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ALGEBRAS STABLY EQUIVALENT TO NAKAYAMA ALGEBRAS OF LOEWY LENGTH AT MOST 4

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Abstract

Two artin algebras Λ and Λ' are said to be stably equivalent if their categories of finitely generated modules modulo projectives are equivalent. In this paper a characterization is given of the artin algebras stably equivalent to Nakayama algebras of Loewy length (at most) four. The proof is an illustration of the technique of using irreducible maps to study problems about stable equivalence.

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Introduction

The purpose of this note is to give a classification of the artin algebras stably equivalent to a Nakayama algebra of Loewy length at most 4, as was announced in Reiten (1977). Before we state our main result, we shall recall some definitions.

An artin algebra Λ is Nakayama (that is, generalized uniserial) if each indecomposable left and right projective Λ-module has a unique composition series. Let mod Λ denote the category of finitely generated left Λ modules. Two artin algebras Λ and Λ' are said to be stably equivalent if the categories $\underline{\text{mod}}\ \Lambda$ and $\underline{\text{mod}}\ \Lambda'$ modulo projectives are equivalent categories. Here the objects of $\underline{\text{mod}}\ \Lambda$ are the same as those of mod Λ, denoted by $\underline{M}$ for an M in mod Λ, and $\text{Hom}(\underline{M}, \underline{N}) = \text{Hom}_\Lambda(M, N)/P(M, N)$, where $P(M, N)$ is the subgroup whose elements are the $f\colon M \to N$ which factor through a projective Λ-module. We denote the image of f in $\text{Hom}(\underline{M}, \underline{N})$ by $\underline{f}$. We further let $\mathbf{r}$ denote the radical of Λ and we denote by $LL(M)$ the Loewy length of a Λ-module M, that is, the smallest $i \geqslant 0$ such that $\mathbf{r}^i M = 0$.

We are now ready to state our classification theorem. We point out that we know from Reiten (1975) that if Λ is stably equivalent to a Nakayama algebra of Loewy length at most 3, then Λ is also Nakayama.

THEOREM. *An indecomposable non-Nakayama algebra Λ is stably equivalent to a Nakayama algebra Λ' of Loewy length (at most) 4 if and only if the following conditions hold.*

The indecomposable projective (injective) Λ-modules which are not uniserial have Loewy length 3, length 4 and simple socle.

The simple Λ-modules $T_1, T_2, \ldots, T_n$, $T_{n+1} = T_1, \ldots$, $n \geq 3$, can be ordered in such a way that no two of the $T_1, \ldots, T_n$ are isomorphic, and such that the following holds.

If P_i, the projective cover of T_i, is not uniserial, then $\operatorname{soc} P_i = T_{i+3}, \mathfrak{r}P_i/\mathfrak{r}^2 P_i \cong T_{i+1} \coprod T_{i+2}$, P_{i+1} is uniserial of length 3, $\operatorname{soc} P_{i+1} = T_{i+4}$, P_{i+2} is uniserial of length 2 or 3. If P_{i+2} has length 3, then $\operatorname{soc} P_{i+2} = T_{i+5}$ and P_{i+3} is not uniserial. If P_{i+2} has length 2, then P_{i+3} is not uniserial or is uniserial of length 3.

If P_i is uniserial of length 3 and T_i is not a summand of any $\mathfrak{r}P_j/\mathfrak{r}^2 P_j$ for P_j not uniserial, then $T_{i+1} = \mathfrak{r}P_i/\mathfrak{r}^2 P_i$, and P_{i+1} is uniserial of length 3 or is not uniserial.

This result contributes to the general problem of understanding what it means for two artin algebras to be stably equivalent. It is known that if Λ is stably equivalent to a self injective algebra, then each indecomposable ring summand of Λ which has Loewy length at least 3 is also self injective (Reiten, 1976). Further, a description is known of the artin algebras stably equivalent to hereditary algebras (Auslander and Reiten (1973, 1975)), and Gabriel and Riedtmann have studied artin algebras stably equivalent to self injective Nakayama algebras over an algebraically closed field.

An important problem in the theory of stable equivalence is whether two stably equivalent algebras have the same number of nonprojective simple modules. In view of the fact that we know that this is the case if one of the algebras is Nakayama (see Reiten, 1978 for a proof of this and a list of other cases where it is known), it is of interest also in connection with this problem to describe the algebras stably equivalent to a Nakayama algebra.

Our proof is also an illustration of the technique of using the irreducible maps introduced in Auslander and Reiten (1977a) to study problems about stable equivalence. We shall devote Section 1 to explaining how we use irreducible maps in this paper together with recalling the necessary background material. It is essentially the method we used in Reiten (1975), which we shall here present from a different point of view. Section 2 is devoted to proving that the conditions are necessary, and Section 3 to proving that they are sufficient.

