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SOLUTIONS FOR DOUBLY RESONANT NONLINEAR NON-SMOOTH PERIODIC PROBLEMS

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Abstract We study a nonlinear second-order periodic problem driven by the scalar p-Laplacian with a non-smooth potential. We consider the so-called doubly resonant situation allowing complete interaction (resonance) with both ends of the spectral interval. Using variational methods based on the non-smooth critical-point theory for locally Lipschitz functions and an abstract minimax principle concerning linking sets we establish the solvability of the problem.

Keywords: p-Laplacian; non-smooth critical-point theory; non-smooth PS-condition; linking sets; minimax principle; double resonance

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1. Introduction

In this paper, we study the solvability of the following nonlinear periodic problem with non-smooth potential:

$$\left.\begin{aligned} -(|x'(t)|^{p-2}x'(t))' &\in \partial j(t,x(t)) \quad \text{a.e. on } T=[0,b],\\ x(0)=x(b), \quad x'(0)&=x'(b), \quad 1<p<\infty. \end{aligned}\right\} \tag{1.1}$$

Here $j : T \times \mathbb{R} \to \mathbb{R}$ is a function which is measurable in $t \in T$, locally Lipschitz in $x \in \mathbb{R}$, and $\partial j(t,x)$ denotes the generalized (Clarke) subdifferential of $j(t,\cdot)$ (see § 2). We focus on the so-called doubly resonant problems. This, roughly speaking, means that asymptotically as $|x| \to \infty$ the ratios $\{u/|x|^{p-2}x\}_{u\in\partial j(t,x)}$, are located between two successive eigenvalues of the negative scalar p-Laplacian with periodic boundary conditions. We allow complete interaction (resonance) with both ends of the spectral interval and we only impose non-uniform non-resonance conditions on the ratios $pj(t,x)/x^p$. To make all these a little more precise, consider the following nonlinear eigenvalue problem:

$$\left.\begin{aligned} -(|x'(t)|^{p-2}x'(t))' &= \lambda|x(t)|^{p-2}x(t) \quad \text{a.e. on } T=[0,b],\\ x(0)=x(b), \quad x'(0)&=x'(b), \quad 1<p<\infty. \end{aligned}\right\} \tag{1.2}$$

A real parameter λ is said to be an eigenvalue of the negative scalar p-Laplacian $-(|x'|^{p-2}x')'$ with periodic boundary conditions (i.e. $x(0)=x(b)$, $x'(0)=x'(b)$), provided problem (1.2) has a non-trivial solution. It is known (see [**15**]) that the eigenvalues of problem (1.2) are $\lambda_n = (2n\pi_p/b)^p$, $n \geqslant 0$, where $\pi_p = 2(p-1)^{1/p}\int_0^1 (1-t^p)^{-1/p}\,\mathrm{d}t$. Note that if $p = 2$ (linear case), then $\pi_p = \pi$ and so we recover the eigenvalues $\lambda_n = (2n\pi/b)^2$, $n \geqslant 0$, of the negative Laplacian with periodic boundary conditions. Also it is interesting to note that if we consider the vector one-dimensional p-Laplacian $-(\|x'\|^{p-2}x')'$, with x being a vector-valued (i.e. $\mathbb{R}^N$-valued, $N > 1$) Sobolev function with periodic boundary conditions, then we have more eigenvalues in addition to the sequence $\{\lambda_n\}_{n\geqslant 0}$ mentioned above. Now we say that problem (1.1) is in double resonance if

$$\lambda_n \leqslant \liminf_{x\to\pm\infty} \frac{u}{|x|^{p-2}x} \leqslant \limsup_{x\to\pm\infty} \frac{u}{|x|^{p-2}x} \leqslant \lambda_{n+1}$$

uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$. Due to the nonlinearity of the differential operator, the lack of convenient orthogonal decomposition of the relevant Sobolev space in terms of the corresponding eigenspaces, and finally the lack of variational characterizations for the higher eigenvalues λ_n, $n \geqslant 2$, we limit ourselves to the beginning of the spectrum, namely the interval $[\lambda_0 = 0,\ \lambda_1 = (2\pi_p/b)^p]$.

