Vert$	3
γ	3	$\langle a\rangle \times \langle \gamma\rangle$	8
$a\gamma$	6	$\Vert$	8
b	2	$\langle a, z_1, b\rangle$	6
ab	2	$\Vert$	6
bz_2	4	$\langle a, bz_2\rangle$	6
abz_2	4	$\Vert$	6
			48

γ. The involutions in $\langle z_1, z_2\rangle$ cannot invert γ. Thus, there is no pair (u, v) such that $uv = a\gamma$. we have proved that $a(a) = 3$.

(2.10) LEMMA. *If* $L \cong \Sigma_4$, *then* $|G| = 2^6 \cdot 3^3 \cdot 7$.

PROOF. We apply Thompson's order-formula and compute

$$\begin{aligned}|G| &= |\mathbf{C}(z_1)| \cdot a(a) + |\mathbf{C}(a)| \cdot a(z_1)\\ &= 192 \cdot 3 + 48 \cdot 240 = 12,096.\end{aligned}$$

(2.11) LEMMA. *If* $L \cong \Sigma_5$ *then* $a(a) = 195$.

PROOF. As we have remarked in (2.7), the group $\langle z_1, z_2\rangle\langle\gamma\rangle$ is isomorphic to A_4 and lies in L' which is isomorphic to A_5. We know that $\langle a, z_1, z_2, b\rangle \in \mathrm{Syl}_2(\mathbf{C}(a))$. The element γ of order 3 is centralized by an involution b' of $L\backslash L'$, and it is clear that $b' \sim a$ in G. Denote by w an element of order 5 of L. The conjugacy classes of $\mathbf{C}(a)$ are listed in Table III.

Put $H_2 = \mathbf{C}(a)$. The roots of a lie in the H_2-classes with the representatives $a, a\gamma, aw$.

Assume that $uv = a$. Then u runs through the 15 elements of $ccl_{\mathbf{C}(a)}(z_1)$. We have $z_1 a \sim a$ in H. Thus, there are precisely 15 pairs (u, v) such that $o(uv) = 2$ and $a \in \langle uv\rangle$.

Assume that $uv = a\gamma$. Clearly, u and v both invert γ. We have $\mathbf{C}(\gamma) \cap H_2 = \langle a, b', \gamma\rangle$, and there is an involution z conjugate to z_1 in L' which inverts γ and centralizes $\langle a, b'\rangle$. Thus, $\mathbf{N}(\langle\gamma\rangle) \cap H_2 = (\langle a, b'\rangle \times \langle\gamma\rangle)\langle z\rangle$.

TABLE III

x	$o(x)$	$\mathbf{C}_{\mathbf{C}(a)}(x)$	$\lvert ccl_{\mathbf{C}(a)}(x)\rvert$
1	1	$\langle a\rangle \times L$	1
a	2	$\lvert\lvert$	1
z_1	2	$\langle a, z_1, z_2, b\rangle$	15
γ	3	$\langle a, b', \gamma\rangle$	20
w	5	$\langle a, w\rangle$	24
$b'\gamma$	6	$\langle a, b', \gamma\rangle$	20
$b' \sim a$	2	$\langle a, z_1', b', \gamma\rangle, z_1' \sim z_1$	10
z_2b	4	$\langle a, z_2b\rangle$	30
az_1	2	$\langle a, z_1, z_2, b\rangle$	15
$a\gamma$	6	$\langle a, b', \gamma\rangle$	20
aw	10	$\langle a, w\rangle$	24
$ab'\gamma$	6	$\langle a, b', \gamma\rangle$	20
ab	2	$\langle a, z_1, b, \gamma'\rangle, \gamma' \sim \gamma$	10
az_2b	4	$\langle a, z_2b\rangle$	30
			240

There are precisely 12 involutions in $\mathbf{N}(\langle\gamma\rangle) \cap H_2$ which invert γ but only three of which are conjugate to z_1 in G. These are $z, \gamma z, \gamma^{-1}z$. We have $z\cdot a\gamma^{-1}z = a\gamma$, $\gamma z\cdot az = a\gamma$, $\gamma^{-1}z\cdot a\gamma z = \gamma^{-1}a\gamma^{-1} = a\gamma$. It follows that there are precisely $3\cdot 20 = 60$ pairs (u, v) such that $o(uv) = 6$ and $a \in \langle uv\rangle$.

