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GENERALIZATIONS OF DECOMPOSITION THEOREMS KNOWN OVER PERFECT RINGS

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Abstract

In this paper we introduce and study the notion of dual continuous (d-continuous) modules. A decomposition theorem for a d-continuous module is proved; this generalizes all known decomposition theorems for quasi-projective modules. Besides we study the structure of d-continuous modules over some special types of rings.

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1.

Bass (1960) proved a decomposition theorem for projective modules over perfect rings. Later Wu and Jans (1967) gave the structure of finitely generated quasi-projective modules over semiperfect rings, which was used by Koehler (1971) to prove a structure theorem for quasi-projective modules over perfect rings. The purpose of this paper is to show that, to some extent, these decomposition theorems can be obtained by intrinsic properties of the modules themselves, and are independent of the ring involved. Let M be a module over any ring R satisfying the following conditions:

(I) For any submodule A of M, $M = M_1 \oplus M_2$ such that $M_1 \subset A$ and $A \cap M_2$ is small in M_2,

(II) If for any submodule N of M, M/N is isomorphic to a summand of M, then N is a summand of M.

A ring R is *perfect* if and only if every quasi-projective R-module satisfies (I) and (II) (Theorem 2.3). A module satisfying (I) and (II) is called *dual continuous* (d-continuous). Example 2.6 shows that a d-continuous module over a perfect ring need not be quasi-projective, and Theorem 2.3 shows that, in general, a projective module need not be d-continuous. Let M be a d-continuous module. Lemma 3.6 shows that for any two summands A and B of M, if $A+B$ is a summand, then $A \cap B$ is also a summand. This result is then used to show that M is perfect, in the sense of Miyashita (1966). Theorem 3.10 gives the structure of the endomorphism ring of M. If $M = N \oplus K$, then Proposition 4.1 shows that any homomorphism $\phi: N \to K/C$ can be lifted to a homomorphism $\psi: N \to K$. As an immediate consequence it follows that if $N \times N$ is d-continuous, then N is quasi-projective. In Theorem 4.7, it is proved that $M = N + N'$ where N' is a summand of M with $\operatorname{Rad} N' = N'$ and $N = \Sigma_{i \in I} \oplus A_i$ where A_i is cyclic indecomposable, the sum of any finitely many A_i's is a summand of M, and if A_i is not quasi-projective, then A_i is not isomorphic to A_j for $j \neq i \in I$. In Section 5, we study the structure of d-continuous modules over some special rings. Theorem 5.4 gives the structure of a d-continuous module over perfect rings. Theorem 5.5 shows that a torsion abelian group is d-continuous if and only if it is quasi-projective.

All rings considered have unities and all modules are unital right modules. Let M be a module. A submodule A of M is called *small in* M (notation $A \underset{S}{\subset} M$) if $A + B \neq M$ for every proper submodule B of M. The sum of finitely many small submodules of M is small. $\operatorname{Rad} M$ will stand for the Jacobson radical of M. It is known that $\operatorname{Rad} M$ is the sum of all small submodules of M. If every proper submodule of M is contained in a maximal submodule (for example if M is finitely generated), then $\operatorname{Rad} M \underset{S}{\subset} M$. A submodule B of M is called a *dual complement* (d-complement) of A in M if B is minimal with the property $A + B = M$. N is called a *d-complement submodule* of M if N is a d-complement for some submodule K of M. Miyashita (1966) called a module M *perfect* if for every pair of submodules N and K of M with $M = N + K$, K contains a d-complement of N. M is called *continuous* (See Utumi (1965)) if it satisfies the following conditions:

(a) Every submodule of M is large in some summand of M,

(b) If a submodule A of M is isomorphic to a summand of M, then A is a summand of M.

A ring R is called (*right*) *perfect* (resp. *semi-perfect*) if every R-module (resp. cyclic R-module) has a projective cover.

2.

In this section characterizations of perfect or semi-perfect rings in terms of d-continuous modules, are established. The following two lemmas are well known:

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LEMMA 2.1. *Let M and N be modules. Let $\phi: M \to N$ be an epimorphism such that* $\operatorname{Ker}\phi$ *is a fully invariant submodule of M. Then, if A is a summand of M, $\phi(A)$ is a summand of N.*

LEMMA 2.2. *Let $\phi: M \to N$ be module homomorphism. If A and B are submodules of M, then $A \underset{S}{\subseteq} B$ implies $\phi(A) \underset{S}{\subseteq} \phi(B)$.*

THEOREM 2.3. *A ring R is (semi-)perfect if and only if every (finitely generated) quasi-projective R-module satisfies the following conditions:*

(I) *For every submodule A of M, $M = M_1 \oplus M_2$ such that $M_1 \subset A$ and $(M_2 \cap A) \underset{S}{\subseteq} M_2$.*

(II) *Every exact sequence $M \to M' \to 0$, with M' a summand of M, splits.*

PROOF. Let M be a quasi-projective module and let $M \xrightarrow{f} M' \to 0$ be an exact sequence with M' a summand of M. Let e denote the natural projection of M onto M'. Since M is quasi-projective, there exists a homomorphism $g: M \to M$ such that $fg = e$. But then $fge = e$, and so the sequence $M \xrightarrow{f} M' \to 0$ splits. This shows that every quasi-projective module (without any condition on the ring) satisfies condition (II).

