A Problem in the Linear Flow of Heat discussed from the point of view of the Theory of Integral Equations.

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In the first section of Kneser's book on *Integral Equations and their Applications to Mathematical Physics*,[1] he applies that theory to the solution of some of the problems which arise in the Linear Flow of Heat. The object of this paper is to illustrate Kneser's use of Integral Equations in the Mathematical Theory of the Conduction of Heat by the discussion of one of the classical problems of Linear Flow which he leaves untouched.

§1. The problem to be solved is the following:—

A thin rod of length l is coated with a substance preventing radiation at the surface. At the ends $x=0$ and $x=l$, radiation takes place into a medium at zero temperature. The initial temperature of the rod is an arbitrary function $f(x)$. To find the temperature at any point of the rod at the time t.

With the usual notation[2] the equations for the temperature are

$$\frac{\partial v}{\partial t}=k\frac{\partial^2 v}{\partial x^2}, \quad (0<x<l) \tag{1}$$

$$-\frac{\partial v}{\partial x}+hv=0, \quad \text{at } x=0, \tag{2}$$

$$\frac{\partial v}{\partial x}+hv=0, \quad \text{at } x=l, \tag{3}$$

and

$$v=f(x) \quad \text{for } t=0, \quad (0<x<l). \tag{4}$$

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[1] Kneser, *Die Integralgleichungen und ihre Anwendungen in der Mathematischen Physik* (Braunschweig, 1911).

[2] Cf., for example, Carslaw, *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, §105, p. 270. In future this book will be referred to as *Fourier's Series*.

It is well known that the case in which radiation takes place at the surface, as well as at the ends, can be reduced to the one named above by a simple substitution.[3] Also, it will be clear from the discussion below that the problem can be solved on similar lines, when the emissivities at the ends of the rod are not the same.

The analysis is made simpler by changing the variable t, so that equation (1) is replaced by

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2} \qquad (1^*)$$

We shall use this form (1*) in the argument which follows.

§ 2. With the usual method of solution we put

$$v=e^{-\lambda t}\phi,$$

where ϕ is a function of x only.

Then we have the equations

$$\frac{d^2\phi}{dx^2}+\lambda\phi=0, \quad (0<x<l) \qquad (5)$$

$$-\frac{d\phi}{dx}+h\phi=0, \quad \text{at } x=0, \qquad (6)$$

and

$$\frac{d\phi}{dx}+h\phi=0, \quad \text{at } x=l. \qquad (7)$$

From (5) it follows that

$$\phi=\mathrm{A}\cos ax+\mathrm{B}\sin ax, \text{ where } \lambda=a^2.$$

Also (6) and (7) are satisfied, if

$$\frac{\mathrm{A}}{a}=\frac{\mathrm{B}}{h}$$

and

$$\tan al=\frac{2ah}{a^2-h^2}. \qquad (8)$$

Hence the expressions $\phi_n=\mathrm{A}_n(a_n\cos a_n x+h\sin a_n x)$ satisfy (5), (6) and (7), provided that a_1, a_2, ... are roots of (8).

It is clear that we can omit the root $a=0$: also that the negative roots are equal in absolute value to the positive ones; so that the functions need only be taken for the positive roots.

[3] Cf. *Fourier's Series*, p. 230.

From (5), (6) and (7) it follows that

$$\int_0^l \phi_m \phi_n dx = 0, \text{ when } m \neq n.$$

Also it is easy to show[4] that

$$\int_0^l (a_n \cos a_n x + h \sin a_n x)^2 dx = \frac{(a_n^2 + h^2)l + 2h}{2}.$$

Hence, if we take $A_n = \sqrt{\dfrac{2}{(a_n^2 + h^2)l + 2h}}$,

so that $$\phi_n = \sqrt{\frac{2}{(a_n^2 + h^2)l + 2h}}(a_n \cos a_n x + h \sin a_n x), \qquad (9)$$

we have $\int_0^l \phi_n^2 dx = 1.$

These functions $\phi_1, \phi_2, \ldots$ will be referred to as the *orthogonal functions* of the problem, and in the form given in (9), where

$$\int_0^l \phi_n^2 dx = 1,$$

they are said to be *normalised.*

Fourier's method of solving such questions amounts in principle to assuming that the arbitrary function $f(x)$ can be expanded in a series of these *normalised orthogonal functions*

$$A_1 \phi_1(x) + A_2 \phi_2(x) + \ldots,$$

and that the series can be integrated term by term.

On this assumption it would follow that

$$A_n \int_0^l \phi_n^2(x) dx = A_n = \int_0^l f(x) \phi_n(x) dx.$$

And finally we would have

$$v = \sum_{n=1}^{\infty} e^{-a_n^2 t} \phi_n(x) \int_0 f(x') \phi_n(x') dx'.$$

§ 3. We proceed to show how the Theory of Integral Equations enables us to avoid the assumptions to which we have referred above. This discussion follows the lines laid down by Kneser.