Assume finally that $uv = aw$; thus, $o(uv) = 10$. Clearly, u and v both invert w. We have $\mathbf{C}(w) \cap H_2 = \langle a, w\rangle$. Thus, $u, v \in \mathbf{C}^*(w) \cap H_2 = (\langle a\rangle \times \langle w\rangle)\langle z\rangle$, where $z \in L'$ and $zwz = w^{-1}$. As $z \in L'$, we have $z \sim z_1$ in L'. The involutions of $\mathbf{C}^*(w) \cap H_2$ conjugate to z_1 in G are precisely the five elements in $\langle w\rangle z$. Clearly, $w^iz\cdot x = aw$ has the solution $x = z^{-1}w^{-i}aw$ and $x \sim a$ in G. It follows that there are precisely $5\cdot 24 = 120$ pairs (u, v) such that $o(uv) = 10$. We conclude $a(a) = 15+60+120 = 195$. The lemma is proved.

(2.12) LEMMA. *If* $L \cong \Sigma_5$ *then* $|G| = |M_{12}|$.

PROOF. Compute

$$|\mathbf{C}(z_1)|\cdot a(a) + |\mathbf{C}(a)|\cdot a(z_1) = 192\cdot 195 + 240\cdot 240$$
$$= 95,040 = 2^6\cdot 3^3\cdot 5\cdot 11.$$

3. The case $L \cong \Sigma_4$

We shall show that the group G with the title property does not exist.

Remember that G has at least two classes of elements of order 3, namely those represented by c and γ. Since $\langle z_1 \rangle$ is a S_2-subgroup of $\mathbf{C}(c)$, and $\langle a \rangle$ is one of $\mathbf{C}(\gamma)$, we see that both $\mathbf{C}(c)$ and $\mathbf{C}(\gamma)$ have normal 2-complements. Note that $\langle z_1, d \rangle$ is a S_2-subgroup of $\mathbf{N}(\langle c \rangle)$ and $\langle a, b' \rangle$ is a S_2-subgroup of $\mathbf{N}(\langle \gamma \rangle)$.

(3.1) LEMMA. *We have $\mathbf{C}(c) = K_c\langle z_1 \rangle$, where K_c is a normal subgroup of $\mathbf{C}(c)$ of order 3^2 or 3^3, and $\mathbf{N}(\langle c \rangle) = K_c\langle z_1, d \rangle$. Also, $\mathbf{C}(\gamma) = K_\gamma\langle a \rangle$, where K_γ is a normal subgroup of $\mathbf{C}(\gamma)$ of order 3^2 or 3^3, and $\mathbf{N}(\langle \gamma \rangle) = K_\gamma\langle a, b' \rangle$.*

PROOF. By a transfer result [3, 7.4.5] both $\mathbf{C}(c)$ and $\mathbf{C}(\gamma)$ have normal 2-complements. These normal 2-complements are normalized by four-groups. Consider for instance $\mathbf{C}(c)$. Put $K = K_c$. Then $K \trianglelefteq K\langle z_1, d \rangle$. The Frattini argument together with [9, 5.1.9] yields that K is a 3-group. As 3^3 divides $|G|$, we get that $|K| \in \{3^2, 3^3\}$.

(3.2) LEMMA. *Denote by T a S_3-subgroup and by S a S_7-subgroup of G. Then $|G: \mathbf{N}(S)| \in \{64, 288\}$ and $|G: \mathbf{N}(T)| \in \{112, 448\}$. If $|G: \mathbf{N}(S)| = 288$, then $\mathbf{C}(S) = S$. A S_3-subgroup of G is not cyclic.*

PROOF. From the order of G we get that G has no proper subgroup of index smaller than 9. Thus, from Sylow's theorem, we get $|G: \mathbf{N}(S)| \in \{36, 64, 288\}$ and $|G: \mathbf{N}(T)| \in \{16, 28, 64, 112, 488\}$. If $|G: \mathbf{N}(S)| = 36$, then $|\mathbf{N}(S)| = 2^4 \cdot 3 \cdot 7$; but then an involution would centralize an element of order 7 which is not the case.

Clearly, T is not cyclic, since $\langle c \rangle$ is not conjugate to $\langle \gamma \rangle$ in G. If we had $|G: \mathbf{N}(T)| \in \{16, 64\}$, then an element of order 7 in G would centralize a Sylow 3-subgroup which contradicts (3.1).