Now, assume that R is (semi-)perfect and let M be a (finitely generated) quasi-projective R-module. Let A be a submodule of M and let

$$Q \xrightarrow{\phi} M \to 0 \quad \text{and} \quad P \xrightarrow{\theta} M/A \to 0$$

be projective covers. Let π denote the natural projection of M onto M/A. We have the row exact diagram

$$\begin{tikzcd} Q \arrow[r, "\phi"] & M \arrow[r] \arrow[d, "\pi"] & 0 \\ P \arrow[r, "\theta"] & M/A \arrow[r] & 0 \end{tikzcd}$$

Since π is onto, we get a splitting epimorphism $h: Q \to P$. Thus

$$Q = Q_1 \oplus Q_2 \quad \text{with} \quad Q_1 = \operatorname{Ker} h \quad \text{and} \quad Q_2 \cong P.$$

Further $Q_1 \subset \operatorname{Ker}\pi\phi$ and $(Q_2 \cap \operatorname{Ker}\pi\phi) \underset{S}{\subseteq} Q_2$. Let $\phi(Q_i) = M_i$, $i = 1, 2$. Since M is quasi-projective, $\operatorname{Ker}\phi$ is a fully invariant submodule of Q (Wu and Jans (1967), Proposition 2.2). Then we get by Lemma 2.1, that $M = M_1 \oplus M_2$. Since $Q_1 \subset \operatorname{Ker}\pi\phi$,

$$M_1 = \phi(Q_1) \subset \operatorname{Ker} \pi = A.$$

Also we have $(M_2 \cap A) \subset \phi(\operatorname{Ker} \pi\phi \cap Q_2)$. Then since $(Q_2 \cap \operatorname{Ker} \pi\phi) \underset{S}{\subsetneq} Q_2$, we get by Lemma 2.2,

$$(M_2 \cap A) \subset \phi(\operatorname{Ker} \pi\phi \cap Q_2) \underset{S}{\subsetneq} \phi(Q_2) = M_2.$$

Hence M satisfies condition (I).

Conversely, assume that every (finitely generated) quasi-projective module satisfies condition (I). Let M be any (finitely generated) R-module. There exists an epimorphism $F \xrightarrow{\alpha} M$ where F is a (finitely generated) free R-module. As F satisfies condition (I), $F = F_1 \oplus F_2$, where

$$F_1 \subset \operatorname{Ker} \alpha \quad \text{and} \quad (F_2 \cap \operatorname{Ker} \alpha) \underset{S}{\subsetneq} F_2.$$

Let $\bar{\alpha}$ denote the restriction of α to F_2. Then it is obvious that $F_2 \xrightarrow{\bar{\alpha}} M \to 0$ is a projective cover. Thus R is (semi-)perfect.

The proof of the above Theorem yields the following

COROLLARY 2.4. *A ring R is semi-perfect if and only if R_R satisfies condition* (I).

Conditions (I) and (II) were found to be the dual to the analogous conditions for continuous rings studied by Utumi (1965) and continuous modules studied by Mohamed and Bouhy (1977). So we give the following

DEFINITION 2.5. A module M is called *dual continuous* (for short d-continuous) if M satisfies conditions (I) and (II) stated in Theorem 2.3.

Theorem 2.3 raises the question whether d-continuous modules over perfect rings are quasi-projective. The following example rules out this possibility.

EXAMPLE 2.6. Let K be a Galois field having a proper subfield F. Consider the matrix ring

$$R = \begin{bmatrix} K & K \\ 0 & F \end{bmatrix}.$$

Let

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Let V be any proper subspace of K_F, and let $M = e_{11}R/e_{12}V$. Then we find that $e_{11}R \to M \to 0$ is a projective cover. But since $e_{12}V \neq e_{11}A$ for any ideal A of R, M is not quasi-projective (Wu and Jans (1967), Theorem 3.1).

However, M is obviously d-continuous. Thus M is an indecomposable d-continuous module over an artinian ring R, which is not quasi-projective.

By Mohamed and Bouhy (1977), every quasi-injective module is continuous. The dual of this result is not, in general true. This can be easily seen by Corollary 2.4. In fact Z_Z, where Z is the ring of integers, is a projective module which is not d-continuous.

3.

In this section some general results on d-continuous modules are proved. We show that any d-continuous module is perfect in the sense of Miyashita (1966). Further the structure of the endomorphism ring of a d-continuous module is given.