1. Background and method

Let Λ be an artin algebra. We recall that a map $f\colon A \to B$ in mod Λ is said to be *irreducible* if f is neither a split monomorphism nor a split epimorphism, and given any commutative diagram

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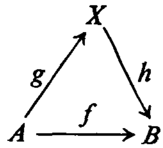

then g is a split monomorphism or h is a split epimorphism (Auslander and Reiten, 1977a). Assume that A and B are indecomposable and not projective. Then we know that $f: A \to B$ is irreducible if and only if $\underline{f}: \underline{A} \to \underline{B}$ is irreducible (Auslander and Reiten, 1977b).

We associate a diagram D with Λ, consisting of the indecomposable objects in mod Λ, together with an arrow $A \to B$ from A to B if there is an irreducible map from A to B, for A and B indecomposable in mod Λ. For an indecomposable nonprojective C in mod Λ, we shall denote by D(C) the part of the diagram where the objects are C and the indecomposable nonprojective objects X such that there is some composition of irreducible maps from X to C which is not zero modulo projectives, and with an arrow $A \to B$ when there is an irreducible map from A to B, for objects A and B in D(C). If we know that there is some nonzero composition of irreducible maps from A to B modulo projectives in D(C), without knowing if there is actually some irreducible map from A to B, we shall indicate this by writing a dotted arrow A ·······> B.

An equivalence F: $\underline{\text{mod}\ \Lambda} \to \underline{\text{mod}\ \Lambda'}$ induces a one–one correspondence between the indecomposable nonprojective objects of mod Λ and mod Λ'. If Λ and Λ' are stably equivalent and C and C' correspond to each other, then we must have that the diagrams D(C) and D(C') correspond to each other. This fact will be the main basis for our method.

We also recall that the category mod Λ modulo injectives, denoted $\overline{\text{mod}\ \Lambda}$, is equivalent to $\underline{\text{mod}\ \Lambda}$ (Auslander and Reiten, 1973). If A and B are indecomposable noninjective modules, then $f: A \to B$ is irreducible if and only if $\bar{f}: \bar{A} \to \bar{B}$ is irreducible (Auslander and Reiten, 1977b). Here the objects in $\overline{\text{mod}\ \Lambda}$ are denoted by $\bar{X}$ for X in mod Λ, and $\bar{f}: \bar{A} \to \bar{B}$ denotes the image of f in Hom $(\bar{A}, \bar{B}) = \text{Hom}_\Lambda(A, B)/I(A, B)$, where I $(A, B$ consists of the maps from A to B which factor through an injective module. We shall denote by D$'$(C) the diagram belonging to an indecomposable noninjective module defined in an analogous way to D(C).

Let now Λ' be an indecomposable Nakayama algebra. It is well known that for every indecomposable Λ'-module C, $C/\mathbf{r}C$ is simple. It is then not hard to see (Auslander and Reiten, 1977a, Section 4) that if A and B are indecomposable in modΛ, then $f: A \to B$ is irreducible if and only if f is an epimorphism with Ker f simple or f is a monomorphism with Im f a maximal submodule of B.

For a Nakayama algebra Λ' of Loewy length 4, we then have the following diagrams associated with the various types of indecomposable nonprojective Λ'-modules, where we assume that Λ' is not local.

(1) Let P be an indecomposable projective Λ'-module of length 4. Then we have

$$\mathrm{D}(P/\mathbf{r}P) = P/\mathbf{r}^3 P \to P/\mathbf{r}^2 P \to P/P\mathbf{r}P;$$

$$\mathrm{D}(P/\mathbf{r}^2 P) = \mathbf{r}P/\mathbf{r}^3 P \begin{array}{c} \nearrow \ \mathbf{r}P/\mathbf{r}^2 P \ \searrow \\ \searrow \ P/\mathbf{r}^3 P \ \nearrow \end{array} P/\mathbf{r}^2 P;$$

$$\mathrm{D}(P/\mathbf{r}^3 P) = \mathbf{r}^2 P/\mathbf{r}^3 P \to \mathbf{r}P/\mathbf{r}^3 P \to P/\mathbf{r}^3 P.$$

(2) If P is indecomposable projective of length 3, we have

$$\mathrm{D}(P/\mathbf{r}P) = P/\mathbf{r}^2 P \to P/\mathbf{r}P;$$
$$\mathrm{D}(P/\mathbf{r}P) = \mathbf{r}P/\mathbf{r}^2 P \to P/\mathbf{r}^2 P.$$

(3) If P is indecomposable projective of length 2, we have

$$\mathrm{D}(P/\mathbf{r}P) = P/\mathbf{r}P.$$

We also point out that if A and B are indecomposable nonprojective Λ'-modules, and $\underline{f}\colon \underline{A}\to\underline{B}$ and $\underline{g}\colon \underline{A}\to\underline{B}$ are not zero, there is some $t\colon A\to A$ such that $\underline{ft} = \underline{g}$ or $\underline{gt} = \underline{f}$ and some $s\colon B\to B$ such that $\underline{sf} = \underline{g}$ or $\underline{sg} = \underline{f}$. We finally point out that we shall often use the easily verified fact that if $f\colon A\to B$ is an epimorphism in mod Λ' and A and B are indecomposable and not projective, then $: \underline{f}\underline{A}\to\underline{B}$ is not zero.