Suppose that $|G: \mathbf{N}(T)| = 28$. Then $|\mathbf{N}(T)| = 2^4 \cdot 3^3$. Assume that $T' \neq \langle 1 \rangle$. Then $|T'| = 3$ and an element of order 3 of G is centralized by a group of order 8 which is not possible. Thus $T' = \langle 1 \rangle$. If T was of type (3, 9), then T had a characteristic subgroup of order 3, and again we get a contradiction to the structures of centralizers of involutions in G. It follows that T is elementary abelian. From (3.1) we get $|\mathbf{C}(c)| = |\mathbf{C}(\gamma)| = 2 \cdot 3^3$. Therefore, in $\mathbf{N}(T)$ there are 8 conjugates of c and 8 conjugates of γ. By a lemma of Burnside [3, 7.1.1], there is an element x of order 3 in T which is not conjugate to c and not to γ in G. Thus, x is not centralized by an involution. It follows that x has 16 conjugates in $\mathbf{N}(T)$; but $8+8+16+1 > |T| = 27$. We have obtained a contradiction also in this case.

Assume finally that $|G : \mathbf{N}(S)| = 288$. Then $|\mathbf{N}(S)| = 2\cdot 3\cdot 7$. Let x be an element of order 3 in $\mathbf{C}(S)$. Then, the order of $\mathbf{C}(x)$ is either $3^2\cdot 7$ or $3^3\cdot 7$; note, that by (3.1), the element x cannot be centralized by an involution. This contradicts the order of $\mathbf{N}(S)$. The lemma is proved.

We shall now rule out all four cases of Lemma (3.2).

Case 1. Here we have $|\mathbf{N}(S)| = 3^3\cdot 7$ and $|\mathbf{N}(T)| = 2^2\cdot 3^3$. By assumption, an element of order 7 is centralized by a group of order 9. If $T' \neq \langle 1\rangle$, then T' is centralized by an involution and by an element of order 7 which is against (3.1). Thus T is abelian. Note that a S_2-subgroup of $\mathbf{N}(T)$ has order 4. Application of (3.1) yields that a S_2-subgroup of $\mathbf{N}(T)$ is conjugate in G to $\langle z_1, d\rangle$ and to $\langle a, b'\rangle$. But this contradicts the fact that $z_1 \sim d \sim z_1 d$ and a is not conjugate to z_1 in G. Case 1 is ruled out.

Case 2. Here we have $|\mathbf{N}(S)| = 3^3\cdot 7$ and $|\mathbf{N}(T)| = 3^3$. From a transfer result of Burnside we get $T' \neq \langle 1\rangle$. By assumption, an element of order 7 centralizes a subgroup of order 9. Thus, we may assume that $\mathbf{C}(S) = S\times R$, where R has order 9 and $R\subset T$. Evidently, $T'\subset R\subset T$. Put $S = \langle\sigma\rangle$ and $T' = \langle\xi\rangle$. As $\mathbf{N}(T) = T$, we see that ξ is not conjugate to its inverse; clearly, ξ is not centralized by an involution. Thus, $|\mathbf{C}(\xi)| = 3^3\cdot 7$. There are six G-conjugates of c and six G-conjugates of γ in T. Therefore, T is generated by elements of order 3. From [3, 5.3.9] it follows that T has exponent 3. Since $|\mathrm{Aut}_G(S)| = 3$, we see that σ is not conjugate to σ^{-1} in G. Let x, y be in $R^{\#}$ and $x\neq y$. Then, $x\sigma$ and $y\sigma$ have order 21 and are not conjugate in G, since such a conjugation would be performed in $\mathbf{C}(\sigma)$ which is abelian. Also, $x\sigma$, $x\sigma^{-1}$, and $y\sigma^{-1}$ lie in three pairwise different G-classes as $|\mathrm{Aut}_G(S)| = 3$. It follows that in $S\times R$ there are representatives for 16 G-classes of elements of order 21. If $x\in R^{\#}$, then $|\mathbf{C}(x\sigma)| = |\mathbf{C}(x\sigma^{-1})| = 3^2\cdot 7$. Our assumptions imply that $|\mathbf{C}(c)| = |\mathbf{C}(\gamma)| = 2\cdot 3^2$. As the centralizers of roots of involutions are known, we may write down the conjugacy classes of G discussed so far, and we see that G has at least 13,056 elements. Since $|G| = 12,096$, we have shown that Case 2 does not occur.