The following Lemma is an immediate consequence of condition (I).

LEMMA 3.1. *Let M be a module with condition* (I). *Then every submodule A of M is of the form $A = N \oplus S$, where N is a summand of M and $S \underset{S}{\subset} M$.*

LEMMA 3.2. *Let M be a module with condition* (II). *If A and B are summands, then any exact sequence $A \xrightarrow{f} B \to 0$ splits. Any summand of M satisfies condition* (II).

PROOF. Let $M = A \oplus A'$. Then

$$B \cong A/\mathrm{Ker}\, f \cong (A \oplus A')/(\mathrm{Ker}\, f \oplus A') = M/(\mathrm{Ker}\, f \oplus A').$$

Then $\mathrm{Ker}\, f \oplus A'$ is a summand of M. Hence $\mathrm{Ker}\, f$ is a summand of A. The other part is obvious.

Hence we have

THEOREM 3.3. *If every finitely generated R-module is d-continuous, then R is semisimple artinian.*

PROOF. Let A be a right ideal of R and let $M = R/A$. The right R-module $R \oplus M$ is finitely generated, hence d-continuous. Then A is a summand of R by the above Lemma. Hence R is completely reducible.

REMARK. If every cyclic R-module is d-continuous, R need not be semisimple, as an example consider $Z/(4)$, where Z is the ring of integers.

The following is due to Miyashita (1966).

LEMMA 3.4. *Let $M = A + B$ where A and B are submodules of M. Then*

(i) *B is a d-complement of A if and only if $(B \cap A) \underset{S}{\subset} B$.*

(ii) *If B is a d-complement of A and if N is a small submodule of M, then $(N \cap B) \underset{S}{\subset} B$.*

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PROPOSITION 3.5. *A summand of a d-continuous module is d-continuous.*

PROOF. Let A be a summand of a d-continuous module M. A satisfies condition (II) by Lemma 3.2. Let N be a submodule of A. Since M is d-continuous, $M = M_1 \oplus M_2$ where $M_1 \subset N$ and $(M_2 \cap N) \underset{S}{\subset} M_2$. Now

$$A = A \cap (M_1 \oplus M_2) = M_1 \oplus (A \cap M_2).$$

Since $A \cap M_2$ is a summand of M and $(M_2 \cap N) \underset{S}{\subset} M$, we get by the above lemma $[(A \cap M_2) \cap (M_2 \cap N)] \underset{S}{\subset} (A \cap M_2)$, that is $[(A \cap M_2) \cap N] \underset{S}{\subset} (A \cap M_2)$. Hence A satisfies condition (I). Therefore A is d-continuous.

LEMMA 3.6. *Let M be a module with condition* (II). *If A, B and A + B are summands of M, then A ∩ B is a summand of M; further A + B = A ⊕ B′ for some summand B′ of B.*

PROOF. Let $N = A + B$. Then N satisfies condition (II) by Lemma 3.2. Also A and B are summands of N. Let $N = B \oplus C$ for some submodule C of N. Now

$$C \cong N/B = (A + B)/B \cong A/(A \cap B).$$

Hence $A \cap B$ is a summand of A by Lemma 3.2, and hence a summand of M. Let $B = (A \cap B) \oplus B'$. Then $A + B = A \oplus B'$.

PROPOSITION 3.7. *A d-continuous module M is perfect (in the sense of Miyashita), and every d-complement submodule of M is a summand.*

PROOF. Let $M = N + L$ for submodules N and L of M. We will show that L contains a d-complement of N. By condition (I)

$$M = A \oplus C; A \subset N \quad \text{and} \quad (N \cap C) \underset{S}{\subset} C.$$

Thus $N = A \oplus (N \cap C)$. Also by Lemma 3.1, $L = B \oplus S$, where B is a summand of M and $S \underset{S}{\subset} M$. So that

$$M = N + L = A + B + (N \cap C) + S.$$

As $(N \cap C) + S$ is small in M, we get $M = A + B$. Then by Lemma 3.6,

$$M = A \oplus B'$$

for some submodule $B' \subset B \subset L$. Define $\phi : C \to B'$ as follows: given $c \in C$, write $c = a + b'$ with $a \in A$ and $b' \in B'$, then let $\phi(c) = b'$. Straightforward calculations give

$$\phi(N \cap C) = N \cap B'.$$

But since $N \cap C \underset{S}{\subset} C$, we get by Lemma 2.2 that $N \cap B' \underset{S}{\subset} B'$. Now since

$M = N + B'$, B' is a d-complement of N by Lemma 3.4. This proves that M is a perfect module.

Let D be a d-complement of a submodule K. By Lemma 3.1 $D = D' \oplus S$, where D' is a summand of M and $S \underset{S}{\subseteq} M$. Now,

$$M = K + D = K + D' + S = K + D'$$

Then minimality of D implies that $D = D'$. This completes the proof.

