ON THE TRIAD EXCISION THEOREM OF BLAKERS AND MASSEY

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The purpose of the present paper is to give a new proof to the triad excision theorem of Blakers and Massey [1], in case $m \geqq 2$ and $n \geqq 2$, by the aid of path spaces and in connection with a recent work of J. P. Serre [2].

1. Preliminary. Let X, A, B be topological spaces such that $X \supset A, B$. By $\Omega_{A,B}(X)$ we denote the totality of paths in X which start A and terminate in B; an element $(\sigma, I) \in \Omega_{A,B}(X)$ is represented by a continuous map $\sigma: I \to X$ of the closed unit interval I into X such that $\sigma(0) \in A$ and $\sigma(1) \in B$. Then $\Omega_{A,B}(X)$ is topologized by the compact open topology.

Let p_s be the projection of $\Omega_{A,B}(X)$ to A such that for $(\sigma, I) \in \Omega_{A,B}(X)$ $p_s(\sigma, I) = \sigma(0)$, and let $p_t : \Omega_{A,B}(X) \to B$ be the projection such that $p_t(\sigma, I) = \sigma(1)$ for $(\sigma, I) \in \Omega_{A,B}(X)$.

In the sequel, it is assumed that for a triad $(X; A, B, x_0)$ and for spaces of paths such as $\Omega_{A,B}(X)$, $\Omega_{A,x_0}(X)$, and so on, X, A, B, $A \cap B$, and spaces of paths are all arcwise connected, and that a reference point of any spaces of paths used, is taken to be an element represented by a constant map $e : I \to x_0$.

The following relations are obvious:

(*a*) $\quad \pi_{i-1}(\Omega_{x_0,x_0}(X), e) \approx \pi_i(X, x_0) \quad$ *for all* $i \geqq 1$,

(*b*) $\quad \pi_{i-1}(\Omega_{A,x_0}(X), e) \approx \pi_i(X, A, x_0) \quad$ *for all* $i \geqq 1$,

(*c*) $\quad$ *A is a deformation-retract of $\Omega_{A,X}(X)$*,

(*d*) $\quad \pi_{i-1}(\Omega_{B,x_0}(X), \Omega_{A\cap B,x_0}(A), e) \approx \pi_i(X; A, B, x_0) \quad$ *for all* $i \geqq 2$

where $(X; A, B, x_0)$ is a triad.

The above isomorphisms (*a*), (*b*) and (*d*) are referred to as *canonical isomorphisms.*

Let (X, A) be a pair of topological spaces, i.e., $X \supset A$. Suppose that X is p-connected for $p \geqq 1$ and (X, A, x_0) is q-connected for $q \geqq 1$, then $\Omega_{A,x_0}(X)$ is $(q-1)$-connected. $(\Omega_{A,X}(X), p_t, X)$ has a fibred structure in the sense of J. P. Serre, the fibre of which is $\Omega_{A,x_0}(X)$. Considering this fibre space, we have the following exact homology sequence with respect to integer coefficients, following J. P. Serre, [2] Chap. III. prop. 5 p. 468;

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$$H_{p+q}(\Omega_{A,x_0}(X)) \xrightarrow{h^*} H_{p+q}(\Omega_{A,X}(X)) \xrightarrow{p_t^*} H_{p+q}(X) \xrightarrow{\Sigma^*} H_{p+q-1}(\Omega_{A,x_0}(X)) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_1(\Omega_{A,x_0}(X)) \longrightarrow H_1(\Omega_{A,X}(X)) \longrightarrow H_1(X) \longrightarrow 0$$

where Σ^* is transgression.

Now, we define homomorphisms

$$c_k^* : H_k(\Omega_{A,x_0}(X) ; G) \longrightarrow H_{k+1}(X, A ; G) \qquad \textit{for all } k \geqq 1$$

by constructing chain maps, where G is an arbitrary coefficient group. For this we use singular cubical homology groups as homology groups defined by J. P. Serre, [2] p. 440.