Case 3. Here we have $|\mathbf{N}(S)| = 2\cdot 3\cdot 7$ and $|\mathbf{N}(T)| = 2^2\cdot 3^3$. If T were abelian then we would get from (3.1) that $\langle z_1, d\rangle$ and $\langle a, b'\rangle$ are conjugate in G which, however, is not the case. Therefore, $T'\neq\langle 1\rangle$ and T' is centralized by an involution. Since $T'\sim\langle c\rangle$ or $T'\sim\langle\gamma\rangle$, there is a four-subgroup V in $\mathbf{N}(T)$. Acting with V on appropriate V-admissible sections of T/T', we see that $T = \Omega_1(T)$. Application of [3, 5.3.9] yields that $\exp(T) = 3$.

Put $\mathbf{N}(T) = T\langle\alpha, \beta\rangle$, where $\langle\alpha, \beta\rangle$ is a four-subgroup of G. Then, all involutions of $\langle\alpha, \beta\rangle$ are conjugate in G but fall into three $\mathbf{N}(T)$-classes. We know that $|\mathbf{C}_T(\alpha)| = |\mathbf{C}_T(\beta)| = |\mathbf{C}_T(\alpha\beta)| = 3$. Without loss of generality we may set $T' = \mathbf{C}_T(\alpha)$. The groups $\mathbf{C}_T(\beta)$ and $\mathbf{C}_T(\alpha\beta)$ are not conjugate in $\mathbf{N}(T)$, since $\beta \nsim \alpha\beta$ holds in $\mathbf{N}(T)$. It follows that a generator for $\mathbf{C}_T(\beta)$ has precisely six conjugates in $\mathbf{N}(T)$; the same is true for a generator for $\mathbf{C}_T(\alpha\beta)$. So far, we have got 14 elements of order 3 in T which are conjugate to an element of order 3 in $\mathbf{C}_G(\alpha)$. There is an element of order 3 in T which is centralized by an involution which is not conjugate to α. Such an element has precisely 12 conjugates in $\mathbf{N}(T)$. It follows that G has precisely two classes of elements of order 3. In particular, either c or γ is 3-central in G.

Writing down the complete table of the conjugacy classes of G, we get $|G| = 10,752$. Therefore, Case 3 is ruled out.

Case 4. Here we have $|\mathbf{N}(S)| == 2 \cdot 3 \cdot 7$ and $\mathbf{N}(T) = T$. By a result of Burnside it is clear that $T' \neq \langle 1\rangle$. Hence $T' = \mathbf{Z}(T)$ has order 3. We have shown above that $|\mathbf{C}(S)| = 7$. Clearly, T' is not centralized by an involution and $|\mathbf{C}(T')| = 3^3$.

Consider $\mathbf{N}(\langle c\rangle)$. This group has order $2^2 \cdot 3^2$ and $\langle z_1, d\rangle$ as a S_2-subgroup with $z_1 \sim d \sim z_1 d$ in G and $dcd = c^{-1}$. From (3.1) and the order of H we get that the S_3-subgroup of $\mathbf{N}(\langle c\rangle)$ is elementary abelian. Let $\langle t, c\rangle$ be the subgroup of order 9 of $\mathbf{N}(\langle c\rangle)$. We may assume that t is 3-central in G. Thus $\langle t\rangle$ is not normalized by a 2-subgroup of G different from $\langle 1\rangle$. In particular, $t \nsim t^{-1}$ in G. It follows that t and t^{-1} each have four conjugates in $\mathbf{N}(\langle c\rangle)$. But c is not 3-central and has precisely two conjugates under the action of $\mathbf{N}(\langle c\rangle)$. Since $1 + 2 + 4 + 4 = 11 > 9$, we have obtained a contradiction which shows that Case 4 does not occur.

Summarizing we get

(3.3) Lemma. *The case* $L \cong \Sigma_4$ *does not occur.*

4. The case $L \cong \Sigma_5$

From (3.3), (2.7), and (2.12) we conclude that $L \cong \Sigma_5$ and that $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$.

It is our aim to determine the structures of all Sylow normalizers and to write down the uniquelydetermined table of the conjugacy classes for G.

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Further, we are interested in the normalizers of certain elementary subgroups of order 9 of G.

(4.1) LEMMA. *We have* $\mathbf{C}(c) = K_c\langle z_1\rangle$, *where* K_c *is a normal subgroup of* $\mathbf{C}(c)$ *of order* 3^2 *or* 3^3, *and* $\mathbf{N}(\langle c\rangle) = K_c\langle z_1, d\rangle$. *The four-group* $\langle a, b'\rangle$ *is a* S_2*-subgroup of* $\mathbf{C}(\gamma)$.