2. Necessity

Throughout this section let Λ be an indecomposable artin algebra stably equivalent to a Nakayama algebra Λ' of Loewy length 4. In this section we shall prove that the conditions stated in the Theorem are necessary.

We denote the simple Λ-modules by T_i and their projective covers by P_i. $X = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ denotes a uniserial module of length 2 with $X/\mathbf{r}X \cong T_1$, $\operatorname{soc} X \cong T_2$. Similarly, $X = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$ denotes a uniserial module of length 3 with $X/\mathbf{r}X \approx T_1$, $\operatorname{soc} X \cong T_3$ and $\mathbf{r}X/\mathbf{r}^2 X \cong T_2$. It will follow from Lemma 1 that X is uniquely determined if it exists.

We denote the simple Λ'-modules by S_i and their projective covers by Q_i, in such a way that if S_i is not projective, then $\mathbf{r}P_i/\mathbf{r}^2 P_i \cong S_{i+1}$. We consider a fixed stable equivalence between mod Λ and mod Λ', and denote by $\leftrightarrow$ the induced correspondence between the indecomposable nonprojective modules over Λ and Λ'.

We start out with the following result which is also proved in Reiten (1975), but we include a proof here for the sake of completeness. Here L denotes the length of a module.

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LEMMA 1. *Let P be an indecomposable projective Λ-module.*

(a) *$L(P) \leqslant 4$ and so if $LL(P) = 4$, then P is uniserial.*

(b) *If $LL(P) = 3$, then $\mathbf{r}^2 P$ is simple and $\mathbf{r}P/\mathbf{r}^2 P$ is either simple or the direct sum of two nonisomorphic simple modules.*

(c) *If $LL(P) = 2$, then P is uniserial.*

(d) *If I is an indecomposable injective Λ-module which is not uniserial, then $LL(I) = 3$, $I/\mathbf{r}I$ is simple and $\mathbf{r}I/\mathbf{r}^2 I$ is either simple or the direct sum of two nonisomorphic simple modules.*

(e) *The indecomposable projective (injective) Λ-modules which are not uniserial of length 2 are injective (projective).*

PROOF. (a) Assume $L(P) \geqslant 5$, and choose a chain of submodules with proper inclusions $(0) = P_0 \subset P_1 \subset P_2 \subset P_3 \subset P_4 \subset P$. Then $P/P_1 \cdots\cdots> P/P_2 \cdots\cdots> P/P_3 \cdots\cdots> P/P_4$ would be part of $\mathrm{D}(P/P_4)$, and this is a contradiction.

(b) If $LL(P) = 3$, we have by (a) that $L(\mathbf{r}^2 P) \leqslant 2$ and $L(\mathbf{r}P/\mathbf{r}^2 P) \leqslant 2$. Assume that $\mathbf{r}^2 P = T_1 \coprod T_2$. If $T_1 \cong T_2$, we would have a commutative diagram

$$\begin{array}{ccc} T_1 & \xrightarrow{\;i\;} & P, \\ \big\downarrow & \nearrow_{j} & \\ T_1 & & \end{array}$$

where $i\colon T_1 \to P$ and $j\colon T_1 \to P$ are inclusion maps, but this is a contradiction. Since then T_1 and T_2 are not isomorphic, we have that

$$\begin{array}{ccccc} P/T_1 & & & & \\ & \searrow & & & \\ & & P/\mathbf{r}^2 P & \dashrightarrow & P/\mathbf{r}P, \\ & \nearrow & & & \\ P/T_2 & & & & \end{array}$$

is part of $\mathrm{D}(P/\mathbf{r}P)$, which is impossible. Hence $L(\mathbf{r}^2 P) = 1$.

(c) Assume $LL(P) = 2$. By (a) we know that $L(\mathbf{r}P) \leqslant 3$. As in (b), we can show that $\mathbf{r}P$ has no repeated summands. Assume first $L(\mathbf{r}P) = 3$, and write

$$\mathbf{r}P = T_1 \coprod T_2 \coprod T_3.$$

We get a contradiction by considering the following part of $\mathrm{D}'(P)$.

$$\begin{array}{ccc} T_1 & \searrow & \\ T_2 & \longrightarrow & P. \\ T_3 & \nearrow & \end{array}$$

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Assume then $L(\mathfrak{r}P) = 2$, and write $\mathfrak{r}P = T_1 \coprod T_2$. Then

$$\begin{array}{ccc} P/T_1 & & \\ & \searrow & \\ & & P/\mathfrak{r}P \\ & \nearrow & \\ P/T_2 & & \end{array}$$

is part of $\mathrm{D}(P/\mathfrak{r}P)$. Since there can be no nonzero composition of maps $P/T_1 \longrightarrow P/T_2 \longrightarrow P/\mathfrak{r}P$ or $P/T_2 \longrightarrow P/T_1 \longrightarrow P/\mathfrak{r}P$, $\mathrm{D}(P/\mathfrak{r}P)$ must be

$$\begin{array}{ccccc} & & P/T_1 & & \\ & \nearrow & & \searrow & \\ C & & & & P/\mathfrak{r}P. \\ & \searrow & & \nearrow & \\ & & P/T_2 & & \end{array}$$

It is easy to see that if $h: X \to P/T_1$ is such that X is indecomposable and $\underline{h}$ is not zero, then h must be an epimorphism. Hence we get $\mathrm{D}(P/T_1) = C \to P/T_1$, and similarly $\mathrm{D}(P/T_2) = C \to P/T_2$. Considering (1), we see that this is a contradiction. Hence we conclude that P is uniserial.