Let (u^k, φ) be a singular cube of $\Omega_{A,x_0}(X)$, then φ defines a map

$$\bar{\varphi} : I \times u^k \longrightarrow X,$$

which gives a singular cube $(I \times u^k, \bar{\varphi})$ of X. By the correspondence

$$c_k : (u^k, \varphi) \longrightarrow (I \times u^k, \bar{\varphi})$$

and by linearity we get a chain homomorphism

$$c_k : C_k(\Omega_{A,x_0}(X)) \longrightarrow C_{k+1}(X).$$

From the following calculations

$$\begin{aligned} d \circ c(u^k, \varphi) &= d(I \times u^k, \bar{\varphi}) \\ &= \left(\sum_{i=1}^{k} (-1)^{i+1} I \times (\lambda_i^0 u^k - \lambda_i^1 u^k) - 0 \times u^k + 1 \times u^k, \bar{\varphi}\right) \\ &= -(I \times du^k, \bar{\varphi}) - (0 \times u^k, \bar{\varphi}) + (1 \times u^k, \bar{\varphi}) \\ &= -c \circ d(u^k, \varphi) - (0 \times u^k, \bar{\varphi}) \end{aligned}$$

where $(1 \times u^k, \bar{\varphi})$ is a degenerate cube and $\bar{\varphi}(0 \times u^k) \subset A$, and from the fact that if (u^k, φ) is degenerate cube, $(I \times u^k, \bar{\varphi})$ is also degenerated, it is concluded that c_k induces the following homomorphism

$$c_k^* : H_k(\Omega_{A,x_0}(X) ; G) \longrightarrow H_{k+1}(X, A ; G).$$

LEMMA 1. *Let (X, x_0) be p-connected for $p \geqq 1$, and let (X, A, x_0) be q-connected for $q \geqq 1$. Then*

i) *c_k^* are isomorphisms onto for $k \leqq p+q-1$,*

ii) *c_{p+q}^* is a homomorphism onto.*

Proof. We consider the following diagram

$$\begin{array}{ccccccccc} H_{p+q}(\Omega_{A,x_0}(X)) & \xrightarrow{h^*} & H_{p+q}(\Omega_{A,X}(X)) & \xrightarrow{p_t^*} & H_{p+q}(X) & \xrightarrow{\Sigma^*} & H_{p+q-1}(\Omega_{A,x_0}(X)) & \xrightarrow{h_*} & \cdots \\ \downarrow c_{p+q}^* & & \wr\, p_s^* & & \wr\, \iota^* & & \downarrow c_{p+q-1}^* & & \\ H_{p+q-1}(X, A) & \xrightarrow{\partial^*} & H_{p+q}(A) & \xrightarrow{i^*} & H_{p+q}(X) & \xrightarrow{j^*} & H_{p+q}(X, A) & \xrightarrow{\partial^*} & \cdots \end{array}$$

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$$\begin{array}{ccccccc} \cdots\longrightarrow H_1(\Omega_{A,x_0}(X)) & \longrightarrow & H_1(\Omega_{A,X}(X)) & \longrightarrow & H_1(X) & \longrightarrow 0. \\ \downarrow c_1^* & & \wr & & \wr & \\ \cdots\longrightarrow H_2(X, A) & \longrightarrow & H_1(A) & \longrightarrow & H_1(X) & \longrightarrow 0. \end{array}$$

Let

$$(u^{i+1}, \varphi)\in C_{i+1}(\Omega_{A,X}(X))$$

be given, then we have

$$\begin{aligned} i\circ p_s(u^{i+1}, \varphi) &= (0\times u^{i+1}, \bar\varphi)\in C_{i+1}(A)\subset C_{i+1}(X), \\ p_t(u^{i+1}, \varphi) &= (1\times u^{i+1}, \bar\varphi)\in C_{i+1}(X), \\ d(I\times u^{i+1}, \bar\varphi) &= -(I\times du^{i+1}, \bar\varphi) - (0\times u^{i+1}, \bar\varphi) \\ &\quad + (1\times u^{i+1}, \bar\varphi). \end{aligned}$$