PROOF. Clearly $\langle z_1\rangle \in \mathrm{Syl}(\mathbf{C}(c))$. Thus $\mathbf{C}(c)$ has a normal 2-complement K_c. Now, $\langle z_1, d\rangle$ acts on K_c and all involutions of $\langle z_1, d\rangle$ are conjugate in G. Thus, the first assertion follows from [9, 5.1.9] and the Frattini argument. As for the second assertion, note that all involutions of $\langle a, b'\rangle$ are conjugate in G.

(4.2) LEMMA. *A Sylow 5-normalizer of* G *is contained in* $\mathbf{C}(a)$.

PROOF. Denote by w an element of order 5 of $\mathbf{C}(a)$. We have $\mathbf{C}(a) \cap \mathbf{C}(w) = \langle a\rangle \times \langle w\rangle$, and so $\langle a\rangle$ is a S_2-subgroup of $\mathbf{C}(w)$. Thus $\mathbf{C}(w)$ possesses a normal 2-complement K. There is an involution $z \in L'$ inverting w. Therefore, from the action of $\langle a, z\rangle$ on K and from [9, 5.1.9] together with the Frattini argument and (4.1) we get that $|K| \in \{5, 3\cdot 5, 3^2\cdot 5\}$. From the structure of $\mathbf{C}(a)$ follows that in G there is precisely one class of elements of order 5. Thus $\mathbf{N}(\langle w\rangle)/\mathbf{C}(w) \cong Z_4$. Since $|\mathbf{C}(w)| = 2\cdot|K|$, we get from Sylow's theorem that $\mathbf{C}(w) = \langle a\rangle \times \langle w\rangle$. The assertion follows.

(4.3) LEMMA. *A Sylow* 11*-normalizer of* G *is a Frobenius group of order* 55.

PROOF. Denote by e an element of order 11 in G. We know that $|\mathbf{C}(e)|$ is neither divisible by 2 nor by 3^3 as $|K_c| \in \{3^2, 3^3\}$. Also, there are no elements of order 55 in G. Thus $|\mathbf{C}(e)| \in \{11, 3\cdot 11, 3^2\cdot 11\}$. From a transfer result we get $|\mathrm{Aut}_G(\langle e\rangle)| \in \{2, 5, 10\}$. Application of Sylow's theorem yields $|\mathbf{N}(\langle e\rangle)| = 5\cdot 11$. The assertion follows.

(4.4) LEMMA. *A* S_3*-subgroup of* G *is nonabelian.*

PROOF. Assume by way of contradiction that a S_3-subgroup T of G is abelian. Since by (4.1) a four-group acts on T, we get that T is elementary abelian. From the above results we get $\mathbf{C}(T) = T$ and $\mathbf{N}(T) \supset T$. Since T cannot have automorphisms of order 5 or 11, we get that $\mathbf{N}(T)/T$ is a 2-group. From the order of $GL(3, 3)$ it follows that $|\mathbf{N}(T)/T| \leq 2^5$. Sylow's theorem yields $|\mathbf{N}(T)| \in \{2^4\cdot 3^3, 2^2\cdot 3^3\}$. A lemma of Burnside implies that any two G-conjugate elements in T are conjugate under the action of $\mathbf{N}(T)$. Let $x, y \in T$ such that $x \sim c$, $y \sim \gamma$ in G. Then $\mathbf{C}(x)$ has order $2 \cdot 3^3$ and lies in $\mathbf{N}(T)$. We consider first the case that $|\mathbf{N}(T)| = 2^4 \cdot 3^3$. Then x has precisely 8 conjugates in T. We have $|\mathbf{C}(y)| = 2^2 \cdot 3^3$, and so, y has 4, 8, or 16 G-conjugates in T. Note that $1 + 8 + 16 = 25 < 27$. In $T^{\#}$ there must be an element t which is not centralized by an involution. Thus t has 16 conjugates in $\mathbf{N}(T)$. But $1 + 8 + 4 + 16 = 29 > 27$ gives a contradiction. Finally, consider the case $|\mathbf{N}(T)| = 2^2 \cdot 3^3$. From (4.1) we get that a S_2-subgroup of $\mathbf{N}(T)$ is conjugate to $\langle z_1, d\rangle$ in G. A result of W. Gaschütz implies that $\mathbf{C}(\gamma) = \langle\gamma\rangle \times X$, where $|X| = 2^2 \cdot 3^2$; remember that $\langle a, b'\rangle$ is a S_2-subgroup of X. Since z_1 is not conjugate to a in G, we see that a S_3-subgroup of $\mathbf{C}(\gamma)$ is not normal in $\mathbf{C}(\gamma)$. If a minimal normal subgroup of X is a 2-group or a group of order 3, then an involution of X is centralized by a subgroup of order 9 of $\mathbf{C}(\gamma)$. This contradiction proves the lemma.