(d) Since Λ and Λ' are stably equivalent, we know that Λ^{op} and Λ'^{op} are also stably equivalent [1]. Further, Λ'^{op} is Nakayama since Λ' is. Hence the indecomposable projective Λ^{op}-modules satisfy (a), (b) and (c). We then use the ordinary duality for artin algebras to finish the proof of (d).

(e) This is a direct consequence of the earlier parts of the lemma.

We shall now go on to investigate the correspondence between the indecomposable nonprojective modules in mod Λ and mod Λ' given by the stable equivalence.

LEMMA 2. *Let P_1 be an indecomposable projective Λ-module which is not uniserial, and let $T_1 = P_1/\mathfrak{r}P_1$. Write $\mathfrak{r}P_1/\mathfrak{r}^2 P_1 = T_2 \coprod T_3$, where T_2 and T_3 are nonisomorphic simple Λ-modules, and write $T_4 = soc\ P_1$. Then we have the following.*

(a) *T_1 corresponds to a Λ'-module of length 2, say $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$, $P_1/soc\, P_1 \leftrightarrow \begin{pmatrix} S_2 \\ S_3 \end{pmatrix}$, $\mathfrak{r}P_1 \leftrightarrow \begin{pmatrix} S_3 \\ S_4 \end{pmatrix}$, and we can choose the numbering of T_2 and T_3 such that $T_2 \leftrightarrow \begin{pmatrix} S_2 \\ S_3 \\ S_4 \end{pmatrix}$, $T_3 \leftrightarrow S_3$, $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \leftrightarrow S_2$ and $\begin{pmatrix} T_1 \\ T_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$. In particular, neither T_2 nor T_3 is isomorphic to T_1.*

(b) *P_2 is uniserial of length 3 and P_3 is uniserial of length 2 or 3.*

(c) *$T_4 \leftrightarrow \begin{pmatrix} S_4 \\ S_5 \end{pmatrix}$ and $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix} \leftrightarrow S_4$. Hence P_4 has length 3 if it is uniserial.*

(d) *If $T_5 = \operatorname{soc} P_2$ and P_4 is not uniserial, then $T_5 \leftrightarrow \begin{pmatrix} S_5 \\ S_6 \\ S_7 \end{pmatrix}$.*

(e) *If P_3 has length 3, then P_4 is not uniserial.*

(f) *T_2 is contained in no other indecomposable module of length 2 than $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$, T_3 in no other than $\begin{pmatrix} T_1 \\ T_3 \end{pmatrix}$, and T_4 in no other than $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$ and $\begin{pmatrix} T_3 \\ T_4 \end{pmatrix}$.*

PROOF. (a) It is not hard to see that we have the diagrams

$$\mathrm{D}(P_1/\operatorname{soc} P_1) = \mathbf{r}P_1 \begin{array}{c} \nearrow T_2 \searrow \\ \searrow T_3 \nearrow \end{array} P_1/\operatorname{soc} P_1,$$

$$\mathrm{D}(T_1) = P_1/\operatorname{soc} P_1 \begin{array}{c} \nearrow \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \searrow \\ \searrow \begin{pmatrix} T_1 \\ T_3 \end{pmatrix} \nearrow \end{array} T_1.$$

All claims in (a) follow by comparing this with the diagrams (1), (2) and (3) in Section 1.

(b) Since $T_2 \leftrightarrow \begin{pmatrix} S_2 \\ S_3 \\ S_4 \end{pmatrix}$, P_2 must be uniserial by (a). And by (1) we know that we have

$$\mathrm{D}(T_2) = A \to B \to T_2.$$

Since we must have $A \leftrightarrow S_4$, $B \leftrightarrow \begin{pmatrix} S_3 \\ S_4 \end{pmatrix}$, it is easy to see that there is no nonzero map modulo projectives from A to an indecomposable module not in the diagram. Since B is not simple, it then follows that $A/\mathbf{r}A$ is simple. We must then have $B \cong \mathbf{r}P_1$ and $A \cong \begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$. This shows that P_2 is uniserial of length 3.

Since $T_3 \leftrightarrow S_3$ and $\mathbf{r}P_1 \leftrightarrow \begin{pmatrix} S_3 \\ S_4 \end{pmatrix}$, we have that $\mathrm{D}(T_3)$ is $C \to \mathbf{r}P_1 \to T_3$ or $\mathbf{r}P_1 \to T_3$. In the first case we must as above have that $C/\mathbf{r}C$ is simple. This shows that P_3 is uniserial of length 2 or 3.