REMARK. Let R be a perfect ring which is not artinian. By Miyashita (1966) every R-module is perfect. Then in view of Theorem 3.3, a perfect R-module need not be d-continuous.

COROLLARY 3.8. *If M is d-continuous, then $M/\operatorname{Rad} M$ is completely reducible.*

PROOF. M is perfect by the above proposition. Then the result follows by Miyashita (1966), Proposition 1.13. However, the corollary is also a consequence of Lemmas 2.1 and 3.1.

COROLLARY 3.9. *Let M_1 be a summand of a d-continuous module M. If M_2 is a d-complement of M_1, then $M = M_1 \oplus M_2$.*

PROOF. By Proposition 3.7, M_2 is a summand of M. Since $M = M_1 + M_2$, $M_1 \cap M_2$ is a summand of M by Lemma 3.6. However, $(M_1 \cap M_2) \underset{S}{\subseteq} M_2$ by Lemma 3.4, and so $M_1 \cap M_2 = 0$.

THEOREM 3.10. *Let M be a d-continuous module. Let $H = \operatorname{Hom}_R(M, M)$ and J denote the Jacobson radical of the ring H. Then*

(i) *H/J is a (von Neumann) regular ring.*
(ii) $J = \{h \in H : \operatorname{Im} h \underset{S}{\subseteq} M\}$.
(iii) *Idempotents modulo J can be lifted.*

PROOF. Let $I = \{h \in H : \operatorname{Im} h \underset{S}{\subseteq} M\}$. It is easy to check that I is an ideal of H. Let $\lambda \in I$. Then $\operatorname{Im} \lambda \underset{S}{\subseteq} M$. Since

$$\operatorname{Im} \lambda + \operatorname{Im}(1 + \lambda) = M,$$

we get $\operatorname{Im}(1 + \lambda) = M$. Then by Lemma 3.2,

$$M = \operatorname{Ker}(1 + \lambda) \oplus M'$$

for some submodule M' of M. So that $(1 + \lambda)$ is right invertible. Since I is an ideal, $\lambda \in J$ and hence $I \subset J$.

Let h be an arbitrary element in H. Then by condition (I), there exists an idempotent $e \in H$ such that

$$eM \subset hM \quad \text{and} \quad [(1-e)M \cap hM] \underset{S}{\subset} (1-e)M.$$

Hence $eh : M \to eM$ is an epimorphism. Again by Lemma 3.2, $\mathrm{Ker}(eh)$ is a summand of M. Write $M = \mathrm{Ker}(eh) \oplus T$ for some submodule T of M. Then the restriction of eh to T is an isomorphism onto eM. As eM is a summand, the inverse isomorphism of eM onto T may be extended to an element $\theta \in H$. Hence $\theta eh = 1_T$. Then for every $t \in T$

$$(h - h\theta eh)(t) = h(t) - h(\theta eh(t)) = h(t) - h(t) = 0.$$

And for every $x \in \mathrm{Ker}(eh)$

$$(h - h\theta eh)(x) = h(x).$$

This proves that

$$\mathrm{Im}(h - h\theta eh) \subset h(\mathrm{Ker}(eh)).$$

Now $h(\mathrm{Ker}(eh)) \subset [(1-e)M \cap hM] \underset{S}{\subset} M$. Thus $\mathrm{Im}(h - h\theta eh) \underset{S}{\subset} M$, and hence $(h - h\theta eh) \in I$. This shows that H/I is a regular ring, so $J \subset I$. Therefore $J = I$. This proves (i) and (ii).

Let a be an idempotent modulo J. Then by (i), $(a - a^2)M \underset{S}{\subset} M$. Now $(a - a^2)M = aM \cap (1-a)M$ and $M = aM + (1-a)M$. Then by 3.7 and 3.9, there exist orthogonal idempotents g and f of H such that

$$gM \subset aM, \quad fM \subset (1-a)M \quad \text{and} \quad gM \oplus fM = M.$$

Since $fM \subset (1-a)M$, $afM \subset (a - a^2)M \subset (1-a)M$. Thus, for every $m \in M$

$$(g - a)m = (g - a)(gm + fm) = gm - agm - afm = (1-a)gm - afm.$$

Hence $(g - a)M \subset (1-a)M$. Also $gM \subset aM$ implies that $(g - a)M \subset aM$. So that

$$(g - a)M \subset [aM \cap (1-a)M] = (a - a^2)M \underset{S}{\subset} M.$$

Therefore $(g - a) \in J$. This completes the proof.

COROLLARY 3.11. *Let M be a d-continuous module. Then M is indecomposable if and only if* $\mathrm{Hom}_R(M, M)$ *is a local ring.*

4.

