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SOME HARDY-TYPE INEQUALITIES FOR THE GENERALIZED BAOUENDI-GRUSHIN OPERATORS*

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Abstract. In this paper, we prove some Hardy-type inequalities for the degenerate operators, $L_{p,\alpha}u = \mathrm{div}_L(|\nabla_L u|^{p-2}\nabla_L u)$, where $\nabla_L u = (\frac{\partial u}{\partial z_1}, \dots, \frac{\partial u}{\partial z_n}, |z|^\alpha \frac{\partial u}{\partial t_1}, \dots, |z|^\alpha \frac{\partial u}{\partial t_m})$. These inequalities are established for the whole space, the pseudo-ball and the external domain of the pseudo-ball. We also give a generalization of a result in [**8**]. Finally, a sharp inequality for $L_\alpha = L_{2,\alpha}$ is obtained.

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1. Introduction. The generalized Baouendi-Grushin operator is of the form

$$L_\alpha = \Delta_z + |z|^{2\alpha}\Delta_t, \tag{1.1}$$

where $\alpha > 0$, $z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$, $t = (t_1, t_2, \dots, t_m) \in \mathbb{R}^m$, Δ_z, Δ_t are the Laplacians on $\mathbb{R}^n$, $\mathbb{R}^m$, respectively.

The related p-degenerate subelliptic operator is

$$L_{p,\alpha} = \mathrm{div}_L(|\nabla_L u|^{p-2}\nabla_L u), \tag{1.2}$$

where $\nabla_L = (\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, |z|^\alpha \frac{\partial}{\partial t_1}, \dots, |z|^\alpha \frac{\partial}{\partial t_m}) = (Z_1, \dots Z_n, Z_{n+1}, \dots, Z_{n+m})$ is the generalized gradient and

$$\mathrm{div}_L(u_1, \dots, u_n, u_{n+1}, \dots, u_{n+m}) = \frac{\partial u_1}{\partial z_1} + \dots + \frac{\partial u_n}{\partial z_n} + |z|^\alpha \frac{\partial u_{n+1}}{\partial t_1} + \dots + |z|^\alpha \frac{\partial u_{n+m}}{\partial t_m}.$$

The operator L_α was first studied by Baouendi [**1**] and Grushin [**6**] when α is a positive integer. It is evident that if α is not an even positive integer then L_α is not a sum of squares of vector fields. This typical class of degenerate elliptic partial differential operators has been widely considered by many authors (see [**4**], [**5**] and references therein). In particular, Garofalo [**5**] established a Hardy-type inequality for L_α by using a representation formula of functions. Zhang and Niu [**8**] obtained a general inequality for $L_{p,\alpha}$ via a Picone identity for the vector fields $\{Z_1, \dots, Z_n, Z_{n+1}, \dots, Z_{n+m}\}$.

For $(z, t) \in \mathbb{R}^n \times \mathbb{R}^m$, define the distance between (z, t) and the origin $(0, 0)$ as

$$d(z, t) = (|z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2)^{\frac{1}{2(\alpha+1)}}.$$

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The homogeneous dimension with respect to the dilation

$$\delta_\epsilon(z, t) = (\epsilon z, \epsilon^{\alpha+1} t), \epsilon > 0$$

is

$$Q = n + m(\alpha + 1).$$

We will give in Section 2 some Hardy-type inequalities on some special domains in $\mathbb{R}^n \times \mathbb{R}^m$. The main approach we use is the appropriate choice of various auxiliary functions and the use of the Picone-type identity in [**8**].

The generalized Picone identities for elliptic operator systems in the Euclidean case and their applications were discussed in [**7**]. In Section 3 we generalize it to the degenerate elliptic operator system. It helps us to derive a Hardy-type inequality for $L_{p,\alpha}$ that is more general than the one in [**8**].

Section 4 is devoted to a sharp Hardy-type inequality for L_α. The method of proof is similar to one used in [**3**].

2. Hardy-type inequalities on some special domains. The open pseudo-ball of radius R and centred at the origin $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is denoted by

$$B_R = \left\{(z, t) \in \mathbb{R}^n \times \mathbb{R}^m \mid d(z, t) < R\right\},$$

and ∂B_R is the boundary of B_R.

For the convenience of the reader, we begin by quoting two known results from [**8**].

LEMMA 2.1. *For the differentiable functions $v > 0$ and $u \geq 0$ on $\Omega \subset \mathbb{R}^{n+m}$, denote*

$$Q_p(u, v) = |\nabla_L u|^p + (p-1)\frac{u^p}{v^p}|\nabla_L v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla_L v|^{p-2}\nabla_L u \cdot \nabla_L v,$$
$$R_p(u, v) = |\nabla_L u|^p - |\nabla_L v|^{p-2}\nabla_L\left(\frac{u^p}{v^{p-1}}\right)\nabla_L v.$$

Then

$$Q_p(u, v) = R_p(u, v) \geq 0.$$

LEMMA 2.2. *For some $\lambda > 0$ and a nonnegative weight function g on Ω, if $0 < v \in C_0^\infty$ satisfying*

$$-L_{p,\alpha} v \geq \lambda g v^{p-1} \qquad on\ \Omega,$$

then for $u \geq 0$,

$$\int_\Omega |\nabla_L u|^p \geq \lambda \int_\Omega g|u|^p.$$

*These are Theorems 3 and 4 in [**8**], respectively.*

The first result is the Hardy inequality on the $B_R\backslash\{(0, 0)\}$.