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(c) If P_3 has length 3, consider the diagram

$$\begin{array}{ccccc} & & \begin{pmatrix} T_3 \\ T_4 \end{pmatrix} & & \\ & \nearrow & & \searrow & \\ T_4 & & & & \mathbf{r}P_1. \\ & \searrow & & \nearrow & \\ & & \begin{pmatrix} T_2 \\ T_4 \end{pmatrix} & & \end{array}$$

Since $\mathbf{r}P_1 \leftrightarrow \begin{pmatrix} S_3 \\ S_4 \end{pmatrix}$, we must have $T_4 \leftrightarrow \begin{pmatrix} S_4 \\ S_5 \end{pmatrix}$. Further, $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$ corresponds to either S_4 or $\begin{pmatrix} S_4 \\ S_5 \\ S_6 \end{pmatrix}$. Since there is a nonzero map from $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$ to T_2, and $T_2 \leftrightarrow \begin{pmatrix} S_2 \\ S_3 \\ S_4 \end{pmatrix}$, we must have $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix} \leftrightarrow S_4$.

If P_3 has length 2, we have $\mathrm{D}(T_3) = \mathbf{r}P_1 \to T_3$. Since the projective cover of S_3 must then have length 2 and $\mathbf{r}P_1 \leftrightarrow \begin{pmatrix} S_3 \\ S_4 \end{pmatrix}$, we get $\mathrm{D}(\mathbf{r}P_1) = \begin{pmatrix} T_2 \\ T_4 \end{pmatrix} \to \mathbf{r}P_1$, and hence $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix} \leftrightarrow S_4$.

Considering $\mathrm{D}(T_2)$, we see that there is no nonprojective indecomposable module mapping onto $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$. Hence any nonzero map from an indecomposable nonprojective Λ-module not isomorphic to $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$ has T_4 as image. It follows that $T_4 \leftrightarrow \begin{pmatrix} S_4 \\ S_5 \end{pmatrix}$. Assume that P_4 is uniserial, and assume that

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ \mathrm{D}(T_4) = A & & & & T_4. \\ & \searrow & & \nearrow & \\ & & C & & \end{array}$$

It is then not hard to see, and it is similar to our previous arguments, that $A/\mathbf{r}A$, $B/\mathbf{r}B$, $C/\mathbf{r}C$ would all be isomorphic to T_4, and $\mathrm{D}(T_4)$ could then not be as assumed. We must then have $\mathrm{D}(T_4) = \begin{pmatrix} T_4 \\ T_5 \end{pmatrix} \to T_4$, so that P_4 has length 3.

(d) Let $T_5 = \operatorname{soc} P_2$ and assume that P_4 is not uniserial. We can then write $\mathbf{r}P_4/\mathbf{r}^2 P_4 = T_5 \coprod T_6$, and we know from (a) that T_5 and T_6 correspond to S_6 and $\begin{pmatrix} S_5 \\ S_6 \\ S_7 \end{pmatrix}$, and that $\begin{pmatrix} T_4 \\ T_5 \end{pmatrix}$ and $\begin{pmatrix} T_4 \\ T_6 \end{pmatrix}$ correspond to S_5 and $\begin{pmatrix} S_4 \\ S_5 \\ S_6 \end{pmatrix}$. There is some non-

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zero map modulo projectives $\begin{pmatrix} T_4 \\ T_6 \end{pmatrix} \to \begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$, but no nonzero map $S_5 \to S_4$. Hence we must have $\begin{pmatrix} T_4 \\ T_6 \end{pmatrix} \leftrightarrow \begin{pmatrix} S_4 \\ S_5 \\ S_6 \end{pmatrix}$ and $\begin{pmatrix} T_4 \\ T_5 \end{pmatrix} \leftrightarrow S_5$. It further follows that $T_6 \leftrightarrow S_6$ and $T_5 \leftrightarrow \begin{pmatrix} S_5 \\ S_6 \\ S_7 \end{pmatrix}$.

(e) Assume that P_3 has length 3. We must then have

$$\mathrm{D}(\mathbf{r}P_1) = T_4 \begin{array}{c} \nearrow \begin{pmatrix} T_3 \\ T_4 \end{pmatrix} \searrow \\ \searrow \begin{pmatrix} T_2 \\ T_4 \end{pmatrix} \nearrow \end{array} \mathbf{r}P_1.$$

If $T_4 \cong T_1$, $P_4 \cong P_1$ is not uniserial. Assume then that $T_4 \ncong T_1$, and let $T_6 = \operatorname{soc} P_3$. We then have that $\begin{pmatrix} T_4 \\ T_5 \end{pmatrix}$ is contained in P_2 and $\begin{pmatrix} T_4 \\ T_6 \end{pmatrix}$ in P_3. If $T_5 \cong T_6$, then P_2 and P_3 would both be contained in the injective envelope of T_5, which is a contradiction to the structure of the indecomposable injective Λ-modules. Hence T_5 and T_6 are not isomorphic, so that $\begin{pmatrix} T_4 \\ T_5 \end{pmatrix}$ and $\begin{pmatrix} T_4 \\ T_6 \end{pmatrix}$ are not isomorphic. Hence P_4 is not uniserial.