This proves

$$i^*\circ p_s^* = \iota^*\circ p_t^*. \tag{α}$$

Next, given

$$(u^i, \varphi)\in C_i(\Omega_{A,x_0}(X)),$$

then we have

$$\begin{aligned} \partial\circ c(u^i, \varphi) &= d(I\times u^i, \bar\varphi) \\ &= -c\circ d(u^i, \varphi) - (0\times u^i, \bar\varphi) \\ &= -p_s\circ h(u^i, \varphi) - c\circ d(u^i, \varphi). \end{aligned}$$

Thus the identity

$$\partial^*\circ c^* = -p_s^*\circ h^* \tag{β}$$

is established.

By J. P. Serre, [2] p. 469, we get the following equivalent homology sequences:

$$\begin{array}{ccccc} H_{i+1}(\Omega_{A,x_0}(X)) & \longrightarrow & H_{i+1}(\Omega_{A,X}(X)) & \longrightarrow & H_{i+1}(\Omega_{A,X}(X), \Omega_{A,x_0}(X)) \\ \wr & & \wr & & \wr\, p_t'^* \\ H_{i+1}(\Omega_{A,x_0}(X)) & \longrightarrow & H_{i+1}(\Omega_{A,X}(X)) & \longrightarrow & H_{i+1}(X) \end{array}$$

$$\begin{array}{ccc} \xrightarrow{\partial^*} H_i(\Omega_{A,x_0}(X)) & \longrightarrow & H_i(\Omega_{A,X}(X)) \\ \wr & & \wr \\ \xrightarrow{\Sigma^*} H_i(\Omega_{A,x_0}(X)) & \longrightarrow & H_i(\Omega_{A,X}(X)) \end{array}$$

for $1\leqq i\leqq p+q-1$, i.e., we have $\sum^* = \partial^*\circ p_t'^{*-1}$.

We now consider the following diagram:

$$\begin{array}{ccc} H_{i+1}(\Omega_{A,X}(X), \Omega_{A,x_0}(X)) & & \\ \downarrow p_t'^* & \searrow \partial^* & \\ H_{i+1}(X) & \xrightarrow{\Sigma^*} & H_i(\Omega_{A,x_0}(X)) \\ & j^*\searrow & \downarrow c_i^* \\ & & H_{i+1}(X, A) \end{array}$$

Let

$$\sum_j (u_j^{i+1},\ \varphi_j) \in Z_{i+1}(\Omega_{A,X}(X),\ \Omega_{A,x_0}(X))$$

be given, then we have

$$p_t'(\sum_j (u_j^{i+1},\ \varphi_j)) = \sum_j (1 \times u_j^{i+1},\ \bar{\varphi}_j) \in Z_{i+1}(X),$$

$$\partial(\sum_j (u_j^{i+1},\ \varphi_j)) = \sum_j (du_j^{i+1},\ \varphi_j) \in Z_i(\Omega_{A,x_0}(X)),$$

$$c \circ \partial(\sum_j (u_j^{i+1},\ \varphi_j)) = \sum_j (I \times du_j^{i+1},\ \bar{\varphi}_j) \in Z_{i+1}(X,\ A).$$

Consider the following chain

$$\sum_j (I \times u_j^{i+1},\ \bar{\varphi}_j) \in C_{i+2}(X),$$

we have

$$d(\sum_j (I \times u_j^{i+1},\ \bar{\varphi}_j)) = -\sum_j (I \times du_j^{i+1},\ \bar{\varphi}_j) - \sum_j (0 \times u_j^{i+1},\ \bar{\varphi}_j) + \sum_j (1 \times u_j^{i+1},\ \bar{\varphi}_j)$$

$$= -(c \circ \partial - p_t')(\sum_j (u_j^{i+1},\ \varphi_j)) - \sum_j (0 \times u_j^{i+1},\ \bar{\varphi}_j),$$

where $\sum_j (0 \times u_j^{i+1},\ \bar{\varphi}_j) \in C_{i+1}(A)$. This proves

$$j^* \circ p_t'^* = c^* \circ \partial^*, \qquad (\gamma)$$

so that

$$c^* \circ \Sigma^* = j^* \circ \iota^* \qquad (\delta)$$

has been established.