THEOREM 2.1. *Let* $\delta(z,t) = d((z,t), \partial B_R)$, *for any* $(z,t) \in B_R$, *and* $\Omega = B_R \setminus \{(0,0)\}$. *Then, for* $p > 1$, *we have*

$$\int_\Omega |\nabla_L u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|z|^{p\alpha}}{d^{p\alpha}} \frac{|u|^p}{\delta^p}, \tag{2.1}$$

for every $u \in C_0^\infty(\Omega)$, *where* $d = d(z,t)$.

Proof. Let $u \geq 0$. It is clear to

$$\begin{aligned} Z_j d &= Z_j(|z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2)^{\frac{1}{2(\alpha+1)}} = d^{-2\alpha-1}|z|^{2\alpha} z_j, \quad j = 1, \ldots, n, \\ Z_j d &= Z_j(|z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2)^{\frac{1}{2(\alpha+1)}} \\ &= d^{-2\alpha-1}|z|^\alpha(\alpha+1)t_j, \quad j = n+1, \ldots, n+m, \end{aligned}$$

therefore

$$\begin{aligned} \nabla_L d &= (d^{-2\alpha-1}|z|^{2\alpha}z_1, \ldots, d^{-2\alpha-1}|z|^{2\alpha}z_n, \\ &\quad \times d^{-2\alpha-1}|z|^\alpha(\alpha+1)t_1, \ldots, d^{-2\alpha-1}|z|^\alpha(\alpha+1)t_m), \\ |\nabla_L d|^2 &= (d^{-2\alpha-1})^2\left[\sum_{j=1}^n (|z|^{2\alpha}z_j)^2 + \sum_{j=n+1}^{n+m} (|z|^\alpha(\alpha+1)t_j)^2\right] \\ &= (d^{-2\alpha-1})^2(|z|^{4\alpha}|z|^2 + |z|^{2\alpha}(\alpha+1)^2|t|^2) \\ &= d^{-4\alpha-2}d^{2(\alpha+1)}|z|^{2\alpha} \\ &= \left(\frac{|z|^\alpha}{d^\alpha}\right)^2, \end{aligned}$$

and

$$|\nabla_L d| = \frac{|z|^\alpha}{d^\alpha}. \tag{2.2}$$

Using the notation

$$\psi_{p,\alpha} = \frac{|z|^{p\alpha}}{d^{p\alpha}},$$

it follows that

$$\begin{aligned} L_{p,\alpha} d &= \operatorname{div}_L(|\nabla_L d|^{p-2}\nabla_L d) \\ &= \psi_{p\alpha}\frac{d^{(\alpha+1)(p-2)}}{d^{(\alpha+1)(p-2)+1}}\left[Q + (\alpha+1)(p-2) + p\alpha\cdot 0 - [(\alpha+1)(p-2)+1]\frac{d^{2(\alpha+1)}}{d^{2(\alpha+1)}}\right] \\ &= \frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{1}{d}[Q + (\alpha+1)(p-2) - (\alpha+1)(p-2) - 1)] \\ &= \frac{(Q-1)|z|^{p\alpha}}{d^{p\alpha+1}}. \end{aligned} \tag{2.3}$$

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Choosing $v=\delta^{\beta}=(R-d)^{\beta}=f(d),\ \beta=\frac{p-1}{p}$, we get

$$\begin{aligned}L_{p,\alpha}(f(d))&=\operatorname{div}_L(|\nabla_L f(d)|^{p-2}\nabla_L f(d))\\&=\operatorname{div}_L(|f'(d)|^{p-2}f'(d)|\nabla_L d|^{p-2}\nabla_L d)\\&=|f'(d)|^{p-2}f'(d)\operatorname{div}_L(|\nabla_L d|^{p-2}\nabla_L d)+\nabla_L(|f'(d)|^{p-2}f'(d))\cdot|\nabla_L d|^{p-2}\nabla_L d\\&=|f'(d)|^{p-2}[f'(d)L_{p,\alpha}d+(p-1)f''(d)|\nabla_L d|^{p}],\end{aligned}\tag{2.4}$$

and then

$$\begin{aligned}-L_{p,\alpha}v&=-\left|-\frac{p-1}{p}\delta^{-\frac{1}{p}}\right|^{p-2}\left[-\frac{p-1}{p}\delta^{-\frac{1}{p}}\frac{(Q-1)|z|^{p\alpha}}{d^{p\alpha+1}}+(p-1)\left(-\frac{p-1}{p^2}\right)\delta^{-\frac{p+1}{p}}\frac{|z|^{p\alpha}}{d^{p\alpha}}\right]\\&=-\left(\frac{p-1}{p}\right)^{p-2}\delta^{-\frac{p-2}{p}}\frac{1}{d^{p\alpha}}\left[-\frac{(p-1)(Q-1)}{p}|z|^{p\alpha}\delta^{-\frac{1}{p}}\frac{1}{d}-\frac{(p-1)^2}{p^2}|z|^{p\alpha}\delta^{-\frac{p+1}{p}}\right]\\&=\left(\frac{p-1}{p}\right)^{p-1}\delta^{-\frac{p-2}{p}}\delta^{-\frac{p+1}{p}}\frac{|z|^{p\alpha}}{d^{p\alpha}}\left[\frac{p-1}{p}+(Q-1)\frac{\delta}{d}\right]\\&=\left(\frac{p-1}{p}\right)^{p-1}\left\{\frac{p-1}{p}+(Q-1)\frac{\delta}{d}\right\}\frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{v^{p-1}}{\delta^{p}}\\&\geq\left(\frac{p-1}{p}\right)^{p}\frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{v^{p-1}}{\delta^{p}}.\end{aligned}$$

From Theorems 2 and 3 in [**8**], we obtain

$$\int_\Omega|\nabla_L u|^p\geq\left(\frac{p-1}{p}\right)^p\int_\Omega\frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{|u|^p}{\delta^p}.$$

For general u, letting $u=u^+-u^-$ yields the desired statement. □

Now we consider the domain $\Omega=\mathbb{R}^{n+m}\backslash B_R$.