Double-resonance problems have been studied in the context of semilinear (i.e. $p = 2$), elliptic equations with Dirichlet boundary conditions and smooth potential (i.e. $j(z,\cdot) \in C^1$). We refer to the works of Berestycki and de Figueiredo [**2**], Cac [**3**], Robinson [**19**], Su [**21**] and the references cited therein.

For the scalar periodic problem, earlier works in this direction deal mostly with semilinear, smooth problems. We refer to the papers of Ahmad and Lazer [**1**], Mawhin [**14**] Fonda and Lupo [**11**], Fabry and Fonda [**10**], Gossez and Omari [**12**] and Omari and Zanolin [**18**]. In [**1**] the ratio $\partial j(t,x)/x$ asymptotically as $|x| \to +\infty$ stays strictly between the zero and the first non-zero eigenvalue. So we do not have resonance. Their approach is variational and uses the Saddle Point Theorem. Soon thereafter, Mawhin [**14**] extended this work to problems in non-variational form and to allow complete resonance at the zero eigenvalue for the ratio $\partial j(t,x)/x$ and non-uniform non-resonance at the first non-zero eigenvalue. His approach is degree theoretic using the Leray–Schauder theory and a Villari-type condition. Fonda and Lupo [**11**] impose at the first eigenvalue an Ahmad–Lazer–Paul-type resonance condition (namely, they assume that $j(t,\cdot)$ satisfies a kind of perturbed sign condition and $\int_0^b j(t,c)\,\mathrm{d}t \to +\infty$ as $|c| \to +\infty$) and at the first non-zero eigenvalue they impose a non-uniform non-resonance condition. Fabry and Fonda [**10**] allow (possible) double resonance as $|x| \to \infty$ for the ratio $\partial j(t,x)/x$ and instead of using non-uniform non-resonance conditions based on the potential $j(t,x)$ (namely for the ratio $j(t,x)/x$), as we do here, they employ a Landesman–Lazer-type condition. Gossez and Omari [**12**] consider a non-variational problem and provide necessary and sufficient conditions for non-resonance (surjectivity) to occur. Finally, Omari and Zanolin [**18**] employ conditions similar to the ones used here and degree-theoretic methods to prove existence of solutions for doubly resonant problems. Recently, there has been increasing interest in periodic problems driven by the scalar p-Laplacian. We mention the works of Del Pino *et al*. [**6**], Fabry and Fayyad [**9**] and Mawhin [**16**]. All these works assume a smooth potential function and only [**9**] deals with the doubly resonant situation, using degree-theoretic methods, the Fučík spectrum (asymmetric nonlinearity) and a Landesman–Lazer-type condition (see also [**10**]).

Our approach is variational, based on the non-smooth critical-point theory for locally Lipschitz functions (see [**4**] and [**13**]). Since this theory uses the generalized subdifferential of locally Lipschitz functions, in the next section, for the convenience of the reader, we recall the basic definitions and facts from the subdifferential calculus of locally Lipschitz functions. We also mention some notions and results from the corresponding non-smooth critical-point theory, which we shall need below. Our main references for these issues are the books of Clarke [**5**] and Denkowski *et al.* [**7**].

2. Mathematical background

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz if, for every $x \in X$, there exists a neighbourhood U of x and a constant $k > 0$ (depending on U), such that $|\varphi(z) - \varphi(y)| \leqslant k\|z - y\|$ for all $z, y \in U$. From convex analysis we know that if $\psi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a convex, lower semicontinuous, proper (i.e. not identically $+\infty$) function, then ψ is locally Lipschitz in the interior of its effective domain $\operatorname{dom}\psi = \{x \in X : \psi(x) < +\infty\}$. In particular then, a continuous, convex function $\psi : X \to \mathbb{R}$ is locally Lipschitz. For a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we define the generalized directional derivative at $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x\\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \to \varphi^0(x; h)$ is sublinear continuous and so $\varphi^0(x; \cdot)$ is the support function of a non-empty, w^*-compact and convex set $\partial\varphi(x)$, defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leqslant \varphi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial\varphi(x)$ is called the generalized (or Clarke) subdifferential of φ at x. If φ is also convex, then the generalized subdifferential coincides with the subdifferential in the sense of convex analysis given by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leqslant \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. Moreover, if $\varphi, \psi : X \to \mathbb{R}$ are two locally Lipschitz functions and $\mu \in \mathbb{R}$, then $\partial(\varphi + \psi) \subseteq \partial\varphi + \partial\psi$ and $\partial(\mu\phi) = \mu\partial\varphi$.