(f) If T_2 is contained in another indecomposable uniserial Λ-module of length 2 than $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$, then T_2 would be contained in a nonuniserial indecomposable projective Λ-module, and T_2 would by (b) correspond to a uniserial Λ'-module of length 2, which is impossible by (a). The argument that T_3 is contained in no other uniserial module of length 2 than $\begin{pmatrix} T_1 \\ T_3 \end{pmatrix}$ is analogous. T_4 is contained in no other uniserial modules of length 2 than $\begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$ and $\begin{pmatrix} T_3 \\ T_4 \end{pmatrix}$, since the injective envelope of T_4 would otherwise not be of the desired type.

LEMMA. 3. *Assume* $T_i \leftrightarrow \begin{pmatrix} S_i \\ S_{i+1} \end{pmatrix}$, P_i *uniserial, and let* $T_{i+1} = \mathbf{r}P_i/\mathbf{r}^2 P_i$.

(a) P_i *has length* 3, $T_{i+1} \leftrightarrow \begin{pmatrix} S_{i+1} \\ S_{i+2} \end{pmatrix}$, *and* $\begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix}$ *is the only uniserial module of length* 2 *in which* T_{i+1} *is contained.*

(b) *If* P_{i+1} *is not uniserial, then* $T_{i+2} = \operatorname{soc} P_i$ *corresponds to a uniserial* Λ'*-module of length* 3.

Proof. (a) Assume first that

$$\mathrm{D}(T_i) = A \begin{array}{c} \nearrow B \searrow \\ \\ \searrow C \nearrow \end{array} T_i,$$

where $B \leftrightarrow S_{i+1}$ and $C \leftrightarrow \begin{pmatrix} S_i \\ S_{i+1} \\ S_{i+2} \end{pmatrix}$. $C/\mathbf{r}C$ has only one copy of T_i. For otherwise there would be maps $u\colon C \to T_i$, $v\colon C \to T_i$ such that for any $t\colon T_i \to T_i$, $\underline{tu} \neq \underline{v}$. If some simple Λ-module $T \not\cong T_i$ was a summand of $C/\mathbf{r}C$, we would have a nonzero $\underline{f}\colon \underline{C} \leftrightarrow \underline{T}$, hence $T \leftrightarrow S_i$. But since there is a nonzero map from $\begin{pmatrix} S_i \\ S_{i+1} \end{pmatrix}$ to $\underline{S}_i$, but not from $\underline{T}_i$ to $\underline{T}$, we get a contradiction. Hence $C/\mathbf{r}C \cong T_i$. In a similar way we also prove that $A/\mathbf{r}A \cong T_i$ and $B/\mathbf{r}B \cong T_i$. But since P_i is uniserial, we then get a contradiction to $\mathrm{D}(T_i)$ being of the above type. Since $T_i \leftrightarrow \begin{pmatrix} S_i \\ S_{i+1} \end{pmatrix}$ we must consequently have $\mathrm{D}(T_i) = A \to T_i$. Similar to the above we can prove that $A/\mathbf{r}A$ is simple. This shows that P_i has length 3, and that $A \cong \begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix}$, where $T_{i+1} \cong \mathbf{r}P_i/\mathbf{r}^2 P_i$. It then follows that $\begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix} \leftrightarrow S_{i+1}$. Since no indecomposable nonprojective Λ-module maps onto $\begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix}$, any nonzero map $X \to \begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix}$, where X is indecomposable and not projective and not isomorphic to $\begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix}$, factors through $T_{i+1} \to \begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix}$. Hence $T_{i+1} \leftrightarrow \begin{pmatrix} S_{i+1} \\ S_{i+2} \end{pmatrix}$.

(b) We assume now that P_{i+1} is not uniserial, $\mathbf{r}P_{i+1}/\mathbf{r}^2 P_{i+1} = T_{i+2} \coprod T_{i+3}$ such that $T_{i+2} = \operatorname{soc} P_i$.

We have seen that $\begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix} \leftrightarrow S_{i+1}$, hence we have

$$\mathrm{D}\left(\begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix}\right) = \begin{pmatrix} T_{i+1} \\ T_{i+3} \end{pmatrix} \to T_{i+1} \to \begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix},$$

so that $\begin{pmatrix} T_{i+1} \\ T_{i+3} \end{pmatrix} \leftrightarrow \begin{pmatrix} S_{i+1} \\ S_{i+2} \\ S_{i+3} \end{pmatrix}$. By (a), T_{i+2} then corresponds to Λ'-module of length 3.