THEOREM 2.2. *Let $\Omega=\mathbb{R}^{n+m}\backslash B_R$ and $1<p<Q$. Then, for every $u\in C_0^\infty(\Omega)$*

$$\int_\Omega|\nabla_L u|^p\geq C\int_\Omega\frac{|z|^{p\alpha}}{d^{p(\alpha+1)}}\frac{|u|^p}{d^p}=C\int_\Omega\frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{|u|^p}{d^{2p}},$$

where $d=d(z,t)$.

Proof. It is enough to prove the claim for $u\geq 0$. Choosing

$$v=\ln\left(\frac{d}{R}\right)^{\alpha}=f(d),\quad \text{where } R<d<+\infty,\ \alpha=p-Q,$$

it follows that

$$f'(d)=\frac{\alpha}{d}=\frac{p-Q}{d},\tag{2.5}$$

and

$$f''(d)=-\frac{\alpha}{d^2}=\frac{Q-p}{d^2}.\tag{2.6}$$

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Thus,

$$\begin{aligned}
-L_{p,\alpha} &= -|f'(d)|^{p-2}[f'(d)L_{p,\alpha}d + (p-1)f''(d)|\nabla_L d|^p] \\
&= -\frac{(Q-p)^{p-2}}{d^{p-2}}\left[\frac{(Q-1)(p-Q)|z|^{p\alpha}}{d^{p\alpha+2}} + (p-1)(Q-p)\frac{|z|^{p\alpha}}{d^{p\alpha+2}}\right] \\
&= -\frac{(Q-p)^{p-2}}{d^{p(\alpha+1)}}|z|^{p\alpha}[-(Q-p)^2] \\
&= (Q-p)^p\frac{|z|^{p\alpha}}{d^{p(\alpha+1)}} \\
&= (Q-p)^p\frac{|z|^{p\alpha}}{d^{p(\alpha+1)}}\frac{v^{p-1}}{v^{p-1}}.
\end{aligned} \tag{2.7}$$

Since

$$\lim_{d\to+\infty}\frac{v^{p-1}}{d^p} = \lim_{d\to+\infty}\left(\ln\left(\frac{d}{R}\right)^{\alpha}\right)^{p-1} d^{-p} = 0,$$

there exists a positive integer $N \geq R$, such that if $d > N$, then $\frac{v^{p-1}}{d^p} < 1$. On the other hand, $\frac{v^{p-1}}{d^p}$ is continuous on the interval $[R, N]$, so we can find a positive constant C' such that $\frac{v^{p-1}}{d^p} \leq C'$. Taking $C'' = \max\{C', 1\}$, it shows that

$$\frac{v^{p-1}}{d^p} \leq C'', \quad \text{i.e., } v^{p-1} \leq C''d^p,$$

if $d \geq R$.

Equation (2.7) becomes

$$-L_{p,\alpha}v > (Q-p)^p\frac{|z|^{p\alpha}}{d^{p(\alpha+1)}}\frac{v^{p-1}}{C''d^p} = C\frac{|z|^{p\alpha}}{d^{p(\alpha+1)}}\frac{v^{p-1}}{d^p},$$

where $C = \frac{(Q-p)^p}{C''}$. Using Theorem 3 in **[8]**, we have

$$\int_\Omega |\nabla_L u|^p \geq C\int_\Omega \frac{|z|^{p\alpha}}{d^{p(\alpha+1)}}\frac{|u|^p}{d^p} = C\int_\Omega \frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{|u|^p}{d^{2p}}.$$ □

THEOREM 2.3. *Let $\Omega = \mathbb{R}^{n+m}\backslash B_R$ and $p \geq Q$. Then the inequality*

$$\int_\Omega |\nabla_L u|^p \geq C\int_\Omega \frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{|u|^p}{d^p\left(\ln\frac{d}{R}\right)^p}, \quad u \in C_0^\infty(\Omega),$$

is valid.

Proof. As above, we only consider $u \geq 0$. The choice

$$v = \left(\ln\frac{d}{R}\right)^{\alpha} = f(d), \quad \text{where } R < d < +\infty,\ 0 < \alpha < 1,$$

yields

$$f'(d) = \frac{\alpha}{d}\left(\ln\frac{d}{R}\right)^{\alpha-1}, \tag{2.8}$$

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$$f''(d) = -\frac{\alpha}{d^2}\left(\ln\frac{d}{R}\right)^{\alpha-1} + \frac{\alpha(\alpha-1)}{d^2}\left(\ln\frac{d}{R}\right)^{\alpha-2}, \tag{2.9}$$