(α), (β) and (δ) show that it holds some commutativity or anti-commutativity in each tetragon of the firstly mentioned diagram. As p_s^* is isomorphism onto by (c) and as ι^* is isomorphism onto induced by identity map, by using "five lemma," we get the first conclusion of this lemma.

(α), (β) and (γ) show that the following diagram is commutative or anti-commutative:

$$\begin{array}{ccc} H_{p+q+1}(\Omega_{A,X}(X),\ \Omega_{A,x_0}(X)) & \xrightarrow{\partial'^*} & H_{p+q}(\Omega_{A,x_0}(X)) \\ \downarrow p'^*_{t,p+q+1} & & \downarrow c^*_{p+q} \\ H_{p+q+1}(X) & \xrightarrow{j^*} & H_{p+q+1}(X,\ A) \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{h^*} & H_{p+q}(\Omega_{A,X}(X)) & \xrightarrow{j'^*} & H_{p+q}(\Omega_{A,X}(X),\ \Omega_{A,x_0}(X)) \\ & \wr\!\wr\ p^*_{s,p+q} & & \wr\!\wr\ p'^*_{t,p+q} \\ \xrightarrow{\partial^*} & H_{p+q}(A) & \xrightarrow{i^*} & H_{p+q}(X). \end{array}$$

By J. P. Serre, [2] Chap. III prop. 5 cor. 1 p. 469, we have

(ε) *$p'^*_{t,p+q}$ is an isomorphism onto, and $p'^*_{t,p+q+1}$ is a homomorphism onto.*

Then, by using a "partial conclusion of five lemma," we get the second conclusion of this lemma. (q.e.d.)

As a collorary of this lemma, we can easily prove the Hurewicz theorem in the relative case.

LEMMA 2. *Let* (X, A, B, x_0) *be a triple, then*

$$\pi_i(\Omega_{A,x_0}(X), \Omega_{B,x_0}(X), e) \approx \pi_i(A, B, x_0) \qquad \textit{for all } i \geqq 1.$$

Proof. Let us consider the following diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \pi_i(\Omega_{A,x_0}(X)) & \xrightarrow{j'} & \pi_i(\Omega_{A,x_0}(X), \Omega_{B,x_0}(X)) & \xrightarrow{\partial'} & \pi_{i-1}(\Omega_{B,x_0}(X)) & \xrightarrow{i'} & \pi_{i-1}(\Omega_{A,x_0}(X)) \to \cdots \\
\wr\!\wr\, k_A & & \downarrow p_s & & \wr\!\wr\, k_B & & \wr\!\wr\, k_A \\
\cdots \longrightarrow \pi_{i+1}(X, A) & \xrightarrow{\partial} & \pi_i(A, B) & \xrightarrow{i} & \pi_i(X, B) & \xrightarrow{j} & \pi_i(X, A) \to \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots \longrightarrow \pi_1(\Omega_{A,x_0}(X)) & \longrightarrow & \pi_1(\Omega_{A,x_0}(X), \Omega_{B,x_0}(X)) & \longrightarrow & \pi_0(\Omega_{B,x_0}(X)) & \longrightarrow & \pi_0(\Omega_{A,x_0}(X)), \\
\wr\!\wr & & \wr\!\wr & & \wr\!\wr & & \wr\!\wr \\
\cdots \longrightarrow \pi_2(X, A) & \longrightarrow & \pi_1(A, B) & \longrightarrow & \pi_1(X, B) & \longrightarrow & \pi_1(X, A),
\end{array}$$

where the upper sequence is a homotopy sequence of the pair $(\Omega_{A,x_0}(X), \Omega_{B,x_0}(X))$ and the lower sequence is a homotopy sequence of the triple (X, A, B, x_0). k_A and k_B are canonical isomorphisms and p_s denotes also the homorphism induced by the projection p_s.