We are now ready to finish the proof that the conditions of the Theorem are necessary. Let Λ be an indecomposable algebra stably equivalent to a Nakayama algebra Λ' of Loewy length 4, and assume that Λ is not Nakayama. The first part follows from Lemma 1. Choose a simple Λ-module T_1 such that P_1 is not uniserial. By Lemma 1(b) we can write $\mathbf{r}P_1/\mathbf{r}^2 P_i = T_2 \coprod T_3$, where T_2 and T_3 are not isomorphic simple modules, and we know by Lemma 2(a) that no two of the simple modules T_1, T_2 and T_3 are isomorphic and that we can choose T_2 such that it corresponds to a Λ'-module of length 3. We further know from Lemma 2(b) that P_2 is uniserial of length 3, P_3 is uniserial of length 2 or 3. If P_3 has length 3, we know by Lemma 2(e) that P_4 is not uniserial, where $T_4 = \operatorname{soc} P_1$. If P_3 has length 2 we know by Lemma 2(c) that P_4 is uniserial of length 3 or is not uniserial.

Assume that P_4 is uniserial. Since T_4 corresponds to a Λ'-module of length 2 by Lemma 2(c), we have, for $T_5 = \mathbf{r}P_4/\mathbf{r}^2 P_4$, that P_5 is uniserial of length 3 or is not uniserial by Lemma 3. In both cases T_5 corresponds to a Λ'-module of length 2 by Lemma 2(a) and Lemma 3(a).

In general, if P_i is not uniserial, we choose the numbering such that

$$\mathbf{r}P_i/\mathbf{r}^2 P_i = T_{i+1} \coprod T_{i+2},$$

where T_{i+1} corresponds to a Λ'-module of length 3, $T_{i+3} = \operatorname{soc} P_i$. We know that P_{i+1} and P_{i+2} are uniserial and that if P_{i+3} is not uniserial, then it is by Lemma 2(d) compatible with our choice of numbering that $T_{i+4} = \operatorname{soc} P_{i+1}$, and $T_{i+5} = \operatorname{soc} P_{i+2}$ if P_{i+2} has length 3. If P_{i+4} is uniserial, we know by Lemma 2(c) that P_{i+4} has length 3 and that T_{i+4} corresponds to a Λ'-module of length 2. So in general if P_j is uniserial of length 3 and T_j corresponds to a Λ'-module of length 2, we choose $T_{j+1} = \mathbf{r}P_j/\mathbf{r}^2 P_j$ and $T_{j+2} = \operatorname{soc} P_j$. Since the T_{i+1} and T_{i+2} above do not correspond to a Λ'-module of length 2 by Lemma 2(a), this is not in conflict with the above choice. Further, we know by Lemma 3(a) that T_{j+1} corresponds to a Λ'-module of length 2 and that P_{j+1} is uniserial of length 3 or is not uniserial. If P_{j+1} is not uniserial, we have by Lemma 3(b) that $T_{j+2} = \operatorname{soc} P_j$ corresponds to a Λ'-module of length 3, so that we have no conflict in our choice here.

We know that no two of the modules T_1, T_2 and T_3 are isomorphic. Let T_i be such that $T_i \cong T_j$, $j < i$, and i is smallest possible. Assume first that $j > 1$. Then T_i and T_j are contained in the same uniserial modules of length 2, either only

$$\begin{pmatrix} T_{i-1} \\ T_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} T_{i-1} \\ T_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T_{i-2} \\ T_i \end{pmatrix}.$$

This will give a contradiction to the minimality of i, and hence $j = 1$. This finishes the proof that our conditions are necessary.

3. Sufficiency

In this section we shall sketch a proof of the fact that the conditions of the main theorem are sufficient. So assume that Λ is an indecomposable algebra satisfying the conditions of the theorem. It follows as a special case of a result of C. M.

Ringel (unpublished) that each indecomposable Λ-module is a factor module of an indecomposable projective module or of the type $\mathfrak{r}Q$, where Q is indecomposable projective.

It is not hard to see that the endomorphism rings of the simple Λ-modules are all isomorphic to the same division ring D. Further, it is also not hard to see that for each indecomposable nonprojective Λ-module M, End () $\cong \underline{M}$D, and if N is also an indecomposable nonprojective Λ-module, then $\operatorname{Hom}(\underline{M}, \underline{N})$ is zero or isomorphic to D as a two sided D-module.

Let n be the number of simple nonisomorphic Λ-modules and let $(r_1, \dots, r_n)$ be the following sequence of numbers, where each r_i is either 3 or 4. If P_i is not uniserial, then let $r_i = r_{i+1} = 4$, and let $r_{i+2} = 3$ if P_{i+2} has length 2, $r_{i+2} = 4$ if P_{i+2} has length 3. The r_j which are not already defined by the above are defined to be 3. We then know that it is possible to find an indecomposable Nakayama algebra Λ' with indecomposable projective modules $Q_1, \dots, Q_n$ such that

$$\mathfrak{r}Q_i/\mathfrak{r}^2 Q_i \cong Q_{i+1}/\mathfrak{r}Q_{i+1}$$

for $i = 1, \dots, n$, where $Q_{n+1} = Q_1$, $L(Q_i) = r_i$ and the endomorphism ring of each simple Λ'-module is isomorphic to D (see Kupisch, 1959 and Murase, 1964). $(r_1, \dots, r_n)$ is then said to be an admissible sequence associated with Λ', and it is unique up to cyclic permutation.

