THE NON-LINEAR THEORY OF SPIRAL STRUCTURE

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ABSTRACT

The main steps of the non-linear theory of spiral structure are described. Near each of the main resonances the basic periodic orbits are calculated, and the sets of non-periodic orbits that follow them are found. A different integral of motion is applicable for each set, besides the Jacobi integral. Then the initial distribution function, f, is expressed as a function of the two integrals and the corresponding angles. The final distribution function is found by averaging over the angles: $f_{fin} = \langle f \rangle$. Then by integrating $\langle f \rangle$ over all velocities we find the response density σ_{resp}. In order that σ_{resp} should be equal to the imposed density, σ_{imp} we must adjust the parameters of the imposed spiral field. The form of σ_{resp} away from resonances can be derived explicitely for tight and open spirals or bars; however near the resonances σ_{resp} can be only calculated numerically. If the imposed field has almost constant amplitude, then the amplitude of the response is very large near the Inner Lindblad Resonance. In the case of a tight spiral the azimuth of the response density maximum with respect to the imposed density maximum tends to zero outside the ILR, while it tends to -90° inside the ILR. One possible self-consistent solution has zero amplitude inside the ILR both in the case of tight spirals and of bars. Finally an important quadrupole term was found near the ILR.

1. INTRODUCTION

The most important non-linear effects in a galaxy are due to its main resonances, namely the Inner and Outer Lindblad Resonances (ILR and OLR) and the Particle Resonance (PR). Near these resonances the basic assumption of the linearized theory is not applicable. In fact, if we write the imposed potential in the form

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$$V = V_o + V_1 , \qquad (1)$$

where V_o is the axisymmetric part, V_1 the spiral part, and the corresponding distribution function is

$$f = f_o + f_1 , \qquad (2)$$

then near the main resonances we have $|f_1| > |f_o|$, and therefore f_1 cannot be considered as a small correction term. This is the opposite of what happens away from resonances. Thus a different approximation scheme has to be used near the resonances, as in other resonant problems of Celestial Mechanics and Stellar Dynamics (Whittaker 1904, Born 1927, Contopoulos 1963). Previous work on the non-linear theory of galactic resonances is contained in the papers of Contopoulos (1970, 1973, 1975 a,b) and Vandervoort (1973, 1975; see also Vandervoort and Monet 1975).

The existence and positions of the main galactic resonances depend on the form of the rotation curve (angular velocity Ω versus r) and on the value of the angular velocity of the spiral pattern Ω_s.

At the ILR and OLR we have

$$\Omega_s = \Omega \mp \kappa/2 , \qquad (3)$$

where κ is the epicyclic frequency, and at the PL

$$\Omega_s = \Omega . \qquad (4)$$

If the galaxy has a sharp increase of density inwards, with a point mass at its center, then the curve $\Omega - \kappa/2$ goes to infinity as $r \to 0$, therefore for any Ω_s we have an ILR (Fig. 1a). However, if the galaxy has a smoother increase of density inwards (e.g. an almost homogeneous nucleus at its center), then $\Omega - \kappa/2$ tends to zero as $r \to 0$, therefore we have, in general, either two ILR's, or no ILR at all (Fig. 1b).

On the other hand we have always a PR, and an OLR, although these resonances may be in the outermost parts of the galaxy.

II. THE DISTRIBUTION FUNCTION

The basic steps in the non-linear theory of spiral structure are the following:

1) Find the appropriate integrals of motion for each

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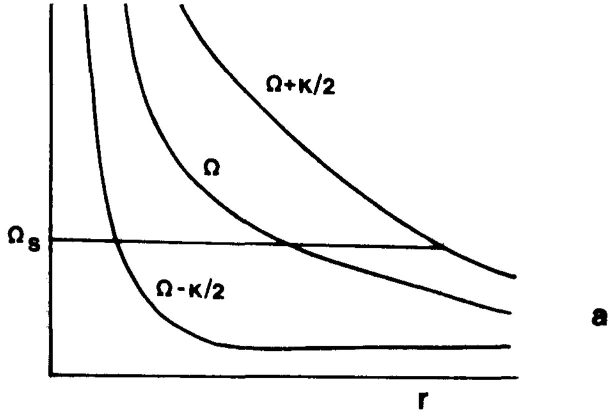