[4] The properties of these functions are worked out at length in the discussion of the problem in *Fourier's Series*, §105.

There exists a continuous function $K(x, \xi)$ which satisfies the equation for steady temperature

$$\frac{d^2v}{dx^2} = 0, \quad (0 < x < l),$$

and the same boundary conditions,

$$-\frac{dv}{dx} + hv = 0, \text{ at } x = 0,$$

$$\frac{dv}{dx} + hv = 0, \text{ at } x = l:$$

while its differential coefficient with regard to x, denoted by $K'(x, \xi)$, is discontinuous for the value $x = \xi$, in such a way that

$$\Big[K'(x, \xi)\Big]_{\xi+0}^{\xi-0} = 1.$$ [5]

Otherwise the derivative is continuous in the interval $0 < x < l$.

This function is called by Hilbert the Green's Function[6] for the case considered. It is obvious that it is the steady temperature at x due to a continuous point source of unit strength at ξ, under the given boundary conditions.

It will be seen that in our case

$$\left.\begin{aligned} K(x, \xi) &= \frac{(1 + h(l - \xi))(1 + hx)}{2h + lh^2}, \quad (0 < x < \xi) \\ K(x, \xi) &= \frac{(1 + h\xi)(1 + h(l - x))}{2h + lh^2}. \quad (\xi < x < l) \end{aligned}\right\} \qquad (10)$$

This function occupies a prominent place in the rest of our argument.

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[5] We use the notation $\Big[F(x)\Big]_a^b$ for $F(b) - F(a)$.

[6] The Green's Functions employed in the applications of Integral Equations to the Conduction of Heat must not be confused with those to which the same term was applied in the case of Instantaneous Point Sources. As a matter of fact, the new Green's Functions can be obtained from the old by integration: and the results given in my paper in these *Proceedings* [Vol. XXI., p. 40, 1903], and in *Fourier's Series*, Chapter XVIII., can be used in the work on Integral Equations.

§4. Having established the existence of the function $K(x, \xi)$, we return to the equations of §2.

There we have $\phi'' + \lambda\phi = 0$, and certain boundary conditions.

Also we have $K''(x, \xi) = 0$, and the same boundary conditions.

Let $F(x) = \phi(x)K'(x, \xi) - \phi'(x)K(x, \xi)$. Then $F(x)$ is discontinuous at $x = \xi$, but is otherwise continuous in the interval $0 < x < l$.

Also $F'(x) = \lambda K(x, \xi)\phi(x)$.

$$\therefore \qquad \int_0^l F'(x)dx = \lambda \int_0^l K(x, \xi)\phi(x)dx.$$

But
$$\int_0^l F'(x)dx = \int_0^{\xi -} F'(x)dx + \int_{\xi+0}^l F'(x)dx$$
$$= \Big[F(x)\Big]_0^l + \Big[F(x)\Big]_{\xi+0}^{\xi-0}$$
$$= \Big[F(x)\Big]_{\xi+0}^{\xi-0},$$

since $\qquad F(0) = F(l) = 0.$

And $\phi(x)$, $\phi'(x)$ are continuous in the interval.

It follows that $\qquad \phi(\xi)\Big[K'(x, \xi)\Big]_{\xi+0}^{\xi-0} = \lambda\int_0^l K(x, \xi)\phi(x)dx.$

Thus $\qquad \phi(\xi) = \lambda\int_0^l K(x, \xi)\phi(x)dx.$

It will be seen from the expressions for $K(x, \xi)$ in § 3 (10) that the Green's Function is a symmetrical function of the two variables, which enter into it.

In other words, $\qquad K(\xi, \eta) = K(\eta, \xi).$[7]

The result we have obtained can therefore be written

$$\phi(x) = \lambda\int_0^l K(x, a)\phi(a)da.$$

Thus the orthogonal functions of §2 *occur as solutions of the Homogeneous Integral Equation whose symmetrical kernel is the Green's Function of the problem considered.*

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[7] For a general proof, cf. Kneser, *loc. cit.*, p. 6.

§ 5. The converse is also true, namely, that every continuous solution of the integral equation

$$\phi(x) = \lambda \int_0^l \mathrm{K}(x, a)\phi(a)da$$

is an orthogonal function of the problem in Linear Flow, which we are examining.

To prove this, we start with the equation

$$\phi(x) = \lambda \int_0^l \mathrm{K}(x, a)\phi(a)da.$$

Since $\mathrm{K}(x, a)$ is finite and continuous, we have

$$\phi'(x) = \lambda \int_0^l \mathrm{K}'(x, a)\phi(a)da.$$

It follows that

$$\pm \phi'(x) + h\phi(x) = \lambda \int_0^l [\pm \mathrm{K}'(x, a) + h\mathrm{K}(x, a)]\phi(a)da.$$

Therefore $\phi(x)$ satisfies the boundary equations, since $\mathrm{K}(x, a)$ does so.