The main purpose of this section is to prove a decomposition theorem for d-continuous modules (Theorem 4.7). We also show that all known decomposition theorems for quasi-projective modules over perfect rings are corollaries to this theorem.

PROPOSITION 4.1. *If $M \oplus N$ is a d-continuous module, then for every submodule A of N any homomorphism $\phi : M \to N/A$ lifts to a homomorphism $\psi : M \to N$.*

PROOF. Let $L = M \oplus N$. Define $\bar{\phi} : L \to N/A$ by

$$\bar{\phi}(m+n) = \phi(m) + \pi(n)$$

where π is the natural homomorphism of N onto N/A. Let $K = \operatorname{Ker} \bar{\phi}$. Given $m \in M$, $\phi(m) = \pi(n)$ for some $n \in N$. Thus

$$\bar{\phi}(m-n) = \phi(m) - \pi(n) = 0.$$

Hence $(m-n) \in K$, and therefore $M \subset (K+N)$. Hence $L = K + N$. By Proposition 3.7, K contains a d-complement P of N. Then

$$L = P \oplus N$$

by Corollary 3.9. Given $m \in M$, $m = p + n$ with $p \in P$ and $n \in N$, define $\psi : M \to N$ by $\psi(m) = n$. Then

$$\phi(m) = \bar{\phi}(m) = \bar{\phi}(p+n) = \bar{\phi}(n) = \pi(n) = \pi\psi(m).$$

Hence we have

COROLLARY 4.2. *If $M \times M$ is d-continuous, then M is quasi-projective.*

REMARK. The above proposition along with Example 2.6 shows that a direct sum of d-continuous modules need not be d-continuous.

LEMMA 4.3. *If M is an indecomposable d-continuous module, then every proper submodule of M is small in M; further if* $\operatorname{Rad} M \neq M$, *then M is cyclic.*

PROOF. The result follows by Lemma 3.1.

The following is an example of an indecomposable d-continuous module which coincides with its radical, and is not quasi-projective.

EXAMPLE 4.4. Let p be a prime number. Consider the discrete valuation ring $Z_{(p)} = \{a/b : a, b$ integers such that $b \neq 0$ and p does not divide $b\}$. Since $Z_{(p)}$ is not complete, the field Q of rational numbers is not quasi-projective as $Z_{(p)}$-module. Every proper $Z_{(p)}$-submodule of Q is of the form $p^n Z_{(p)}$ where n is an integer, and hence is small in Q. Consequently $\operatorname{Rad}(Q) = Q$ as a $Z_{(p)}$-module. Q is obviously an indecomposable d-continuous $Z_{(p)}$-module.

Now, we prove

PROPOSITION 4.5. *Let M be a d-continuous module. Let A be a cyclic indecomposable summand and let M_1 be a finitely generated summand of M.*

Then either $M_1 + A$ is a summand of M or $M_1 + A = M_1 \oplus yR$ with $y \in \operatorname{Rad} M$ and A is isomorphic to a summand of M_1.

PROOF. If $A \subset M_1$, then $M_1 + A = M_1$ and we have nothing to prove. So assume that $A \not\subset M_1$. Let $M = M_1 \oplus M_2$ for some submodule M_2 of M. Then

$$M_1 + A = M_1 \oplus [(M_1 + A) \cap M_2].$$

So that

$$(M_1 + A) \cap M_2 \cong (M_1 + A)/M_1 \cong A/(A \cap M_1).$$

Hence there is an epimorphism $\phi: A \to (M_1 + A) \cap M_2$. Let $x \in [(M_1 + A) \cap \operatorname{Rad} M]$, and write $x = x_1 + x_2$, with $x_1 \in M_1$ and $x_2 \in [(M_1 + A) \cap M_2]$. Then it is clear that $x_i \in \operatorname{Rad} M_i$, $i = 1, 2$. Let $x_2 = \phi(a)$ for some $a \in A$. If $a \in \operatorname{Rad} A$, then $x_2 = \phi(a) \in \operatorname{Rad}[(M_1 + A) \cap M_2] \subset \operatorname{Rad}(M_1 + A)$. Hence

$$x = x_1 + x_2 \in [\operatorname{Rad} M_1 + \operatorname{Rad}(M_1 + A)] = \operatorname{Rad}(M_1 + A).$$

Now we consider two cases:

(i) For every $x \in [(M_1 + A) \cap \operatorname{Rad} M]$, $x = x_1 + \phi(a)$ with $a \in \operatorname{Rad} A$. Then

$$[(M_1 + A) \cap \operatorname{Rad} M] \subset \operatorname{Rad}(M_1 + A) \subset [(M_1 + A) \cap \operatorname{Rad} M].$$

Hence $\operatorname{Rad}(M_1 + A) = [(M_1 + A) \cap \operatorname{Rad} M]$. Now by Lemma 3.1, $M_1 + A = P + S$, where P is a summand of M and $S \underset{S}{\subseteq} M$. Then

$$S \subset [(M_1 + A) \cap \operatorname{Rad} M] = \operatorname{Rad}(M_1 + A).$$

As $M_1 + A$ is finitely generated, $S \underset{S}{\subseteq} (M_1 + A)$. So that $M_1 + A = P$, a summand of M.