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, a point $x \in X$ is a critical point of φ if $0 \in \partial\varphi(x)$. Then $c = \varphi(x)$ is a critical value of φ. It is easy to see that if $x \in X$ is a local extremum of φ (i.e. a local minimum or a local maximum), then x is a critical point of φ (i.e. $0 \in \partial\varphi(x)$).

In the classical (smooth) critical-point theory, central role plays a compactness-type condition, known as the Palais–Smale (PS) condition. In the present non-smooth setting this condition takes the following form.

A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the non-smooth PS condition if any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $m(x_n) = \inf[\|x^*\| : x^* \in \partial\varphi(x_n)] \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

If $\varphi \in C^1(X)$, then as we already mentioned $\partial\varphi(x_n) = \{\varphi'(x_n)\}$, $n \geqslant 1$, and so we see that the above definition coincides with the classical one (see [**17**, p. 130] and [**8**, p. 171]).

The geometric notion that follows is important in critical-point theory (see [**20**, p. 116] and [**8**, p. 178]).

Definition 2.1. Let Y be a Hausdorff topological space and let E_1, D be two non-empty subsets of Y with D closed. We say that E_1 and D link in Y if

(a) $E_1 \cap D = \emptyset$,

(b) there exists a closed set $E \supseteq E_1$ such that for any $\eta \in C(E, Y)$ with $\eta|_{E_1} = \mathrm{id}_{E_1}$, we have $\eta(E) \cap D \neq \emptyset$.

Using this geometric notion, Kourogenis and Papageorgiou [**13**] proved the following abstract minimax principle.

Theorem 2.2. ***If*** *X is a reflexive Banach space, E_1 and D are two non-empty subsets of X with D closed, E_1 and D link in X, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the non-smooth PS condition, $\sup_{E_1} \varphi < \inf_D \varphi$,* ***then*** *$\varphi$ has a critical point x, with critical value $c = \varphi(x) \geqslant \inf_D \varphi$ given by*

$$c = \inf_{\eta\in\Gamma} \sup_{v\in E} \varphi(\eta(v)),$$

where $E \supseteq E_1$ is as in the definition of linking sets and $\Gamma = \{\eta \in C(E, X) : \eta|_{E_1} = \mathrm{id}_{E_1}\}$. Moreover, if $c = \inf_D \varphi$, then $x \in D$.

Remark 2.3. With suitable choices of the sets E_1 and D, from this abstract result, we derive non-smooth versions of the Mountain Pass Theorem, the Saddle Point Theorem and the Generalized Mountain Pass Theorem (see [**13**]).

3. The existence theorem

Our hypotheses on the non-smooth potential $j(t, x)$ are as follows.

$\mathbf{H}(\boldsymbol{j})$. $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot, 0) \in L^1(T)$ and

(i) for all $x \in \mathbb{R}$, $t \to j(t, x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t, x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $\alpha_r \in L^1(T)_+$ such that for almost all $t \in T$, all $|x| \leqslant r$ and all $u \in \partial j(t, x)$, we have $|u| \leqslant \alpha_r(t)$;

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(iv) there exist $\vartheta_1, \vartheta_2 \in L^\infty(T)$ with $0 \leqslant \vartheta_1(t)$ a.e. on T, $\vartheta_2(t) \leqslant \lambda_1 = (2\pi_p/b)^p$ a.e. on T, these two inequalities are strict on sets of positive measure and

$$0 \leqslant \liminf_{x\to\pm\infty} \frac{u}{|x|^{p-2}x} \leqslant \limsup_{x\to\pm\infty} \frac{u}{|x|^{p-2}x} \leqslant \lambda_1$$

uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$,

and

$$\vartheta_1(t) \leqslant \liminf_{x\to\pm\infty} \frac{pj(t,x)}{|x|^p} \leqslant \limsup_{x\to\pm\infty} \frac{pj(t,x)}{|x|^p} \leqslant \vartheta_2(t)$$

uniformly for almost all $t \in T$.