Firstly, we prove that (k_A, p_s, k_B) is a homomorphism of the sequences, i.e., that $\partial \circ k_A = p_s \circ j'$, $i \circ p_s = k_B \circ \partial'$, $j \circ k_B = k_A \circ i'$.

The identity $j \circ k_B = k_A \circ i'$ is obvious.

Let $\alpha \in \pi_i(\Omega_{A,x_0}(X))$ be given such that a map $f : (E^i, \dot{E}^i) \longrightarrow (\Omega_{A,x_0}(X), e)$ represents α, then

$$k_A \circ f = \bar{f} : (E^i \times I, E^i \times 0, E^i \times 1 \cup \dot{E}^i \times I) \longrightarrow (X, A, x_0)$$

is defined by f canonically. The map

$$\partial \circ k_A \circ f = \bar{f} | (E^i \times 0, \dot{E}^i \times 0) \longrightarrow (A, x_0) \subset (A, B)$$

is identical to the map $p_s \circ j' \circ f$, which proves the identity

$$\partial \circ k_A = p_s \circ j'.$$

Secondly, if $\beta \in \pi_i(\Omega_{A,x_0}(X), \Omega_{B,x_0}(X))$ is represented by a map

$$g : (E^{i-1} \times I, E^{i-1} \times 0, E^{i-1} \times 1 \cup \dot{E}^{i-1} \times I) \longrightarrow (\Omega_{A,x_0}(X), \Omega_{B,x_0}(X), e),$$

g defines canonically a map

$$\begin{aligned}\bar{g} : (&E^{i-1} \times I \times I', E^{i-1} \times I \times 0', E^{i-1} \times 0 \times 0', \\ &E^{i-1} \times 1 \times I' \cup E^{i-1} \times I \times 1' \cup \dot{E}^{i-1} \times I \times I') \longrightarrow (X, A, B, x_0).\end{aligned}$$

Then $i\circ p_S\circ g$ and $k_B\circ\partial'\circ g$ are the following restrictions of $\bar{g}$ respectively:

$$i\circ p_S\circ g=\bar{g}\,|\,(E^{i-1}\times I\times 0',\ E^{i-1}\times 0\times 0',\ E^{i-1}\times 1\times 0'\cup \dot{E}^{i-1}\times I\times 0')\longrightarrow(A,\ B,\ x_0)\subset(X,\ B,\ x_0),$$

$$k_B\circ\partial'\circ g=\bar{g}\,|\,(E^{i-1}\times 0\times I',\ E^{i-1}\times 0\times 0',\ E^{i-1}\times 0\times 1'\cup \dot{E}^{i-1}\times 0\times I')\longrightarrow(X,\ B,\ x_0).$$

A homotopy between two maps $i\circ p_S\circ g$ and $k_B\circ\partial'\circ g$ will be given in $(E^{i-1}\times I\times I')$ as follows:

$$G_\theta(E^{i-1}\times I\times I')=\begin{cases}\bar{g}\,|\,(E^{i-1}\times t\times 2\theta t) & 0\leqq\theta\leqq 1/2,\\ \bar{g}\,|\,(E^{i-1}\times(2-2\theta)t\times t) & 1/2\leqq\theta\leqq 1.\end{cases}$$

This proves the identity

$$i\circ p_S=k_B\circ\partial'.$$

It follows that $(k_A,\ p_S,\ k_B)$ is a homomorphism of the sequences. Since k_A and k_B are isomorphisms and since $(k_A,\ p_S,\ k_B)$ is a homomorphism of the sequences it is concluded in virtue of "five lemma" that p_S also is isomorphism. (q.e.d.)