(ii) For some $x \in [M_1 + A) \cap \operatorname{Rad} M]$, $x = x_1 + \phi(a)$, with $a \notin \operatorname{Rad} A$. Then by Lemma 4.3, $aR = A$. Hence

$$(M_1 + A) \cap M_2 = \phi(A) = \phi(a)R.$$

Let $y = \phi(a)$. Then $y \in \operatorname{Rad} M_2 \subset \operatorname{Rad} M$, and

$$M_1 + A = M_1 \oplus [(M_1 + A) \cap M_2] = M_1 \oplus yR.$$

Let $M = A \oplus B$ for some submodule B of M. Then

$$M = A \oplus B = M_1 + A + B = M_1 + yR + B = (M_1 + B) + yR.$$

Since $yR \underset{S}{\subseteq} M$, $M = M_1 + B$. Hence

$$A \cong M/B = (M_1 + B)/B = M_1/(M_1 \cap B).$$

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As M_1 and B are summands of M, $M_1 \cap B$ is a summand of M by Lemma 3.6. This completes the proof.

COROLLARY 4.6. *Let A and B be nonzero cyclic indecomposable summands of a d-continuous module M. If $A \cap B \neq 0$, then $A \cong B$.*

PROOF. We may assume that $A \neq B$. So $A \cap B \neq A$ and hence $(A \cap B) \underset{s}{\subset} A$ by Lemma 4.3. By the above theorem, either $A + B$ is a summand or $A \cong B$. If $A + B$ is a summand, then by Lemma 3.6, $A \cap B$ is also a summand of M. But then $A \cap B = 0$, a contradiction. This completes the proof.

THEOREM 4.7. *Let M be a d-continuous module. Then $M = N + N'$ where N' is a summand of M with* $\operatorname{Rad} N' = N'$ *and $N = \sum_{i \in I} \oplus A_i$ where A_i is cyclic indecomposable, the sum of any finitely many A_i's is a summand of M, and if A_i is not quasi-projective, then A_i is not isomorphic to A_j for $i \neq j \in I$.*

PROOF. If $\operatorname{Rad} M = M$, we have nothing to prove. So assume that $\operatorname{Rad} M \neq M$. Hence M contains a maximal submodule T. By Proposition 3.7, T has a d-complement T'. Obviously, T' is indecomposable and $\operatorname{Rad} T' \neq T'$. Then by Lemma 4.3, T' is a cyclic module. Thus M has nonzero cyclic indecomposable summands. Let $\{A_\lambda : \lambda \in \Lambda\}$ be the set of all cyclic indecomposable summands of M. By Zorn's Lemma we can find a maximal subset K of Λ with the property that $\sum_{j \in J} A_j$ is a summand of M for any finite subset J of K. Let $N = \sum_{k \in K} A_k$. Again by Proposition 3.7, N has a d-complement N' which is a summand of M. We claim that $\operatorname{Rad} N' = N'$. Suppose not. Since N' is a d-continuous module by Proposition 3.5, then the above argument shows that N' contains a nonzero cyclic indecomposable summand A. Thus A is also a summand of M. Since N' is a d-complement of N, $(N \cap N') \underset{s}{\subset} M$ by Lemma 3.4. Thus $A \not\subset N$ and so $A \neq A_k$ for any $k \in K$. Maximality of K then implies the existence of a finite subset J of K such that $(\sum_{j \in J} A_j) + A$ is not a summand. Hence by Proposition 4.5, we get

$$\left(\sum_{j \in J} A_j\right) + A = \left(\sum_{j \in J} A_j\right) \oplus yR$$

with $y \in \operatorname{Rad} M$. Hence $N + A = N + yR$. Now $N' = A \oplus B$ for some submodule B of N'. Then

$$M = N + N' = N + A + B = N + yR + B = (N + B) + yR.$$

Since $yR \underset{s}{\subset} M$, $M = N + B$. But then minimality of N' implies $N' = B$. Hence $A = 0$, a contradiction. Thus $\operatorname{Rad} N' = N'$.