Remark 3.1. In hypothesis H(j) (iv), the first set of inequalities imply the double-resonance situation with complete resonance at both ends. It should be pointed out that none of the semilinear, 'smooth' papers on elliptic Dirichlet problems, mentioned in § 1, allowed for complete resonance at both ends. They always had partial resonance (non-uniform non-resonance) in at least one of the two endpoints. In hypothesis H(j) (iv), the second set of inequalities imply that the ratio $pj(t,x)/|x|^p$ satisfies certain non-uniform non-resonance conditions at $\pm\infty$. In [**12**], the authors discuss how the limits of $u/(|x|^{p-2}x)$ and $pj(t,x)/|x|^p$ are related in the context of smooth, time-invariant potential (in [**12**], $p = 2$). Note that condition H(j) (iv) implies that $\int_0^b j(t,c)\,\mathrm{d}t \to +\infty$ as $|c| \to +\infty$, $c \in \mathbb{R}$. Hence hypotheses H(j) remain true if we add to $\partial j(t,x)$ an $L^\infty(T)$ function $h(\cdot)$. So our existence result is in fact a surjectivity result (we thank the referee for this last observation).

In what follows,

$$W^{1,p}_{\mathrm{per}}(T) = \{x \in W^{1,p}(T) : x(0) = x(b)\}.$$

Note that $W^{1,p}(T)$ is embedded compactly in $C(T)$ and so the pointwise evaluations of x at $t = 0$ and $t = b$ make sense. Also, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}_{\mathrm{per}}(T), W^{1,p}_{\mathrm{per}}(T)^*)$. Let $\varphi : W^{1,p}_{\mathrm{per}}(T) \to \mathbb{R}$ be the energy function defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))\,\mathrm{d}t, \quad x \in W^{1,p}_{\mathrm{per}}(T).$$

We know that φ is locally Lipschitz (see [**7**, p. 616]).

Proposition 3.2. ***If*** *hypotheses H(j) hold,* ***then*** *φ satisfies the non-smooth PS condition.*

Proof. Consider a sequence $\{x_n\}_{n\geqslant1} \subseteq W^{1,p}_{\mathrm{per}}(T)$ such that

$$|\varphi(x_n)| \leqslant M_1 \quad \text{for some } M_1 > 0, \text{ all } n \geqslant 1, \qquad \text{and} \qquad m(x_n) \to 0 \quad \text{as } n \to \infty.$$

Exploiting the fact that $\partial\varphi(x_n) \subseteq W^{1,p}_{\mathrm{per}}(T)^*$ is weakly compact and that the norm in a Banach space is weakly lower semicontinuous, from the Weierstrass theorem, we know that we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geqslant 1$. We have

$$x_n^* = A(x_n) - u_n, \quad n \geqslant 1.$$

Here $u_n \in L^1(T)$, $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T (see [**7**, p. 617]) and $A : W^{1,p}_{\mathrm{per}}(T) \to W^{1,p}_{\mathrm{per}}(T)^*$ is the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\,\mathrm{d}t \quad \text{for all } x, y \in W^{1,p}_{\mathrm{per}}(T).$$

It is easy to check that A is demicontinuous, monotone, hence maximal monotone.

We claim that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$ is bounded. Suppose for the moment that this is not true. Then by passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$ as $n \to \infty$. Set $y_n = x_n/\|x_n\|$, $n \geqslant 1$. Then, at least for a subsequence, we have

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\mathrm{per}}(T) \quad \text{and} \quad y_n \to y \quad \text{in } C_{\mathrm{per}}(T) \quad \text{as } n \to \infty$$