and then

$$\begin{aligned}
-L_{p,\alpha}v &= -|f'(d)|^{p-2}[(p-1)f''(d)|\nabla_L d|^p + f'(d)L_{p,\alpha}d] \\
&= -\left|\frac{\alpha}{d}\left(\ln\frac{d}{R}\right)^{\alpha-1}\right|^{p-2}\left\{(p-1)\frac{|z|^{p\alpha}}{d^{p\alpha}}\left(-\frac{\alpha}{d^2}\left(\ln\frac{d}{R}\right)^{\alpha-1}\right.\right. \\
&\quad \left.\left. + \frac{\alpha(\alpha-1)}{d^2}\left(\ln\frac{d}{R}\right)^{\alpha-2}\right) + \frac{(Q-1)|z|^{p\alpha}}{d^{p\alpha+1}}\frac{\alpha}{d}\left(\ln\frac{d}{R}\right)^{\alpha-1}\right\} \\
&= -\frac{\alpha^{p-2}}{d^{p-2}}\left(\ln\frac{d}{R}\right)^{(\alpha-1)(p-2)}\left\{(p-1)\frac{|z|^{p\alpha}}{d^{p\alpha+2}}\left(-\alpha\left(\ln\frac{d}{R}\right)^{\alpha-1}\right.\right. \\
&\quad \left.\left. + \alpha(\alpha-1)\left(\ln\frac{d}{R}\right)^{\alpha-2}\right) + \frac{(Q-1)|z|^{p\alpha}\alpha}{d^{p\alpha+2}}\left(\ln\frac{d}{R}\right)^{\alpha-1}\right\} \\
&= -\frac{|z|^{p\alpha}\alpha^{p-1}}{d^{p(\alpha+1)}}\left(\ln\frac{d}{R}\right)^{\alpha(p-1)-p}\left[(\alpha-1)(p-1) + (Q-p)\ln\frac{d}{R}\right].
\end{aligned} \tag{2.10}$$

Noting $-\alpha^{p-1}(Q-p)\ln\frac{d}{R} \geq 0$, we get from (2.10) that

$$-L_{p,\alpha}v \geq -\alpha^{p-1}(\alpha-1)(p-1)\frac{v^{p-1}}{(\ln\frac{d}{R})^p}\frac{|z|^{p\alpha}}{d^{p(\alpha+1)}} = C\frac{v^{p-1}}{(\ln\frac{d}{R})^p}\frac{|z|^{p\alpha}}{d^{p(\alpha+1)}},$$

where $C = -\alpha^{p-1}(\alpha-1)(p-1)$. Using Theorem 3 in [**8**] gives

$$\int_\Omega |\nabla_L u|^p \geq C\int_\Omega \frac{|z|^{p\alpha}}{d^{p(\alpha+1)}}\frac{|u|^p}{(\ln\frac{d}{R})^p} = C\int_\Omega \frac{|z|^{p\alpha}}{d^{p\alpha}}\frac{|u|^p}{d^p(\ln\frac{d}{R})^p}.$$

This ends the proof. □

We do not know how to prove Hardy-type inequalities for L_α or $L_{p,\alpha}$ on bounded or unbounded domains in $\mathbb{R}^n \times \mathbb{R}^m$ other than those discused above. This is an open question.

3. Generalized Picone-type identities and Hardy-type inequalities. Consider the following system of degenerate subelliptic operators

$$p_\gamma[u] = \nabla_L \cdot (a(x)\Phi(\nabla_L u)) + c(x)\phi(u) \tag{3.1}$$

$$P_\gamma[v] = \nabla_L \cdot (A(x)\Phi(\nabla_L v)) + C(x)\phi(v), \tag{3.2}$$

where $\phi(s) = |s|^{\gamma-1}s$, $s \in \mathbb{R}$, $\Phi(\xi) = |\xi|^{\gamma-1}\xi$, $\xi \in \mathbb{R}^n \times \mathbb{R}^m$, $\gamma > 0$, ∇_L is the generalized gradient. The boundary $\partial\Omega$ of the domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ is piecewise smooth.The functions $a(x), c(x), A(x)$ and $C(x)$ in (3.1) and (3.2) satisfy respectively: $a \in C^1(\bar{\Omega}; \mathbb{R}_+,)$, $A \in C^1(\bar{\Omega}; \mathbb{R}_+)$, $c \in C(\bar{\Omega}; \mathbb{R})$, $C \in C(\bar{\Omega}; \mathbb{R})$.

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Define

$$D_{p_\gamma(\Omega)} = \left\{u \in C^1(\bar{\Omega};\mathbb{R}) : a(x)\Phi(\nabla_L u) \in C^1(\Omega;\mathbb{R}) \bigcap C(\bar{\Omega};\mathbb{R})\right\};$$

$$D_{P_\gamma(\Omega)} = \left\{v \in C^1(\bar{\Omega};\mathbb{R}) : A(x)\Phi(\nabla_L v) \in C^1(\Omega;\mathbb{R}) \bigcap C(\bar{\Omega};\mathbb{R})\right\},$$

respectively.

The following is a generalized Picone-type identity.

LEMMA 3.1. *Suppose that* $u \in D_{p_\gamma(\Omega)}$, $v \in D_{P_\gamma(\Omega)}$ *and* $v \neq 0$ *on* Ω. *Then*

$$\begin{aligned}
&\nabla_L \cdot \left\{\frac{u}{\phi(v)}\left[\phi(v)a(x)\Phi(\nabla_L u) - \phi(u)A(x)\Phi(\nabla_L v)\right]\right\}\\
&\quad = (a(x) - A(x))\,|\nabla_L u|^{\gamma+1} + (C(x) - c(x))\,|u|^{\gamma+1}\\
&\qquad + A(x)\left[|\nabla_L u|^{\gamma+1} + \gamma\left|\frac{u}{v}\nabla_L v\right|^{\gamma+1} - (\gamma+1)\left|\frac{u}{v}\nabla_L v\right|^{\gamma-1}\nabla_L u\cdot\left(\frac{u}{v}\nabla_L v\right)\right]\\
&\qquad + \frac{u}{\phi(v)}\left\{\phi(v)p_\gamma[u] - \phi(u)P_\gamma[v]\right\}.
\end{aligned} \tag{3.3}$$