(4.5) Lemma. *A S_3-subgroup T of G is nonabelian of exponent* 3 *and* $|\mathbf{N}(T)| = 2^2 \cdot 3^3$.

Proof. We know that $\mathbf{C}(T) = T' = \mathbf{Z}(T)$ from (4.1). For the order of $|\mathbf{N}(T)|$ we get the following possibilities from Sylow's theorem: 3^3, $2^2 \cdot 3^3$, $2^4 \cdot 3^3$. The case $|\mathbf{N}(T)| = 2^4 \cdot 3^3$ is not possible as an element of order 3 is not centralized by a group of order 8.

Let us assume $|\mathbf{N}(T)| = 3^3$. Put $T' = \langle\xi\rangle$. Then, by a lemma of Burnside, ξ is not inverted in G, and so, ξ is neither conjugate to c nor to γ in G. Thus, in G there are at least four classes of elements of order 3.

To obtain a contradiction we write down the table for the conjugacy classes of G obtained so far, and we get at least 105,600 elements in G. Thus, the case that $|\mathbf{N}(T)| = 3^3$ is ruled out.

We are left with the case $|\mathbf{N}(T)| = 2^2 \cdot 3^3$. Since $T' \neq \langle 1\rangle$, we get $T' \sim \langle c\rangle$ or $T' \sim \langle\gamma\rangle$. From the structure of $\mathbf{C}(a)$ it follows that a S_2-subgroup of $\mathbf{N}(\langle\gamma\rangle)$ is elementary of order 8. Thus, a S_2-subgroup of $\mathbf{N}(T)$ is elementary abelian of order 4. From the orders of the centralizers of involutions it follows that T is generated by elements of order 3. Since the nilpotency class of T is 2, application of [3, 5.3.9] yields that T has exponent 3. The lemma is proved.

(4.6) Lemma. *The element c is* 3*-central.*

Proof. Assume that there is a S_3-subgroup T of G such that $T' = \langle\gamma\rangle$. Then $\mathbf{C}(\gamma) = T\langle a, b'\rangle$. Consider the factor group $X = T\langle a, b'\rangle/\langle\gamma\rangle$.

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From the orders of the centralizers of involutions and by the stabilizing-chain argument [3, 5.3.2], we get that no involution of X centralizes an element of order 3 of X. As this is not possible, we get that c is 3-central.

We are able to write down the uniquely determined table of the conjugacy classes of the simple group G of order 95,040.

TABLE IV

x	$o(x)$	$\lvert \mathbf{C}_G(x) \rvert$	$\lvert ccl_G(x) \rvert$
1	1	$95,040$	1
z_1	2	$2^6 \cdot 3$	495
a	2	$2^4 \cdot 3 \cdot 5$	396
$z_3 a$	4	2^5	2970
$z_2 z_3 a$	4	2^5	2970
bd	8	2^3	11880
$z_3 bd$	8	2^3	11880
c	3	$2 \cdot 3^3$	1760
$z_1 c$	6	$2 \cdot 3$	15840
γ	3	$2^2 \cdot 3^2$	2640
$a\gamma$	6	$2^2 \cdot 3$	7920
w	5	$2 \cdot 5$	9504
aw	10	$2 \cdot 5$	9504
e_1	11	11	8640
e_2	11	11	8640
			95,040

We see that Table IV is identical with the table for the Mathieu group M_{12}.

(4.7) LEMMA. *Let $T \in \mathrm{Syl}_3(G)$ such that $T' = \langle c \rangle$. Then, $\mathbf{N}(T) = T\langle z_1, d \rangle$ and $\mathbf{N}(T)$ contains elementary abelian subgroups M_1, M_2 such that $\mathbf{N}(M_i)$ is a splitting extension of M_i by $GL(2, 3)$ for $i = 1, 2$. Further, M_1 is not conjugate to M_2 in G. There are two elementary abelian subgroups of order 9 in T which are conjugate in $\mathbf{N}(T)$ and have only two 3-central elements each.*