(recall that $W^{1,p}_{\mathrm{per}}(T)$ is embedded compactly in $C_{\mathrm{per}}(T)$). From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$, we have

$$\begin{aligned} &\left|\langle A(x_n), y_n - y\rangle - \int_0^b u_n(t)(y_n - y)(t)\,\mathrm{d}t\right| \leqslant \varepsilon_n\|y_n - y\| \quad \text{with } \varepsilon_n \downarrow 0, \\ \Rightarrow\ &\left|\langle A(y_n), y_n - y\rangle - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}}(y_n - y)(t)\,\mathrm{d}t\right| \leqslant \frac{\varepsilon_n}{\|x_n\|^{p-1}}\|y_n - y\|. \end{aligned} \tag{3.1}$$

By virtue of hypothesis H(j) (iii) and (iv), for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t, x)$, we have

$$|u| \leqslant \alpha(t) + c|x|^{p-1} \quad \text{with } \alpha \in L^1(T)_+,\ c > 0.$$

So we can write that

$$\begin{aligned} &\frac{|u_n(t)|}{\|x_n\|^{p-1}} \leqslant \frac{\alpha(t)}{\|x_n\|^{p-1}} + c|y_n(t)|^{p-1} \quad \text{a.e. on } T, \\ \Rightarrow\ &\left\{\frac{u_n}{\|x_n\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^1(T) \quad \text{is uniformly integrable.} \end{aligned} \tag{3.2}$$

Hence it follows that

$$\int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}}(y_n - y)(t)\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

So if we pass to the limit as $n \to \infty$ in (3.1), we obtain

$$\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0. \tag{3.3}$$

But A being maximal monotone, it is generalized pseudomonotone (see [**8**, p. 58]) and so from (3.3) it follows that

$$\begin{aligned} &\langle A(y_n), y_n\rangle \to \langle A(y), y\rangle, \\ \Rightarrow\ &\|y_n'\|_p \to \|y'\|_p. \end{aligned}$$

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Since $y_n' \xrightarrow{w} y'$ in $L^p(T)$ and $L^p(T)$ has the Kadec–Klee property (being uniformly convex), we deduce that $y_n' \to y'$ in $L^p(T)$ and so $y_n \to y$ in $W^{1,p}_{\mathrm{per}}(T)$.

Recall that $\{u_n/\|x_n\|^{p-1}\}_{n\geqslant 1} \subseteq L^1(T)$ is uniformly integrable. So, by the Dunford–Pettis theorem, we assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h \quad \text{in } L^1(T).$$

Given $\varepsilon > 0$ and $n \geqslant 1$, we introduce the set

$$C^+_{\varepsilon,n} = \left\{ t \in T : x_n(t) > 0,\ -\varepsilon \leqslant \frac{u_n(t)}{|x_n(t)|^{p-2}x_n(t)} \leqslant \lambda_1 + \varepsilon \right\}.$$

Note that $x_n(t) \to +\infty$ for all $t \in \{y > 0\}$. So if $\chi_{\varepsilon,n} = \chi_{C^+_{\varepsilon,n}}$, because of hypothesis H(j) (iv) we have that

$$\chi_{\varepsilon,n}(t) \to 1 \quad \text{a.e. on } \{y > 0\}.$$

We have

$$\begin{aligned} \chi_{\varepsilon,n}(t)\frac{u_n(t)}{\|x_n\|^{p-1}} &= \chi_{\varepsilon,n}(t)\frac{u_n(t)}{x_n(t)^{p-1}} y_n(t)^{p-1} \\ \Rightarrow \quad -\varepsilon\chi_{\varepsilon,n}(t)y_n(t)^{p-1} &\leqslant \chi_{\varepsilon,n}(t)\frac{u_n(t)}{\|x_n\|^{p-1}} \leqslant \chi_{\varepsilon,n}(t)(\lambda_1+\varepsilon)y_n(t)^{p-1}. \end{aligned}$$

Taking weak limits in $L^1(\{y > 0\})$, we obtain

$$-\varepsilon y(t)^{p-1} \leqslant h(t) \leqslant (\lambda_1 + \varepsilon)y(t)^{p-1} \quad \text{a.e. on } \{y > 0\}.$$

Since $\varepsilon > 0$ was arbitrary, let $\varepsilon \downarrow 0$ to obtain

$$0 \leqslant h(t) \leqslant \lambda_1 y(t)^{p-1} \quad \text{a.e. on } \{y > 0\}. \tag{3.4}$$