Now, we show that N is a direct sum of a subfamily of $\{A_k : k \in K\}$. By

Zorn's Lemma, we can find a maximal subset I of K such that $\Sigma_{i\in I} A_i$ is direct. Suppose that for some $k \in K$, $A_k \not\subset (\Sigma_{i\in I} A_i)$. Let J be a finite subset of I. Then $[A_k \cap (\Sigma_{j\in J} A_j)] \subsetneq A_k$. So that $[A_k \cap (\Sigma_{j\in J} A_j)] \underset{S}{\subset} M$ by Lemma 4.3. By our choice of K, $(\Sigma_{j\in J} A_j)$ and $A_k + (\Sigma_{j\in J} A_j)$ are summands of M. So that $A_k \cap (\Sigma_{j\in J} A_j)$ is a summand of M. Hence $A_k \cap (\Sigma_{j\in J} A_j) = 0$. This shows that $A_k + (\Sigma_{i\in I} \oplus A_i)$ is direct, which is a contradiction to the maximality of I. Hence $A_k \subset (\Sigma_{i\in I} \oplus A_i)$ and so $N = \Sigma_{i\in I} \oplus A_i$.

The last statement of the Theorem follows by Corollary 4.2.

Corollary 4.8. *If M is a d-continuous module such that every proper submodule is contained in a maximal submodule, then M is a direct sum of cyclic indecomposable modules.*

Proof. Using the notation in Theorem 4.7, we find that $N' = 0$.

Corollary 4.9. *If M is a d-continuous finitely generated module, then M is a direct sum of cyclic indecomposable modules; moreover* $\mathrm{Hom}_R(M, M)$ *is a semiperfect ring.*

Proof. Let $H = \mathrm{Hom}_R(M, M)$. Then H has a bounded number of orthogonal indecomposable idempotents. Then by Theorem 3.10, $H/\mathrm{Rad}\, H$ is semisimple artinian. The result now follows by Bass (1960).

Corollary 4.10. *A projective d-continuous module is a direct sum of cyclic indecomposable modules.*

Proof. Using the notation in Theorem 4.7, N' is a projective module. Thus $\mathrm{Rad}\, N' = N'$ implies that $N' = 0$ by Bass (1960).

Corollary 4.11. *Let R be a perfect ring. Then every quasi-projective R-module M is a direct sum of cyclic indecomposable modules.*

Proof. By Theorem 2.3, M is d-continuous. Also by Bass (1960) $\mathrm{Rad}\, B \neq B$ for every nonzero R-module B. The result now follows by Theorem 4.7.

5.

In this section we determine the structure of d-continuous modules over some special rings. We start with the following

Proposition 5.1. *Let M be any module and A, B be two small submodules of M such that $M/A \oplus M/B$ is d-continuous, then $M/A \cong M/B$.*

Proof. Let $\phi: M/A \to M/(A+B)$, *and* $\pi: M/B \to M/(A+B)$ be natural homomorphisms. By Proposition 4.1 there exists a homomorphism

$\eta : M/A \to M/B$ such that $\phi = \pi\eta$. Then $\operatorname{Im}\eta + \operatorname{Ker}\pi = M/B$ and $\operatorname{Ker}\pi \subset_{s} M/B$ give $\operatorname{Im}\eta = M/B$, so that η is an epimorphism and by Lemma 3.2, it splits. However $\operatorname{Ker}\eta \subset (A+B)/A \subset_{s} M/A$. Hence $\operatorname{Ker}\eta = 0$. This proves that $M/A \cong M/B$.

The following corollary is an immediate consequence:

COROLLARY 5.2. *If $M_1 \oplus M_2$ is d-continuous and if M_1 and M_2 have isomorphic projective covers, then $M_1 \cong M_2$.*

LEMMA 5.3. *Let R be a right perfect ring e_1, e_2 be two indecomposable idempotents of R such that $e_1R/e_1A_1 \oplus e_2R/e_2A_2$ is d-continuous for some right ideals A_1, A_2 of R, then there exists an ideal B of R such that $e_1B = e_1Re_1A_1$ and $e_2B = e_2Re_2A_2$.*

PROOF. Without loss of generality we can suppose that $A_1 \subset \operatorname{Rad} R$, $A_2 \subset \operatorname{Rad} R$. Consider any $e_2xe_1 \in e_2Re_1$. Define

$$\phi : e_1R/e_1A_1 \to e_2R/(e_2A_2 + e_2xe_1A_1)$$

by

$$\phi(e_1r + e_1A_1) = e_2xe_1r + (e_2A_2 + e_2xe_1A_1).$$

Let $\pi : e_2R/e_2A_2 \to e_2R/(e_2A_2 + e_2xe_1A_1)$ be the natural homomorphism. By Proposition 4.1 there exists a homomorphism $\eta : e_1R/e_1A_1 \to e_2R/e_2A_2$ such that $\phi = \pi\eta$. Let

$$\eta(e_1 + e_1A_1) = e_2ye_1 + e_2A_2.$$

Then $(e_2xe_1 - e_2ye_1) \in (e_2A_2 + e_2xe_1A_1)$ and $e_2ye_1A_1 \subset e_2A_2$. So for some $a_1 \in A_1$ and $a_2 \in A_2$, $e_2xe_1(1+a_1) = e_2ye_1 + e_2a_2$. Hence

$$e_2xe_1(1+a_1)A_1 \subset e_2ye_1A_1 + e_2a_2A_1 \subset e_2A_2.$$

As A_1 is quasi-regular $(1+a_1)A_1 = A_1$. Therefore $e_2xe_1A_1 \subset e_2A_2$ and hence $e_2Re_1A_1 \subset e_2A_2$. Similarly $e_1Re_2A_2 \subset e_1A_1$. Let $B = Re_1A_1 + Re_2A_2$ then B is an ideal of R such that $e_1B = e_1Re_1A_1$, $e_2B = e_2Re_2A_2$.