PROOF. Clearly $\mathbf{N}(T) = T\langle z_1, d \rangle$. Every involution of $\langle z_1, d \rangle$ is conjugate to z_1 in G and centralizes a subgroup of order 3 of T; clearly $T = \mathbf{C}_T(z_1) \cdot \mathbf{C}_T(d) \cdot \mathbf{C}_T(z_1 d)$. Put $\mathbf{C}_T(d) = \langle r \rangle$, $\mathbf{C}_T(z_1 d) = \langle s \rangle$. Then

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$c \sim r \sim s$ holds in G. Put $M_1 = \langle c, r\rangle$ and $M_2 = \langle c, s\rangle$. Evidently, M_i is normal in $\mathbf{N}(T)$, and so, $M_i^\#$ consists only of 3-elements which are 3-central. In T there are 12 conjugates of γ, and this implies that $M_1 \cup M_2$ contains all the 14 3-central elements of T. Clearly, $\langle c, rs\rangle \sim \langle c, rs^{-1}\rangle$ via d and M_1 is not conjugate M_2 by a result of Burnside.

Since M_i possesses precisely four subgroups of order 3, we get that M_i is normalized by precisely four S_3-subgroups of G, one of which is T. Moreover, $\langle z_1, d\rangle$ normalizes M_i. Since $\mathbf{C}(M_i) = M_i$, we see that $\mathbf{N}(M_i)$ is a $(2, 3)$-group. As $|\mathbf{N}(M_i):\mathbf{N}(M_i)\cap \mathbf{N}(T)| = 4$, we get $|\mathbf{N}(M_i)| = 2^4 \cdot 3^3$. It follows that $\mathbf{N}(M_i)$ is a splitting extension of M_i by $GL(2, 3)$ as T has exponent 3; $i = 1, 2$.

5. The identification of G with M_{12}

In what follows we shall change our notation completely, because we are going to find generators and relations for G as given in [11, p. 421]. So, we shall use from now on only the structural information obtained for G so far.

There is an elementary abelian subgroup M of order 9 of G, the normalizer of which is a splitting extension of M of $GL(2, 3)$. Since $\mathbf{C}(M) = M$, we get that $\mathbf{N}(M)$ is uniquely determined.

Studying Todd's presentation for M_{12}, we see that we may put $\mathbf{N}(M) = \langle a^2c, aca\rangle\langle a, b, e, f\rangle$ so that the relations between the generators of $\mathbf{N}(M)$ are those of Todd. Then, we have $M = \langle a^2c, aca\rangle$ and $\langle a, b, e, f\rangle \cong GL(2, 3)$.

Clearly, $\langle a, b, e, f\rangle$ lies in $\mathbf{C}(a^2) = H$ and a^2 is a 2-central involution. Since ef is an element of order 3 in H, it follows $f \in H\backslash O_2(H)$. We know that $\langle a, b\rangle$ is a normal quaternion subgroup of H. We may thus add a generator d for H so that Todd's relations hold between the generators of H. We get $H = \langle a, b, d, e, f\rangle$ and $\langle d, e, f\rangle \cong \Sigma_4$.

Thus, to prove $G \cong M_{12}$, it suffices to show that $(cd)^3 = 1$.

Consider the diagram

$$\underset{a^2}{\circ} \!-\!\!-\!\!-\! \underset{c}{\circ} \overset{m}{-\!\!-\!\!-} \underset{d}{\circ} \!-\!\!-\!\!-\! \underset{e}{\circ} \!-\!\!-\!\!-\! \underset{f}{\circ}\,.$$

All relations represented by the diagram are known except $(cd)^m = 1$. It is easy to see that all involutions occurring in the diagram are 2-central in G.

We have $\langle a^2, c, d\rangle \subseteq \mathbf{C}(f) = H_f$. Since $o(a^2c) = 3$, we get that both a^2 and c are contained in $H_f\backslash O_{2,3}(H_f)$. The table of conjugacy classes of H shows that all involutions of $H\backslash O_2(H)$ are conjugate in H. It follows that

$\mathbf{C}(a^2) \cap H_f = \langle f, x_2, a^2 \rangle$ is elementary abelian of order 8, where $\langle f, x_2, x_3 \rangle$ is the normal elementary abelian subgroup of order 8 of H_f.

Now d lies in $\langle f, x_2, a^2 \rangle$. Assume first that $d \in \langle f, x_2 \rangle$. Then $cd \in \langle f, x_2, x_3 \rangle c$, and so the order of cd is 2 or 4.