Further, we may write

$$\phi'(x) = \lambda \int_0^{x-0} \mathrm{K}'(x, a)\phi(a)da + \lambda \int_{x+0}^l \mathrm{K}'(x, a)\phi(a)da.$$

Then it is clear that

$$\begin{aligned}\phi''(x) &= \lambda \int_0^l \mathrm{K}''(x, a)\phi(a)da + \lambda[\mathrm{K}'(x, x-0) - \mathrm{K}'(x, x+0]\phi(x) \\ &= \lambda[\mathrm{K}'(x, x-0) - \mathrm{K}'(x, x+0)]\phi(x),\end{aligned}$$

since $\mathrm{K}''(x, a) = 0$.

It will be seen from the expressions for $\mathrm{K}(x, \xi)$, for $x \gtreqless \xi$, in § 3 (10) that

$$\mathrm{K}'(x, x-0) = \mathrm{K}'(x+0, x)$$

and

$$\mathrm{K}'(x, x+0) = \mathrm{K}'(x-0, x).$$

It follows that

$$\begin{aligned}\phi''(x) &= -\lambda[\mathrm{K}'(x-0, x) - \mathrm{K}'(x+0, x)]\phi(x) \\ &= -\lambda\phi(x),\end{aligned}$$

since

$$\Big[\mathrm{K}'(x, \xi)\Big]_{\xi+0}^{\xi-0} = 1$$

for any value of ξ in the interval $0 < \xi < l$.

Thus we have $\qquad \phi''(x) + \lambda\phi(x) = 0.$

Therefore we have shown that all the orthogonal functions of our problem in the Linear Flow of Heat are solutions of the Integral Equation

$$\phi(x) = \lambda\int_0^l \mathrm{K}(x, a)\phi(a)da,$$

and that all the continuous solutions of this equation are orthogonal functions of that problem.

The solution of the integral equation and the corresponding value of λ will be denoted by $\phi_n(x)$ and λ_n.

§6. *Now, in the Theory of Integral Equations it is proved that if the series of normalised orthogonal functions*

$$\frac{\phi_1(x)\phi_1(y)}{\lambda_1} + \frac{\phi_2(x)\phi_2(y)}{\lambda_2} + \dots$$

is uniformly convergent in the region

$$0 \leqq x \leqq l$$
$$0 \leqq y \leqq l,$$

the sum of this series is equal to the Green's Function $\mathrm{K}(x, y)$, *the symmetrical kernel of the Integral Equation.*[8]

In the problem we are examining we have seen, §2 (9), that the *normalised orthogonal functions* are

$$\phi_n(x) = \sqrt{\frac{2}{(a_n^2 + h^2)l + 2h}}(a_n \cos ax + h \sin a_n x),$$

the values of a being the positive roots of the equation

$$\tan al = \frac{2ah}{a^2 - h^2}.$$

And we have $\qquad \lambda_n = a_n^2.$

But it is easy to show that

$$(n-1)\pi < a_n l < n\pi.$$

[8] Cf. Bôcher, *An Introduction to the Study of Integral Equations* (*Camb. Math. Tracts, No. 10*), *p. 58*, where a more exact statement of this fundamental theorem is given.

Also Kneser, *loc. cit.*, pp. 33–4.

Hence the order of the term

$$\frac{\phi_n(x)\phi_n(y)}{\lambda_n}$$

becomes $\frac{1}{n^2}$, as n increases.

Thus the series

$$\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}$$

converges uniformly in the given interval.

Therefore, by the theorem at the beginning of this paragraph,

$$K(x, y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}.$$

§ 7. This result can be verified independently as follows :—

In a paper on *The Use of Green's Functions in the Theory of Conduction of Heat*,[9] I have shown that the temperature at time t due to an *instantaneous* point source of unit strength at the point ξ at time t' is given by

$$v = 2\sum_{1}^{\infty} e^{-a_n{}^2(t-t')} \frac{(h\sin a_n x + a_n \cos a_n x)(h \sin a_n \xi + a_n \cos a_n \xi)}{(a_n{}^2 + h^2)l + 2h}.$$

Now the steady temperature, $K(x, \xi)$, would be obtained by placing a continuous source supplying unit quantity of heat per unit time at the point ξ, and acting from $-\infty$ up to the time t.

The temperature due to this continuous source will be found by integrating the above expression with regard to t' from $t' = -\infty$ to $t' = t$. Also we can integrate the series term by term.