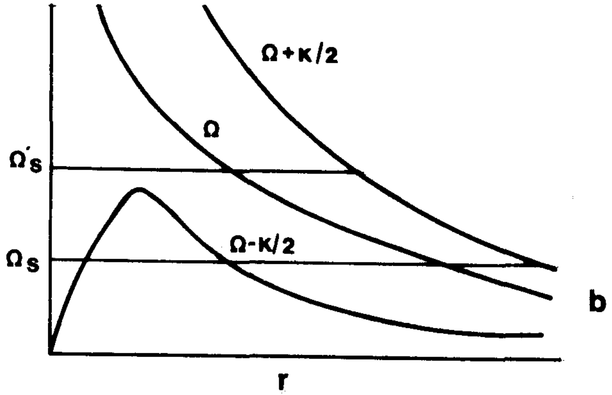

Fig. 1. The positions of the ILR, PR, and OLR are at the intersections of the curves $\Omega - \varkappa/2$, Ω, and $\Omega + \varkappa/2$ by the line Ω_s = const. The two possible forms of the curve $\Omega - \varkappa/2$ are described in the text.

resonant (and non-resonant) case.

2) Express the distribution function f in terms of these integrals.

3) Calculate the response density σ_{resp} by integrating f over all velocities, and

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4) Solve the self-consistency equation

$$\sigma_{resp} = \sigma_{imp}, \tag{5}$$

which, in fact, consists of two equations, one referring to the agreement of amplitudes and one to the agreement of phases.

The first step has already been studied by Contopoulos (1975, ordinary spirals) and Contopoulos and Mertzanides (1977, bars).

We have now a computer program that gives the basic periodic orbits, for every assumed axisymmetric and spiral model, near (and far from) each resonance, and the sets of non-periodic orbits following them.

Near the resonances the energy and angular momentum are not approximate integrals. On the other hand the Jacobi integral is always an exact integral of motion. Furthermore a new (resonant) integral was derived, which has a different form for each of the above sets of orbits.

The appropriate form of the distribution function is found as follows. The initial distribution function, f , is expressed as a function of the unperturbed integrals of motion, namely the energy and the angular momentum. Such is the case, e.g., with the Schwarzschild distribution.

This function is now expressed in terms of the integrals of motion valid in each case and the "corresponding angles". The definition of the "corresponding angles" θ_1 , θ_2 , is given in a forthcoming paper. The basic property of these angles is that they vary linearly in time. Therefore after a long time they phase-mix, and we can take the final distribution function as the average of the initial distribution function over the angles θ_1 , θ_2:

$$f_{fin} = \langle f \rangle . \tag{6}$$

The form of f_{fin} is different inside, close, or outside each resonance.

In Fig. 2a,b,c we give $\langle f \rangle$ near the ILR in a particular spiral field of the form (1). The form of $\langle f \rangle$ depends on the value of hamiltonian, H, or of the corresponding radius, r, of the unperturbed circular orbits. If the unperturbed circular orbit is inside, or a little outside the ILR, there is only one population of orbits around the periodic orbits x_1 (Fig. 2a). A little further outside the ILR a second

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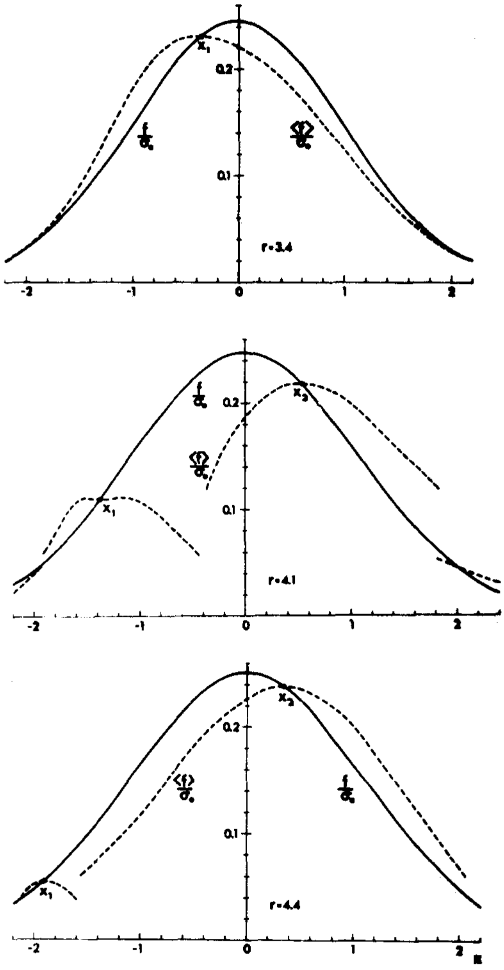