Let $(X\ ;\ A,\ B,\ x_0)$ be a triad, then $(\Omega_{X,x_0}(X)\ ;\ \Omega_{A,x_0}(X),\ \Omega_{B,x_0}(X),\ e)$ is also a triad, where $\Omega_{A,x_0}(X)\cap\Omega_{B,x_0}(X)=\Omega_{A\cap B,x_0}(X)$. The following lemma can be proved easily by considering homotopy sequences of each triads and by the above lemma and by "five lemma."

LEMMA 3. *Let* $(X\ ;\ A,\ B,\ x_0)$ *be triad, then*

$$\pi_i(X\ ;\ A.\ B,\ x_0)\approx\pi_i(\Omega_{X,x_0}(X)\ ;\ \Omega_{A,x_0}(X),\ \Omega_{B,x_0}(X),\ e)\qquad \textit{for all } i\geqq 2.$$

LEMMA 4. *Let* $(X\ ;\ A,\ B,\ x_0)$ *be a triad such that* $X=(\mathit{Int}\ A)\cup(\mathit{Int}\ B)$, *and let* $(A,\ A\cap B)$ *be n-connected* $(n\geqq 1)$, *then* $(X,\ B)$ *is n-connected.*

Proof. Let $\alpha\in\pi_m(X,\ B)$ be represented by a map

$$f:(E^m,\ E^{m-1},\ J^{m-1})\longrightarrow(X,\ B,\ x_0),$$

where $m\leqq n$. If we put $U=f^{-1}(\mathit{Int}\ A)$ and $V=f^{-1}(\mathit{Int}\ B)$, then $\{U,\ V\}$ is an open covering of E^m.

We subdivide E^m simplicially such that the mesh of this subdivision is smaller than the Lebesgues number of $\{U,\ V\}$. Let K and L_1 be maximal subcomplexes contained in U and V respectively. Let us put $L=L_1+E^{m-1}+J^{m-1}$ and $M=K\cap L$, then we have $K\cup L=E^m$. Let

$$g:(K,\ M)\longrightarrow(A,\ A\cap B)$$

be a restriction of f. As K is m-dimensional, $m\leqq n$, and as $(A,\ A\cap B)$ is n-connected, g is deformable into $A\cap B$ relative to M. Denoting this deforma-

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tion by g_t, we have

$$g_0 = g,$$
$$g_1(K) \subset A \cap B,$$
$$g_t | M = g | M \qquad \text{for } 0 \leqq t \leqq 1.$$

We define a deformation f_t of f as follows:

$$f_t | K = g_t \qquad \text{for } 0 \leqq t \leqq 1,$$
$$f_t | L = f | L \qquad \text{for } 0 \leqq t \leqq 1.$$

This gives a deformation of f into B relative to L, which establishes the lemma. (q.e.d.)

2. Proof of the triad excision theorem of Blakers and Massey.

Now we proceed to prove a theorem of A. L. Blakers and W. S. Massey, [1] p. 192, in case $m, n \geqq 2$. The theorem is stated as follows.

THEOREM. *Let* $(X ; A, B, x_0)$ *be a triad which satisfies the following conditions:*

(*a*) $X = (Int\ A) \cup (Int\ B)$:

(*b*) $(A, A \cup B)$ *is* m*-connected,* $m \geqq 2$, *and* $(B, A \cap B)$ *is* n*-connected,* $n \geqq 2$;

then the triad $(X ; A, B)$ *is* $(m+n)$*-connected.*

A triad with the condition (*a*) is said to be *proper* by a denomination of S. Eilenberg and N. E. Steenrod, [3] p. 34. From Lemma 4 (X, A) is n-connected, $n \geqq 2$, and (X, B) is m-connected, $m \geqq 2$. Therefore $\Omega_{X,x_0}(X)$, $\Omega_{A,x_0}(X)$, $\Omega_{B,x_0}(X)$ and $\Omega_{A\cap B,x_0}(X)$ are all arcwise connected. If $(X ; A, B, x_0)$ is proper, it is obvious that $(\Omega_{X,x_0}(X) ; \Omega_{A,x_0}(X), \Omega_{B,x_0}(X), e)$ is also a proper triad. Thus, from Lemma 3 it is sufficient for us to consider the triad $(\Omega_{X,x_0}(X) ; \Omega_{A,x_0}(X), \Omega_{B,x_0}(X), e)$ instead of the given triad. As $\Omega_{X,x_0}(X)$ is contractible, it is sufficient to prove the theorem in a special case where X is contractible.