Proof. A direct calculation shows that

$$(P_1)\begin{cases}
\phi(s) \in C^1(\mathbb{R}\backslash\{0\};\mathbb{R})\\
s\phi'(s) \in C(\mathbb{R};\mathbb{R})\\
s\phi'(s) = \gamma\phi(s),\ s \in \mathbb{R}\\
\phi(s)\phi(t) = \phi(st),\ s \in \mathbb{R}\ \text{ and }\ t \in \mathbb{R}\\
\phi(-s) = -\phi(s),\ s \in \mathbb{R}\\
\phi(s) > 0,\ s > 0,
\end{cases}$$

$$(P_2)\begin{cases}
\Phi(\xi) \in C^1\left(\mathbb{R}^{n+m}\backslash\{0\};\mathbb{R}^{n+m}\right)\bigcap C\left(\mathbb{R}^{n+m};\mathbb{R}^{n+m}\right)\\
\Phi(-\xi) = -\Phi(\xi),\ \xi \in \mathbb{R}^{n+m}\\
\phi(s)\Phi(\xi) = \Phi(s\xi),\ s \in \mathbb{R}\ \text{ and }\ \xi \in \mathbb{R}^{n+m}.
\end{cases}$$

Evidently, we have

$$\begin{aligned}
up_\gamma[u] &= u\nabla_L\cdot(a(x)\Phi(\nabla_L u)) + c(x)u\phi(u)\\
&= \nabla_L\,(ua(x)\Phi(\nabla_L u)) - \nabla_L u\cdot(a(x)\Phi(\nabla_L u)) + c(x)u\phi(u),
\end{aligned}$$

$$\nabla_L\left(\frac{u}{\phi(v)}\phi(v)a(x)\Phi(\nabla_L u)\right) = a(x)\nabla_L u\cdot\Phi(\nabla_L u) - c(x)u\phi(u) + \frac{u}{\phi(v)}\left(\phi(v)p_\gamma[u]\right), \tag{3.4}$$

and

$$\begin{aligned}
&\nabla_L\cdot\left(u\phi(u)\frac{A(x)\Phi(\nabla_L v)}{\phi(v)}\right)\\
&\quad = \nabla_L u\cdot\left(\phi(u)\frac{A(x)\Phi(\nabla_L v)}{\phi(v)}\right) + u\phi'(u)\nabla_L u\cdot\frac{A(x)\Phi(\nabla_L v)}{\phi(v)}\\
&\qquad + u\phi(u)\left(-\frac{\phi'(v)}{\phi^2(v)}\nabla_L v\right)\cdot(A(x)\Phi(\nabla_L v)) + u\frac{\phi(u)}{\phi(v)}\nabla_L\cdot(A(x)\Phi(\nabla_L v))
\end{aligned}$$

$$
\begin{aligned}
&= A(x)\phi\left(\frac{u}{v}\right)\nabla_L u\cdot\Phi(\nabla_L v)+\gamma A(x)\phi\left(\frac{u}{v}\right)\nabla_L u\cdot\Phi(\nabla_L v)\\
&\quad -\gamma\frac{u}{v}\phi\left(\frac{u}{v}\right)A(x)\nabla_L v\cdot\Phi(\nabla_L v)+\frac{u}{\phi(v)}(\phi(u)P_\gamma[v])-C(x)u\phi(u)\\
&= A(x)\nabla_L u\cdot\Phi(\nabla_L u)-C(x)u\phi(u)-A(x)\bigg\{\nabla_L u\cdot\Phi(\nabla_L u)\\
&\quad +\gamma\left(\frac{u}{v}\nabla_L v\right)\cdot\Phi\left(\frac{u}{v}\nabla_L v\right)-(\gamma+1)\nabla_L u\cdot\Phi\left(\frac{u}{v}\nabla_L v\right)\bigg\}\\
&\quad +\frac{u}{\phi(v)}(\phi(u)P_\gamma[v]). \qquad (3.5)
\end{aligned}
$$

We immediately obtain

$$
\begin{aligned}
&\nabla_L\cdot\left\{\frac{u}{\phi(v)}\left[\phi(v)a(x)\Phi(\nabla_L u)-\phi(u)A(x)\Phi(\nabla_L v)\right]\right\}\\
&= (a(x)-A(x))\,\nabla_L u\cdot\Phi(\nabla_L u)+(C(x)-c(x))\,u\phi(u)\\
&\quad + A(x)\bigg\{\nabla_L u\cdot\Phi(\nabla_L u)+\gamma\left(\frac{u}{v}\nabla_L v\right)\cdot\Phi\left(\frac{u}{v}\nabla_L v\right)\\
&\quad -(\gamma+1)\nabla_L u\cdot\Phi\left(\frac{u}{v}\nabla_L v\right)\bigg\}+\frac{u}{\phi(v)}\left(\phi(v)p_\gamma[u]-\phi(u)P_\gamma[v]\right)\\
&= (a(x)-A(x))\,|\nabla_L u|^{\gamma+1}+(C(x)-c(x))\,|u|^{\gamma+1}\\
&\quad + A(x)\left\{|\nabla_L u|^{\gamma+1}+\gamma\left|\frac{u}{v}\nabla_L v\right|^{\gamma+1}-(\gamma+1)\left|\frac{u}{v}\nabla_L v\right|^{\gamma-1}\nabla_L u\cdot\left(\frac{u}{v}\nabla_L v\right)\right\}\\
&\quad +\frac{u}{\phi(v)}\left(\phi(v)p_\gamma[u]-\phi(u)P_\gamma[v]\right).
\end{aligned}
$$

The result follows. □

In particular, taking $a = c = C = 0$, $\gamma = p-1 > 0$, $A(x) > 0$, yields the following corollary.