a

b

c

Fig. 2. The unperturbed and the perturbed distribution functions, f and $\langle f\rangle$ *(both divided by the unperturbed density* σ_0*) along the axis x. The corresponding circular unperturbed orbits are. a) r = 3.4 inside the ILR, b) r = 4.1 outside the ILR, c) r = 4.4, further outside the ILR.*

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population of orbits appears around the periodic orbit x_2 (Fig. 2b); in this case we speak of two populations of trapped orbits, while there is a third population of orbits, that are not trapped around either x_1 or x_2. For still larger r the second population becomes the dominant one (Fig. 2c). In such cases we can ignore the population around x_1, and consider the third population as a continuation of the second population.

The various forms of $f_{fin} = \langle f \rangle$ are given by a computer program, and are then used to calculate the response density σ_{resp}.

III. THE RESPONSE DENSITY

The integration of $\langle f \rangle$ over all velocities, to give the response density

$$\sigma_{resp} = \int \langle f \rangle \, d\bar{v}, \qquad (7)$$

must be done by taking into account the form of $\langle f \rangle$ appropriate for each population.

If the imposed spiral potential is of the form

$$V_1 = A \cos (2\theta - \Phi), \qquad (8)$$

where $A = A(r)$ is the amplitude, $\Phi = \Phi(r)$ a phase angle, and θ the azimuth in a frame rotating with angular velocity Ω_s, we write the response density in the form

$$\sigma_{resp} = \sigma_o - X_{resp} \cos(2\theta - \Phi - Z_{resp}) + Q_4, \qquad (9)$$

where Q_4 is a quadrupole term.

This is to be compared with the imposed density, which is written in a similar form

$$\sigma_{resp} = \sigma_o - X_{imp} \cos(2\theta - \Phi - Z_{imp}) + Q_4', \qquad (10)$$

where, usually, Z_{imp} is quite small.

Very near the ILR the forms of X_{resp} and Z_{resp} involve the calculation of complicated integrals that can be found only by means of a computer. However further away, inside or outside the ILR, where only one population of orbits is

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dominant, the integration can be performed explicitely and gives

$$\sigma_{resp} = \sigma_o + \sigma_o \left\{ \left[\frac{\Omega+\Omega_s}{r(\Omega-\Omega_s)} x_m - \frac{\sigma_o'}{\sigma_o} x_m - x_m' \right] \cos(2\theta-u) - -u' x_m \sin(2\theta-u) \right\}. \quad (11)$$

Here accents indicate derivatives with respect to r, x_m is the periodic orbit x_1 inside the resonance, or x_2 outside the resonance, and

$$u = \Phi + q_+ , \quad (12)$$

where q_+ is found if we analyze the potential (8) into components; namely the most important component near the ILR is proportional to $\cos(\theta_1 - 2\theta_2 + \Phi + q_+)$ (Contopoulos 1975, Appendix B).

The form (11) of the σ_{resp} is derived in a simple way

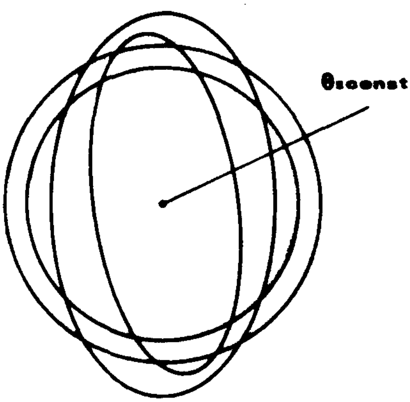

Fig. 3. The unperturbed circular periodic orbits are transformed into two almost elliptical periodic orbits after the spiral field is introduced.

in the Appendix by considering only the behaviour of the periodic orbits (Fig. 3).