Proof. As (X, A) is n-connected from Lemma 4, and as X is contractible, A is $(n-1)$-connected. Thus, by Lemma 1 it is seen that

(1) $$c_i^* : H_i(\Omega_{A\cap B,x_0}(A) ; Z) \approx H_{i+1}(A, A \cap B ; Z) \qquad \text{for } 0 < i \leqq m+n-2,$$

(2) $$c_{m+n-1}^* : H_{m+n-1}(\Omega_{A\cap B,x_0}(A) ; Z) \longrightarrow H_{m+n}(A, A \cap B ; Z)$$ *is a homomorphism onto.*

As (X, B) is m-connected and X is contractible, we have, from the same Lemma 1,

(3) $$c_i'^* : H_i(\Omega_{B,x_0}(X) ; Z) \approx H_{i+1}(X, B ; Z) \qquad \text{for all } i > 0.$$

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From (1), (3) and from the excision theorem in homology theory we have

(4) $$l_i^* : H_i(\Omega_{A\cap B, x_0}(A) ; Z) \approx H_i(\Omega_{B, x_0}(X) ; Z) \quad \text{for } 0 < i \leqq m+n-2.$$

Next, we consider the following diagram. The commutativity of this diagram is easily seen:

$$\begin{array}{ccc} H_{m+n-1}(\Omega_{A\cap B, x_0}(A) ; Z) & \xrightarrow{l^*_{m+n-1}} & H_{m+n-1}(\Omega_{B, x_0}(X) ; Z) \\ \downarrow c^*_{m+n-1} & & \wr\wr\, c'^*_{m+n-1} \\ H_{m+n}(A, A\cap B ; Z) & \overset{e^*_{m+n}}{\approx} & H_{m+n}(X, B ; Z) \end{array}$$

Since e^*_{m+n} is an excision isomorphism, and since c'^*_{m+n-1} is an isomorphism by (3) and since c^*_{m+n-1} is a homomorphism onto by (2), we have

(5) $$l^*_{m+n-1} : H_{m+n-1}(\Omega_{A\cap B, x_0}(A) ; Z) \to H_{m+n-1}(\Omega_{B, x_0}(X) ; Z)$$

is a homomorphism onto.

By (4) and (5), and by considering the homology sequence of the pair $(\Omega_{B, x_0}(X), \Omega_{A\cap B, x_0}(A))$ we can prove

(6) $$H_i(\Omega_{B, x_0}(X), \Omega_{A\cap B, x_0}(A) ; Z) \approx 0 \qquad \text{for } 0 < i \leqq m+n-1.$$

From (6) and from the Hurewicz theorem in the relative case where $\pi_1(\Omega_{B, x_0}(X)) \approx 1$, $\pi_1(\Omega_{A\cap B, x_0}(A)) \approx 1$, $(\Omega_{B, x_0}(X), \Omega_{A\cap B, x_0}(A), e)$ is $(m+n-1)$-connected. This is equivalent to the fact that $(X ; A, B, x_0)$ is $(m+n)$-connected. (q.e.d.)

In an analoguous way as above we can also prove the theorem corresponding to the case where $m \geqq 2$, $n=1$, and $(A, A\cap B)$ is $(m+1)$-simple. But it is unnecessarily too long for us to put down here the proof, so that it is omitted.

We can also prove quite analogously as above a generalization of the triad excision theorem, which has been announced by J. C. Moore [4].

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