Arguing in a similar fashion, we also show that

$$\lambda_1|y(t)|^{p-2}y(t) \leqslant h(t) \leqslant 0 \quad \text{a.e. on } \{y < 0\}. \tag{3.5}$$

Finally, from (3.2), it is clear that

$$h(t) = 0 \quad \text{a.e. on } \{y = 0\}. \tag{3.6}$$

From (3.4), (3.5) and (3.6), it follows that there exists $\xi \in L^\infty(T)$ such that $0 \leqslant \xi(t) \leqslant \lambda_1$ a.e. on T and $h(t) = \xi(t)|y(t)|^{p-2}y(t)$ a.e. on T. From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$, we have

$$\left| \langle A(y_n), v\rangle - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} v(t)\,\mathrm{d}t \right| \leqslant \varepsilon_n\|v\| \quad \text{for all } v \in W^{1,p}_{\mathrm{per}}(T),\ \text{with } \varepsilon_n \downarrow 0.$$

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Passing to the limit as $n \to \infty$ and using the fact that $A(y_n) \xrightarrow{w} A(y)$ in $W^{1,p}_{\mathrm{per}}(T)^*$, we obtain

$$\langle A(y), v\rangle = \int_0^b h(t)v(t)\,\mathrm{d}t \quad \text{for all } v \in W^{1,p}_{\mathrm{per}}(T). \tag{3.7}$$

If by $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair

$$(W_0^{1,p}(T), W^{-1,q}(T) = W_0^{1,p}(T)^*)\left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

and since $(|y'(t)|^{p-2}y')' \in W^{-1,q}(T)$ (see [**7**, p. 362]) via Green's identity (integration by parts), we obtain

$$\langle A(y), v\rangle = \langle -(|y'|^{p-2}y')', v\rangle_0 \quad \text{for all } v \in W_0^{1,p}(T). \tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned} &\langle -(|y'|^{p-2}y')', v\rangle_0 = \int_0^b h(t)v(t)\,\mathrm{d}t = \langle h, v\rangle_0 \quad \text{for all } v \in W_0^{1,p}(T), \\ \Rightarrow \quad &-(|y'(t)|^{p-2}y'(t))' = h(t) \quad \text{a.e. on } T, \quad y(0) = y(b) \quad (\text{since } y \in W^{1,p}_{\mathrm{per}}(T)). \end{aligned} \tag{3.9}$$

Also from (3.7), again via Green's identity, we have

$$\begin{aligned} |y'(b)|^{p-2}y'(b)v(b) - |y'(0)|^{p-2}y'(0)v(0) - \int_0^b (|y'(t)|^{p-2}y'(t))'v(t)\,\mathrm{d}t& \\ = \int_0^b h(t)v(t)\,\mathrm{d}t \quad \text{for all } v \in W^{1,p}_{\mathrm{per}}(T)&, \\ \Rightarrow \quad |y'(0)|^{p-2}y'(0)v(0) = |y'(b)|^{p-2}y'(b)v(b) \quad \text{for all } v \in W^{1,p}_{\mathrm{per}}(T) \quad (\text{see } (3.9))&, \\ \Rightarrow \quad |y'(0)|^{p-2}y'(0) = |y'(b)|^{p-2}y'(b).& \end{aligned}$$

Because $r \to |r|^{p-2}r$ is a homeomorphism on $\mathbb{R}$, we infer that

$$y'(0) = y'(b).$$

So, finally, we have

$$\left.\begin{aligned} -(|y'(t)|^{p-2}y'(t))' = \xi(t)|y(t)|^{p-2}y(t) \quad \text{a.e. on } T = [0,b], \\ y(0) = y(b), \quad y'(0) = y'(b), \quad 0 \leqslant \xi(t) \leqslant \lambda_1, \quad \text{a.e. on } T. \end{aligned}\right\} \tag{3.10}$$

We consider the three distinct possible cases concerning the weight function $\xi \in L^\infty(T)_+$.