The response density (11) is composed of various

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components, namely:

a) The term $\frac{\sigma_o(\Omega+\Omega_s)}{2(\Omega-\Omega_s)}x_m\cos(2\theta-u)$ is due to the fact that a star stays longer near apocentron than near pericentron inside the particle resonance (the opposite is true outside the PR).

b) The term $-\frac{\sigma_o'}{\sigma_o}x_m\cos(2\theta-u)$ is due to the fact that the unperturbed matter of the initial circular ring is moving outwards, where the local density is smaller, or inwards, where the local density is larger.

c) The term $-\sigma_o x_m'\cos(2\theta-u)$ is due to the crowding of periodic orbits, which is largest near the resonance.

d) The term $-\sigma_o u' x_m \sin(2\theta-u)$ is due to the differential rotation of the major axes of successive periodic orbits in the spiral field (Fig. 3).

If the imposed spiral field is tight (trailing) then the most important term is the last one. In fact then $q_+ \simeq -\pi/2$ and $q' \simeq 0$, thus

$$u' \simeq \Phi' = k, \qquad (13)$$

where kr is absolutely large. If we disregard then the first three terms of the response (11) we find that the spiral density maxima are 45° ahead of the major axes of the perturbed orbits in the trailing case. Thus outside the ILR (and inside the PR) the response density is in phase with the potential minima, which approximately coincide with the imposed density maxima. Inside the ILR, on the other hand, the response is completely out of phase, i.e. the response density maxima are near the imposed density minima. This means that there cannot be self-consistency unless the amplitude of the spiral wave goes to zero inside the ILR. These results are consistent with the results of the linearized theory of density waves in the asymptotic case (large k ; Lin, Yuan and Shu 1969).

The situation is different for open spirals or bars. In the case of a bar, if the imposed potential has an almost constant amplitude, the orbits between the ILR and the neighbourhood of the PR are mainly elongated along the bar, while they are elongated perpendicularly to the bar inside the ILR. However the density maxima are along the bar both inside and outside the ILR. One only sees that the response is very large near the ILR.

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A detailed discussion of the various cases will be given in a future paper.

The main conclusions of the present study are the following:

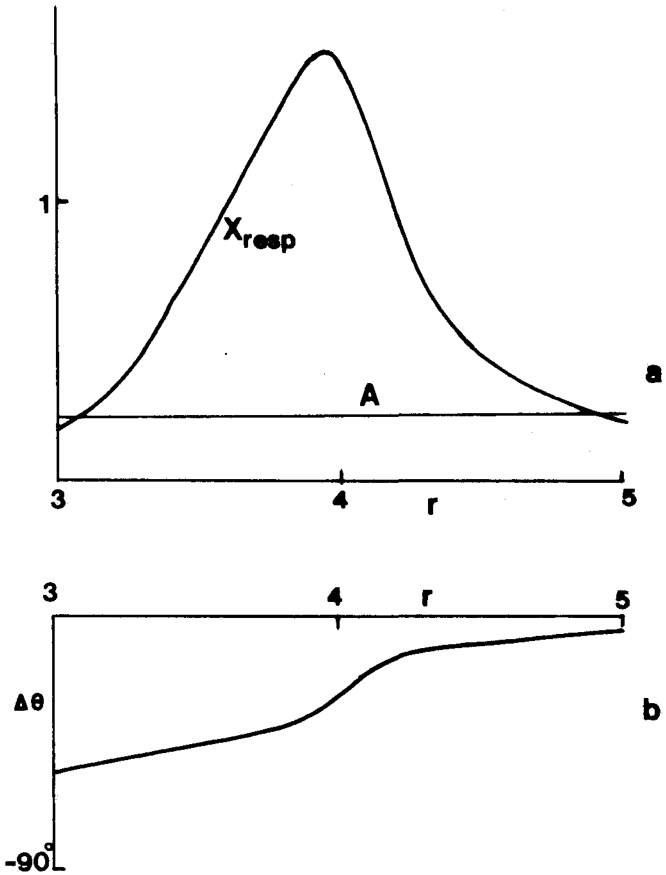

Fig. 4. The amplitude and the azimuth of the response density in the case of an imposed field of almost constant amplitude, A,*and wave number* $k \simeq -1.3$ *near the ILR* ($r \simeq 3.7$ kpc, *in this model*).