COROLLARY 3.1. *Let $v > 0$ and $u \geq 0$ be differentiable. Then*

$$
L(u, v) = R(u, v) \geq 0, \qquad (3.6)
$$

where

$$
L(u, v) = A(x)\left[|\nabla_L u|^p+(p-1)\left|\frac{u}{v}\nabla_L v\right|^p-p\left|\frac{u}{v}\nabla_L v\right|^{p-2}\nabla_L u\cdot\left(\frac{u}{v}\nabla_L v\right)\right],
$$

$$
R(u, v) = A(x)|\nabla_L u|^p-\nabla_L\left(\frac{u^p}{v^{p-1}}\right)\cdot A(x)\left|\frac{u}{v}\nabla_L v\right|^{p-2}\nabla_L v.
$$

LEMMA 3.2. *Suppose that $0 < v \in C^\infty(\Omega)$ satisfies*

$$
-L_{p,\alpha}v \geq \lambda g v^{p-1},
$$

for some $\lambda > 0$ and weight function g on Ω, where

$$
-L_{p,\alpha}v = -\nabla_L\cdot(A(x)|\nabla_L v|^{p-2}\nabla_L v).
$$

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Then, for any $u \in C_0^\infty(\Omega)$, one has

$$\int_\Omega A(x)|\nabla_L u|^p \geq \lambda \int_\Omega g|u|^p. \tag{3.7}$$

Proof. Let Ω_0 be a compact subset of Ω and let $\phi \geq 0$ be differentiable. We have, by Corollary 3.1,

$$\begin{aligned}
0 &\leq \int_{\Omega_0} L(\phi, v) \leq \int_\Omega L(\phi, v) = \int_\Omega R(\phi, v) \\
&= \int_\Omega A(x)|\nabla_L \phi|^p - \nabla_L \left(\frac{\phi^p}{v^{p-1}}\right) \cdot A(x)|\nabla_L v|^{p-2}\nabla_L v \\
&= \int_\Omega A(x)|\nabla_L \phi|^p + \frac{\phi^p}{v^{p-1}} \nabla_L \cdot \left(A(x)|\nabla_L v|^{p-2}\nabla_L v\right) \\
&\leq \int_\Omega A(x)|\nabla_L \phi|^p - \frac{\phi^p}{v^{p-1}} \lambda g v^{p-1} \\
&= \int_\Omega A(x)|\nabla_L \phi|^p - \lambda g \phi^p.
\end{aligned}$$

For general u, let $\phi = u^+ - u^-$. The proof is easily completed. □

THEOREM 3.1. *Let $A(z,t) = \frac{p-Q}{\beta p}\left|\frac{p-Q}{\beta p}\right|^{p-2} d^{(p-1)(1-\beta-\frac{Q}{p})}$, with $\beta < 0$, $1 < p < Q$. Then for any $u \in C_0^\infty(\mathbb{R}^{n+m}\backslash(0,0))$, we have*

$$\int_{\mathbb{R}^{n+m}\backslash(0,0)} A(z,t)\,|\nabla_L \phi|^p \geq \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{R}^{n+m}\backslash(0,0)} \frac{|z|^{p\alpha}}{d^k}\,|u|^p, \tag{3.8}$$

where $k = Q - \frac{Q}{p} + p(\alpha + \beta) + (1 - \beta)$.

Proof. Choosing $v = d^\beta$ and noting that

$$\nabla_L v = (Z_1 d^\beta, \ldots, Z_n d^\beta, Z_{n+1} d^\beta, \ldots, Z_{n+m} d^\beta) = \beta d^{\beta-1} \nabla_L d,$$

it follows that

$$|\nabla_L v| = |\beta|\, d^{\beta-1}\, |\nabla_L d| = -\beta d^{\beta-1}\, |\nabla_L d|,$$

and

$$\begin{aligned}
&-\nabla_L (A(z,t)|\nabla_L v|^{p-2}\nabla_L v) \\
&\quad = -\nabla_L \cdot \left(\frac{p-Q}{\beta p}\left|\frac{p-Q}{\beta p}\right|^{p-2} d^{(p-1)\left(1-\beta-\frac{Q}{p}\right)}\, |\beta|^{p-2}\, d^{(\beta-1)(p-1)}\, |\nabla_L d|^{p-2}\, \beta \nabla_L d\right) \\
&\quad = -\nabla_L \cdot \left(\frac{p-Q}{p}\left|\frac{p-Q}{p}\right|^{p-2} d^{-\frac{Q}{p}(p-1)}\, |\nabla_L d|^{p-2}\, \nabla_L d\right) \\
&\quad = \left(\frac{Q-p}{p}\right)^{p-1} \nabla_L \cdot \left(|z|^{\alpha(p-2)} d^{-\frac{Q}{p}(p-1)-\alpha(p-2)} \nabla_L d\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \left(\frac{Q-p}{p}\right)^{p-1} \Big\{\alpha(p-2)|z|^{\alpha(p-2)-1} d^{-\frac{Q}{p}(p-1)-\alpha(p-2)} \cdot \nabla|z| \cdot \nabla_L d \\
&\quad + \left[-\frac{Q}{p}(p-1)-\alpha(p-2)\right] d^{-\frac{Q}{p}(p-1)-\alpha(p-2)-1} |z|^{\alpha(p-2)} |\nabla_L d|^2 \\
&\quad + |z|^{\alpha(p-2)} d^{-\frac{Q}{p}(p-1)-\alpha(p-2)} L_\alpha d \Big\},
\end{aligned} \tag{3.9}
$$