THEOREM 5.4. *Let M be a d-continuous module over a perfect ring R. Then $M = N \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_t$ such that*

(a) *N is quasi-projective.*

(b) *$N_1, N_2, \cdots, N_t$ are indecomposable and d-continuous, such that none of N_i is quasi-projective, for any $j \neq i$ the projective covers of N_i and N_j are non-isomorphic, further given any indecomposable summand K of N, N_i and K are non-isomorphic.*

(c) *There exist right ideals $A_1, A_2, \cdots, A_t$ of R, an ideal B of R, and* *indecomposable idempotents*, $e_1, e_2, \cdots, e_t$ *with the following properties*:

(i) $N_i \cong e_iR/e_iA_i$, $e_iRe_iA_i = e_iB$.

(ii) *For any indecomposable summand* K *of* N, $K \cong eR/eB$ *for some indecomposable idempotent* e *of* R.

PROOF. By Corollary 4.8

$$M = \sum_{\alpha \in \Lambda} \oplus N_\alpha$$

where N_α are indecomposable d-continuous modules. By Corollary 5.2, if any two A_α's have isomorphic projective covers, then these A_α's are isomorphic and quasi-projective. Consequently, using Koehler (1971), Theorem 1.10, we can write

$$M = N \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_t$$

where N is quasi-projective and N_i satisfy (b).

Since R is perfect, $N_i \cong e_iR/e_iA_i$ for some right ideal A_i of R. By Koehler (1971), there exist finitely many indecomposable orthogonal idempotents $f_1, f_2, \cdots, f_u$ and an ideal C of R, such that $f_1R/f_1C \oplus \cdots \oplus f_uR/f_uC$ is isomorphic to a summand of N and any indecomposable summand K of N is isomorphic to f_jR/f_jC for some $j = 1, 2, \cdots, u$. Now

$$f_1R/f_1C \oplus \cdots \oplus f_uR/f_uC \oplus e_1R/e_1A_1 \oplus \cdots \oplus e_tR/e_tA_t$$

being isomorphic to a summand of M, is d-continuous by Proposition 3.5. Lemma 5.3 shows that there exists an ideal B of R such that

$$f_iC = f_iB \quad \text{for} \quad i = 1, 2, \cdots, u$$

and $e_jB = e_jRe_jA_j$ for $j = 1, 2, \cdots, t$. This proves (c).

After this we determine the structure of a d-continuous torsion abelian group. We prove the following:

THEOREM 5.5. *Any torsion abelian group is* d*-continuous if and only if it is quasi-projective.*

PROOF. Let G be a torsion abelian group. Without loss of generality we can suppose that G is a p-group for some prime number p. If G is quasi-projective, then by Fuchs and Rangaswamy (1970) there exists a positive integer n, such that G is a direct sum of copies of $Z/(p^n)$. Consequently G is a quasi-projective module over the artinian ring $Z/(p^n)$ and hence by Theorem 2.3, G is d-continuous.

Conversely let G be d-continuous. Consider the quasi-cyclic p-group (Fuchs (1960)) which we denote by Z_{p^∞}. For any proper subgroup K of Z_{p^∞},

$Z_{p^\infty} \cong Z_{p^\infty}/K$. But K is not a summand of Z_{p^∞}, hence Z_{p^∞} is not d-continuous. Consequently G is a reduced abelian group. By Fuchs (1960), Corollary 24.3,

$$G = G_1 \oplus C_1$$

where C_1 is a cyclic group of order say p^n. Again G_1 being d-continuous gives $G_1 = G_2 \oplus C_2$ where C_2 is a cyclic group of order say p^m. We claim $m = n$. On the other hand let $m > n$. By Proposition 3.5

$$Z/(P^m) \oplus Z/(P^n) \cong C_1 \oplus C_2$$

is d-continuous. Now $(P^n)/(P^m)$ is such that $Z/(P^m)/((P^n)/(P^m)) \cong Z/(P^n)$, but $(P^n)/(P^m)$ is not a summand of $Z/(P^m)$. This is a contradiction. Consequently $n = m$. Then by Fuchs (1960), Theorem 24.5, we get G is a direct sum of copies of $Z/(P^n)$. Consequently, by Fuchs and Rangaswamy (1970), G is quasi-projective. This proves the theorem.

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