Assume next that $d \in \langle f, x_2, a^2 \rangle \backslash \langle f, x_2 \rangle$. Then, $d = a^2 x$ with $x \in \langle f, x_2 \rangle$. It follows $cd \in ca^2 \langle f, x_2, x_3 \rangle$, and as $o(ca^2) = o(a^2 c) = 3$, we get that $o(cd) \in \{3, 6\}$. If $o(cd) = 6$, then the structure of H shows that $(cd)^3 = f$.

We have thus obtained the following possibilities for m: $m \in \{2, 4, 3, 6\}$.

First we shall eliminate the possibility $m = 2$. This is easy as the assumption $o(cd) = 2$ implies that in G the subgroup $\langle d, e, f \rangle$ which is isomorphic to Σ_4 is centralized by the element $a^2 c$ of order 3. But this contradicts the results of Table IV.

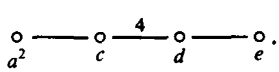
Next, we shall rule out the case $m = 6$. Assume that $o(cd) = 6$. Put $A = \langle c, d, e, f \rangle$. The generators of A respect the diagram

plus the additional relation $(cd)^3 f = 1$. Compute

$$(cd)^{edec} = c^{edec} d^{edec} = c^{dec} e = cedcdece = ecdcdc = efd.$$

It follows that the element efd of order 6 lies in the subgroup $\langle d, e, f \rangle$ of G which is isomorphic to Σ_4. This is a contradiction which shows that the case $m = 6$ is not possible.

Finally, we treat the case $o(cd) = 4$. Put $A = \langle a^2, c, d, e \rangle$ and $Y = \langle a^2, c, d \rangle$. The generators of A respect the diagram



Therefore, A is an epimorphic image of the Coxeter group F_4 (see, for example, [2, Table 10]) and so, $|A|$ divides $2^6 \cdot 3^2$. We know that Y has order divisible by $2^3 \cdot 3$ and that Y is an epimorphic image of $Z_2 \times \Sigma_4$; see [2, Table 10]. Thus, Y is isomorphic to Σ_4 or to $Z_2 \times \Sigma_4$. The case $|A| = 24$ is not possible as then $Y = A$. But in Σ_4 there is no element of order 6. If $|A| = 2^4 \cdot 3$, then $A \cong Z_2 \times \Sigma_4$ which would imply $e \in \mathbf{Z}(A)$ as e centralizes the element $a^2 c$ of order 3 of A. Thus, if 3^2 does not divide $|A|$, then $|A| = 2^6 \cdot 3$ as $|\mathbf{N}_A(\langle a^2 c \rangle)| = 2^2 \cdot 3$; note that $a^2 c$ is 3-central in G. If $|A| = 2^6 \cdot 3$, then $|O_2(A)| = 2^5$, and the element $a^2 c$ centralizes the involution e which must lie in $O_2(A)$. This contradicts the fact that de has order 3.

Assume now that 3^2 divides $|A|$. Then $2^3 \cdot 3^2$ divides $|A|$ and since Σ_4 is present, we see that A is not 3-closed.

First suppose that $|A| = 2^3 \cdot 3^2$. A Sylow 2-subgroup of A is dihedral. If $O_3(A) = \langle 1 \rangle$, then $O_2(A)$ is a four-group which is centralized by a group of order 3, contradicting $O_3(A) = \langle 1 \rangle$. Thus, $O_3(A)$ has order 3. It follows that $A \cong (Z_3 \times A_4)Z_2$ with $A_4 Z_2 \cong \Sigma_4$. But then, a non-2-central involution of G would have a root of order 4 which is not the case.

We have proved that $2^4 \cdot 3^2$ divides $|A|$. Let X be a minimal normal subgroup of A. Then A is not a 3-group as A is not 3-closed and no element of order 3 of G is centralized by a subgroup of order 8. Hence, $|X| \in \{2, 2^2, 2^3\}$. From the structures of centralizers of involutions of G we get that X is a four-group. As X is centralized by an element of order 3 we get that all elements of $X^\#$ are non-2-central. Since a non-2-central involution of G has no roots of order 4, we see that all involultions of A centralize X. This implies that $\langle a^2, c \rangle$ centralizes X. But this is a contradiction, since $a^2 c$ is 3-central. The case $m = 4$ has been ruled out.

It remains $o(cd) = 3$. Thus G possesses elements a, b, c, d, e, f which satisfy the Todd relations for a presentation of the simple Mathieu group M_{12}. Since $|G| = |M_{12}|$, we get $G \cong M_{12}$, and we have reached our final goal.

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