To finish our proof it is then sufficient to show that Λ is stably equivalent to the Λ' chosen above. Since $n \geqslant 3$, we have that End $(\underline{A}) \cong D$ for each indecomposable nonprojective Λ'-module A, and that Hom $(\underline{A}, \underline{B})$ is zero or isomorphic to D as a two sided D-module if A and B are indecomposable nonprojective Λ'-modules. By the above remarks about Λ, it will then suffice to define a correspondence F between the indecomposable nonprojective modules over Λ and Λ', which has the following properties. If A, B and C are indecomposable nonprojective Λ-modules, then $\operatorname{Hom}(\underline{A}, \underline{B}) \neq 0$ if and only if $\operatorname{Hom}(\underline{FA}, \underline{FB}) \neq 0$. And if $f\colon A \to B$, $g\colon B \to C$, $f'\colon FA \to FB$, $g'\colon FB \to FC$ are such that $\underline{f}, \underline{g}, \underline{f}', \underline{g}'$ are not zero, then $\underline{gf}$ is zero if and only if $\underline{g}'\underline{f}'$ is zero.

We shall explain the correspondence F below. It is not hard to check that the desired properties hold, so that F can be extended to an equivalence $F\colon \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda'$.

(1) Assume that P_i is not uniserial. Then we define

$$T_i \to \begin{pmatrix} S_i \\ S_{i+1} \end{pmatrix}, \quad T_{i+1} \to \begin{pmatrix} S_{i+1} \\ S_{i+2} \\ S_{i+3} \end{pmatrix},$$

$$T_{i+2} \to S_{i+2}, \quad \begin{pmatrix} T_i \\ T_{i+1} \end{pmatrix} \to S_{i+1},$$

$$\begin{pmatrix} T_i \\ T_{i+2} \end{pmatrix} \to \begin{pmatrix} S_i \\ S_{i+1} \\ S_{i+2} \end{pmatrix}, \quad \begin{pmatrix} T_{i+1} \\ T_{i+3} \end{pmatrix} \to S_{i+3},$$

$$P_i/\operatorname{soc} P_i \to \begin{pmatrix} S_{i+1} \\ S_{i+2} \end{pmatrix} \quad \text{and} \quad \mathfrak{r}P_i \to \begin{pmatrix} S_{i+2} \\ S_{i+3} \end{pmatrix}.$$

If P_{i+2} has length 3, then $\begin{pmatrix} T_{i+2} \\ T_{i+3} \end{pmatrix}$ is also not projective. In this case we also have $r_{i+2} = 4$ and so we define

$$\begin{pmatrix} T_{i+2} \\ T_{i+3} \end{pmatrix} \to \begin{pmatrix} S_{i+2} \\ S_{i+3} \\ S_{i+4} \end{pmatrix}.$$

(2) Assume that P_j is uniserial of length 3 and that T_j is not a submodule of any $P/\operatorname{soc} P$, for a non uniserial indecomposable projective Λ-module P. Then,

$$T_j \to \begin{pmatrix} S_j \\ S_{j+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T_j \\ T_{j+1} \end{pmatrix} \to S_{j+1}.$$

We end this section with some comments to our theorem.

(i) It follows from our theorem that the structure of the indecomposable non Nakayama algebras Λ stably equivalent to a Nakayama algebra Λ' of Loewy length at most 4 is given by the following data. Let α denote a triple (P_1, P_{i+1}, P_{i+2}) of indecomposable projective Λ-modules, where P_i is not uniserial, P_{i+1} and P_{i+2} are uniserial of length 3. Let β denote a triple (P_i, P_{i+1}, P_{i+2}), where P_i is not uniserial, P_{i+1} is uniserial of length 3 and P_{i+2} is uniserial of length 2. Let γ denote (P_i), where P_i is uniserial of length 3 and $P_i/\mathfrak{r}P_i$ is not a summand of any $\mathfrak{r}P/\mathfrak{r}^2 P$ for an indecomposable nonuniserial projective Λ-module P. To each α corresponds the part of the admissible sequence for Λ' (4, 4, 4), to β (4, 4, 3) and to γ (3). So the structure of Λ is given by a cycle of α's, β's and γ's, with the requirement that no γ follows an α.

(ii) Let Λ be stably equivalent to an indecomposable Nakayama algebra Λ' of Loewy length 4. If some maximal number of consecutive 4 in the admissible sequence for Λ' is $3i+1$, then by the above Λ must be Nakayama. Further it is easy to see that Λ' is selfinjective if and only if all entries in the admissible sequence are 4. Hence it also follows from the above that if Λ' is selfinjective and the number of simple Λ'-modules is not divisible by 3, then Λ is Nakayama. And if Λ' is not selfinjective and some maximal number of consecutive 4 in the admissible sequence for Λ' is not $3i+2$, then Λ is Nakayama.

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