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1) The amplitude of the density response is very large near the ILR (Fig. 4a).

2) The azimuth difference $\Delta\theta$ between the response density maxima and the imposed density maxima tends to zero outside resonance (Fig. 4b). Near resonance $\Delta\theta$ becomes large negative and remains so inside the resonance. For tight spirals $\Delta\theta \rightarrow -90^{\circ}$ well inside resonance. Only in the case of bars $\Delta\theta$ is always zero (or -90°).

3) One may have a self-consistent solution with zero amplitude inside resonance. This is applicable both to the case of a tight spiral and to the case of an open spiral or a bar.

4) In the case of a bar there may be another solution if we assume the amplitude to increase considerably inwards. Such a solution would have a singularity at the center, unless there is a second ILR (Fig. 1b).

5) A global spiral solution should not have any singularities, either near the center or very far from the center. It is expected that these boundary conditions would be satisfied only for some values of the angular velocity of the spiral pattern Ω_s. These are the analogues of the eigenvalues of the linearized problem.

6) In the response (11) we have omitted the quandrupole term Q_4 of eq. (9). This term is particularly important in the region very near resonance, where the periodic orbits x_1 and x_2 deviate considerably from circles.

Such an important quadrupole term was found by Crane (1975) in the isophotes of the SB0 galaxy NGC 2950. The fact that the quadrupole term is maximum in the region where the ellipticity of the orbits is maximum indicates that it is, in fact, connected with the ILR and constitutes an independent check of the non-linear theory of density waves.

APPENDIX

We consider that the orbits between the circles r_c and $r_c + \Delta r_c$ in the axisymmetric field are transformed to the orbits between two neighbouring resonant periodic orbits (Fig. 3)

$$r = r_c + x_m \cos(2\theta - u), \qquad \text{(A1)}$$

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and

$$r+\Delta r= r_c+\Delta r_c\left[1+x_m' \cos(2\theta-u)+u x_m \sin(2\theta-u)\right], \qquad (A2)$$

in the spiral field.

Away from resonances the angular momentum is approximately conserved, hence

$$r^2(\theta'+\Omega_s) = r_c^2\Omega_c \ , \qquad (A3)$$

where θ' is the angular velocity in the rotating frame. The flux of matter through an axis θ =const. in the elliptical ring is the same as in the unperturbed (circular) ring, because the period is the same in first order approximation. Thus

$$\sigma r\theta'\Delta r = \sigma_c r_c \theta_c' \Delta r_c , \qquad (A4)$$

where $\theta'=r_c\Omega_c$, and we derive, in first approximation,

$$\sigma= \sigma_c \left[1+ \frac{(\Omega_c+\Omega_s)}{r_c(\Omega_c-\Omega_s)} x_m \cos(2\theta-u)-x_m' \cos(2\theta-u)- -u x_m \sin(2\theta-u)\right]. \qquad (A5)$$

If we compare the density σ with the unperturbed density σ_o at distance r (and not with the density σ_c at distance r_c) we must use also the relation

$$\sigma_o = \sigma_c+\sigma_c' x_m \cos(2\theta-u), \qquad (A6)$$

and derive finally

$$\sigma = \sigma_o \left\{1+ \left[\frac{(\Omega_c + \Omega_s)}{r_c(\Omega_c-\Omega_s)} x_m - \frac{\sigma_o'}{\sigma_o} x_m - x_m'\right] \cos(2\theta-u) - u x_m \sin(2\theta-u)\right\} \qquad (A7)$$

If we replace now r_c and Ω_c in the first order terms by r and Ω we find eq. (11).

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