where we have used (2.2). Otherwise, using the formulae

$$
\begin{aligned}
\nabla_L|z| \cdot \nabla_L d &= (Z_1|z|, \ldots, Z_n|z|, Z_{n+1}|z|, \ldots, Z_{n+m}|z|) \\
&\quad \cdot (d^{-2\alpha-1}|z|^{2\alpha} z_1, \ldots, d^{-2\alpha-1}|z|^{2\alpha} z_n, d^{-2\alpha-1}|z|^{\alpha}(\alpha+1)t_1, \\
&\quad \ldots, d^{-2\alpha-1}|z|^{\alpha}(\alpha+1)t_m) \\
&= \frac{d^{-2\alpha-1}}{|z|}|z|^{2\alpha}|z|^2 \\
&= \frac{|z|^{2\alpha+1}}{d^{2\alpha+1}},
\end{aligned}
$$

$L_\alpha d = \frac{(Q-1)|z|^{2\alpha}}{d^{2\alpha+1}}$ and (2.2), (3.9) becomes

$$
\begin{aligned}
&-\nabla_L(A(z,t)|\nabla_L v|^{p-2}\nabla_L v) \\
&= \left(\frac{Q-p}{p}\right)^{p-1} \Big\{\alpha(p-2)|z|^{\alpha(p-2)-1} \frac{|z|^{2\alpha+1}}{d^{2\alpha+1}} d^{-\frac{Q}{p}(p-1)-\alpha(p-2)} \\
&\quad + \left[-\frac{Q}{p}(p-1)-\alpha(p-2)\right] d^{-\frac{Q}{p}(p-1)-\alpha(p-2)-1} \frac{|z|^{2\alpha}}{d^{2\alpha}} |z|^{\alpha(p-2)} \\
&\quad + |z|^{\alpha(p-2)} d^{-\frac{Q}{p}(p-1)-\alpha(p-2)} \frac{(Q-1)|z|^{2\alpha}}{d^{2\alpha+1}} \Big\} \\
&= \left(\frac{Q-p}{p}\right)^{p-1} \left\{|z|^{p\alpha} d^{-Q+\frac{Q}{p}-\alpha p-1}\left[\alpha(p-2)+\frac{Q}{p}-Q-\alpha p+2\alpha+Q-1\right]\right\} \\
&= \left(\frac{Q-p}{p}\right)^{p-1} \left\{|z|^{p\alpha} d^{-Q+\frac{Q}{p}-\alpha p-1-\beta(p-1)} v^{p-1}\left(\frac{Q}{p}-1\right)\right\} \\
&= \left(\frac{Q-p}{p}\right)^{p} |z|^{p\alpha} d^{-\left[Q-\frac{Q}{p}+p(\alpha+\beta)+1-\beta\right]} v^{p-1}.
\end{aligned}
$$

Hence we obtain the claim by Lemma 3.2. □

REMARK. If $\beta = 1 - \frac{Q}{p}$, then (3.8) is (1.4) in **[8]**.

4. A sharp Hardy-type inequality for L_α. We first give a generalization of Theorem 8.3.4 in **[2]**.

LEMMA 4.1. *If there exists a function* $0 < f \in C^\infty$ *such that* $V(x) \geq \frac{L_\alpha f}{f}$ *a.e. on* $\mathbb{R}^n \times \mathbb{R}^m$*, then* $\operatorname{Spec}(H)$ *(the spectrum of* H*)* $\subseteq [0,\infty)$*, where* $Hf(x) = -L_\alpha f(x) + V(x)f(x)$.

Proof. Let $W = \frac{L_\alpha f}{f}$, $\varphi \in C_0^\infty$ and $\psi = \frac{\phi}{f}$. Then

$$\begin{aligned}
Q(\phi) &= \int_{\mathbb{R}^{n+m}} (|\nabla_L \phi|^2 + V|\phi|^2)\, dx \\
&\geq \int_{\mathbb{R}^{n+m}} \left(|\nabla_L \phi|^2 + \frac{L_\alpha f}{f}|\phi|^2\right) dx \\
&\geq \int_{\mathbb{R}^{n+m}} (-\psi L_\alpha f - 2\nabla_L f \nabla_L \psi - f L_\alpha f + W f \psi) f \overline{\psi}\, dx \\
&= \int_{\mathbb{R}^{n+m}} \left(-\psi L_\alpha f - 2\nabla_L f \nabla_L \psi - f L_\alpha f + \frac{L_\alpha f}{f} f \psi\right) f \overline{\psi}\, dx \\
&= \int_{\mathbb{R}^{n+m}} (-2f\overline{\psi}\nabla_L f \nabla_L \psi - f^2 L_\alpha \psi\, \overline{\psi})\, dx \\
&= \int_{\mathbb{R}^{n+m}} (-\overline{\psi}\, \nabla_L f^2\, \nabla_L \psi - (L_\alpha \psi)(f^2\, \overline{\psi}))\, dx \\
&= \int_{\mathbb{R}^{n+m}} (-\overline{\psi}\, \nabla_L f^2\, \nabla_L \psi + \nabla_L \psi\, \nabla_L (f^2\, \psi))\, dx \\
&= \int_{\mathbb{R}^{n+m}} \nabla_L \psi (-\overline{\psi} \nabla_L f^2 + \nabla_L (f^2\, \overline{\psi}))\, dx \\
&= \int_{\mathbb{R}^{n+m}} (\nabla_L \psi\, f^2\, \nabla_L \overline{\psi})\, dx \\
&= \int_{\mathbb{R}^{n+m}} f^2\, |\nabla_L \psi|^2\, dx.
\end{aligned}$$