Thus we have

$$\begin{aligned} K(x, \xi) &= \sum_{1}^{\infty} \phi_n(x)\phi_n(\xi) \int_{-\infty}^{t} e^{-a_n{}^2(t-t')} dt' \\ &= \sum_{1}^{\infty} \phi_n(x)\phi_n(\xi) \int_{0}^{\infty} e^{-a_n{}^2\gamma} d\gamma \\ &= \sum_{1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{a_n{}^2} \\ &= \sum_{1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}. \end{aligned}$$

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[9] *Proc. Edin. Math. Soc.*, Vol. XXI., p. 40, 1903. Also *Fourier's Series*, p. 386.

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§ 8. We are now able to establish the following theorem :—

Let $\psi(x)$ be any function of x such that the interval $0<x<l$ can be broken up into a finite number of segments in which $\psi(x)$ is continuous, while in each of them the function tends towards a limiting value at the ends. Also let $f(x)$ be defined by the equation

$$f(x)=\int_0^l \mathrm{K}(x, a)\psi(a)da.$$

Then the function $f(x)$ can be represented by the series of normalised orthogonal functions

$$\mathrm{A}_1\phi_1(x)+\mathrm{A}_2\phi_2(x)+\ldots$$

where
$$\mathrm{A}_n=\int_0^l f(\beta)\phi_n(\beta)d\beta.$$

We have
$$f(x)=\int_0^l \mathrm{K}(x, a)\psi(a)da$$
$$=\int_0^l \sum_1^\infty \frac{\phi_n(x)\phi_n(a)}{\lambda_n}\,.\,\psi(a)da$$
$$=\sum_1^\infty \frac{\phi_n(x)}{\lambda_n}\int_0^l \phi_n(a)\psi(a)da,$$

since this series can be integrated term by term.

Also
$$\int_0^l f(\beta)\phi_n(\beta)d\beta=\int_0^l \phi_n(\beta)\left[\int_0^l \mathrm{K}(\beta, a)\psi(a)da\right]d\beta$$
$$=\int_0^l \psi(a)\left[\int_0^l \mathrm{K}(a,\beta)\phi_n(\beta)d\beta\right]da$$
$$=\frac{1}{\lambda_n}\int_0^l \psi(a)\phi_n(a)da.$$

Thus we have shown that

$$f(x)=\sum_1^\infty \mathrm{A}_n\phi_n(x),\ \text{ where }\ \mathrm{A}_n=\int_0^l f(\beta)\phi_n(\beta)d\beta.$$

§ 9. The limitations which have been imposed upon the function $f(x)$ in the theorem of § 8 can be shown to leave that function still very general. *Indeed, any function $f(x)$ which, with its first derivative is continuous in the interval $0<x<l$, while its second derivative, if discontinuous, is discontinuous only in the sense in which $\psi(x)$ was discontinuous in that theorem, can be represented by an integral of the type*

$$\int_0^l \mathrm{K}(x, a)\psi(a)da,$$

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provided that $f(x)$ satisfies the boundary conditions to which $K(x, a)$ *is subjected.*

To prove this we have only to note that

$$\frac{d}{dx}[f(x)K'(x, \xi) - f'(x)K(x,\xi)] = -f''(x)K(x, \xi),$$

and $$\int_0^l \frac{d}{dx}[f(x)K'(x,\xi) - f'(x)K(x,\xi)]dx = -\int_0^l f''(x)K(x, \xi)dx.$$

Hence, under the conditions named above,

$$f(\xi)\Big[K'(x,\xi)\Big]_{\xi+0}^{\xi-0} = -\int_0^l f''(x)K(x,\xi)dx.$$

Therefore $$f(x) = -\int_0^l K(x, a)f''(a)da,$$

since $K(x,\xi)$ is symmetrical, and $\Big[K'(x,\xi)\Big]_{\xi+0}^{\xi-0} = 1.$

Therefore we have proved that under these very general conditions the function $f(x)$ can be expanded in the series

$$A_1\phi_1(x) + A_2\phi_2(x) + \ldots,$$

and that the coefficients are obtained by term-by-term integration in the usual Fourier's method.

Kneser's argument also shows[10] that the expansion is possible when the condition of continuity is removed, and when $f(x)$ does not satisfy the same boundary conditions as the Green's Function. But it is beyond the purpose of this paper to enter into these details, so that the exact statement of the condition under which the expansion is shown to be possible is omitted.

§ 10. It was pointed out at the end of § 2 that, if we are in a position to state that the arbitrary function $f(x)$, expressing the initial temperature, can be represented by the series

$$A_1\phi_1(x) + A_2\phi_2(x) + \ldots,$$

the temperature at the time t is given by

$$v = \sum_1^\infty A_n e^{-\lambda_n t}\phi_n(x).$$

We have thus shown how the Theory of Integral Equations fills up the gap in Fourier's discussion of this problem.

[10] Cf. Kneser, *loc. cit.* p. 20.