Since C_0^∞ is a form core, the conclusion is proved. □

THEOREM 4.1. *Suppose that* $u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$. *Then*

$$\left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^{n+m}} \frac{|z|^{2\alpha}}{d^{2(\alpha+1)}} u^2(z,t)\, dz\, dt \leq \int_{\mathbb{R}^{n+m}} |\nabla_L u(z,t)|^2\, dz\, dt. \tag{4.1}$$

Proof. Note

$$L_\alpha f = \Delta_z f + |z|^{2\alpha} \Delta_t f = \sum_{j=1}^n \frac{\partial^2 f}{\partial z_j^2} + |z|^{2\alpha} \sum_{j=1}^m \frac{\partial^2 f}{\partial t_j^2}. \tag{4.2}$$

For $s > 0$ and $k < 0$, choosing

$$f = [s + |z|^{2(\alpha+1)} + (\alpha+1)^2 |t|^2]^k.$$

A direct calculation implies

$$\frac{\partial f}{\partial z_j} = 2k(\alpha+1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{k-1} |z|^{2\alpha} z_j, \tag{4.3}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial z_j^2} &= 2k(\alpha+1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{k-1} |z|^{2\alpha} \\
&\quad \times \left\{2(k-1)(\alpha+1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{-1} |z|^{2\alpha} z_j^2 + 2\alpha|z|^{-2} z_j^2 + 1\right\},
\end{aligned} \tag{4.4}$$

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$$\frac{\partial f}{\partial t_j} = 2k(\alpha+1)^2 t_j \left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{k-1}, \tag{4.5}$$

$$\frac{\partial^2 f}{\partial t_j^2} = 2k(\alpha+1)^2 \left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{k-1} \times \left\{2(k-1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{-1}(\alpha+1)^2 t_j^2 + 1\right\}. \tag{4.6}$$

Combining (4.4) and (4.6) with (4.2) leads to

$$\begin{aligned} L_\alpha f &= 2k(\alpha+1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{k-1} |z|^{2\alpha} \\ &\quad \times \left\{ \sum_{j=1}^{n} 2(k-1)(\alpha+1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{-1} |z|^{2\alpha} z_j^2 \right. \\ &\quad + \sum_{j=1}^{m} 2(k-1)(\alpha+1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{-1} (\alpha+1)^2 t_j^2 \\ &\quad \left. + 2\alpha|z|^{-2} z_j^2 + 1 + (\alpha+1)\right\} \\ &= 4k(k-1)(\alpha+1)^2\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{k-2} |z|^{2\alpha}\left[|z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right] \\ &\quad + 2k(\alpha+1)\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^{k-1} |z|^{2\alpha}\left[m(\alpha+1) + n + 2\alpha\right], \end{aligned} \tag{4.7}$$

$$\begin{aligned} \frac{L_\alpha f}{f} &= \frac{\left[4k^2(\alpha+1)^2 + 2k(\alpha+1)(m(\alpha+1)+n-2)\right]}{\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^2} \\ &\quad \times |z|^{2\alpha}\left[|z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right] \\ &\quad + \frac{2k(\alpha+1)|z|^{2\alpha}\left[m(\alpha+1)+n+2\alpha\right]s}{\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^2}. \end{aligned} \tag{4.8}$$

Denote

$$\begin{aligned} g(k) &= 4(\alpha+1)^2k^2 + 2(\alpha+1)\left[m(\alpha+1)+n-2\right]k \\ &= 4(\alpha+1)^2k^2 + 2(\alpha+1)(Q-2)k. \end{aligned} \tag{4.9}$$

Clearly, taking $k = -\frac{Q-2}{4(\alpha+1)}$, one has

$$\begin{aligned} \min g(k) &= 4(\alpha+1)^2\left[-\frac{Q-2}{4(\alpha+1)}\right]^2 + 2(\alpha+1)(Q-2)\left[-\frac{Q-2}{4(\alpha+1)}\right] \\ &= \frac{(Q-2)^2}{4} - \frac{(Q-2)^2}{2} \\ &= -\left(\frac{Q-2}{2}\right)^2, \end{aligned} \tag{4.10}$$

and

$$\frac{L_\alpha f}{f} = -\left(\frac{Q-2}{2}\right)^2 \frac{\left[|z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]|z|^{2\alpha}}{\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^2} + \frac{2k(\alpha+1)|z|^{2\alpha}\left[m(\alpha+1)+n+2\alpha\right]s}{\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^2} \leq 0. \tag{4.11}$$

By Lemma 4.1, we have

$$\int_{\mathbb{R}^{n+m}} \left[\left(\frac{Q-2}{2}\right)^2 \frac{\left[|z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]|z|^{2\alpha}}{\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^2} - \frac{2k(\alpha+1)s|z|^{2\alpha}\left[m(\alpha+1)+n+2\alpha\right]}{\left[s + |z|^{2(\alpha+1)} + (\alpha+1)^2|t|^2\right]^2}\right]u^2(z,t)\,dz\,dt \leq \int_{\mathbb{R}^{n+m}} |\nabla_L u(z,t)|^2 dz\,dt \tag{4.12}$$

Letting $s \to 0$, we deduce the claim. □

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