Case 1 ($\boldsymbol{\xi \equiv 0}$). In this case from (3.10) we have that $y = c \in \mathbb{R}$. Note that $\|y\| = 1$ (since $y_n \to y$ in $W^{1,p}_{\mathrm{per}}(T)$ and $\|y_n\| = 1$, $n \geqslant 1$) and so $c \neq 0$ (i.e. y is the normalized eigenfunction for the simple eigenvalue $\lambda_0 = 0$). Assume that $c > 0$ (the analysis is similar if $c < 0$). From the mean-value theorem for locally Lipschitz functions (see [**7**, p. 609]) and parts (iii) and (iv) of hypothesis H(j), we see that for almost all $t \in T$ and all $x \in \mathbb{R}$, we have

$$
\begin{aligned}
& |j(t,x)| \leqslant \hat{\alpha}(t) + \hat{c}|x|^p \quad \text{with } \hat{\alpha} \in L^1(T)_+,\ \hat{c} > 0, \\
\Rightarrow \quad & \frac{|j(t,x_n(t))|}{\|x_n\|^p} \leqslant \frac{\hat{\alpha}(t)}{\|x_n\|^p} + \hat{c}|y_n(t)|^p \quad \text{a.e. on } T, \\
\Rightarrow \quad & \left\{ \frac{j(\cdot,x_n(\cdot))}{\|x_n\|^p} \right\}_{n\geqslant 1} \subseteq L^1(T) \quad \text{is uniformly integrable}\,.
\end{aligned}
$$

Thus because of the Dunford–Pettis theorem, we may assume that

$$
\frac{j(\cdot,x_n(\cdot))}{\|x_n\|^p} \xrightarrow{w} g \quad \text{in } L^1(T).
$$

Arguing as for the sequence $\{u_n/\|x_n\|^{p-1}\}_{n\geqslant 1}$ and using the second set of limit inequalities in hypothesis H(j) (iv), we establish that

$$
g(t) = \eta(t)|y(t)|^p \quad \text{a.e. on } T,
$$

with $\eta \in L^\infty(T)$, $\vartheta_1(t) \leqslant p\eta(t) \leqslant \vartheta_2(t)$ a.e. on T. From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(T)$ we have that

$$
\left| \frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}\,\mathrm{d}t \right| \leqslant \frac{M_1}{\|x_n\|^p}.
$$

Passing to the limit as $n \to \infty$ and recalling that $\|y_n'\|_p \to 0$, we obtain

$$
|c|^p \int_0^b \eta(t)\,\mathrm{d}t = 0.
$$

But $\int_0^b \eta(t)\,\mathrm{d}t > 0$ and so $|c|^p \int_0^b \eta(t)\,\mathrm{d}t > 0$, a contradiction.

Case 2 (both sets $\{0 < \xi\}$ and $\{\xi < \lambda_1\}$ have positive measure). Consider the weighted nonlinear eigenvalue problem

$$
\left.
\begin{aligned}
-(|y'(t)|^{p-2}y'(t))' = \lambda\xi(t)|y(t)|^{p-2}y(t) \quad \text{a.e. on } T, \\
y(0) = y(b), \quad y'(0) = y'(b), \quad \lambda \in \mathbb{R}.
\end{aligned}
\right\} \tag{3.11}
$$

Since $\xi \geqslant 0$, $\lambda_0 = 0$ is the first eigenvalue and because of the strict monotonicity of the eigenvalues on the weight, for the first non-zero eigenvalue $\lambda_1(\xi) > 0$ of (3.11), we have $\lambda_1(\xi) > 1$. Therefore, from (3.10) it follows that $y \equiv 0$, a contradiction to the fact that $\|y\| = 1$.

Case 3 ($\xi = \lambda_1$). From (3.10) it follows that y is an eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$. So

$$
\|y'\|_p^p = \lambda_1\|y\|_p^p \quad \text{and} \quad y(t) \neq 0 \quad \text{a.e. on } T \tag{3.12}
$$

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(see [**15**]; in fact, $y(\cdot)$ has isolated zeros). From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\text{per}}(T)$, we have

$$\frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}\,\mathrm{d}t \leqslant \frac{M_1}{\|x_n\|^p}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\|y'\|_p^p \leqslant \int_0^b \eta(t)|y(t)|^p\,\mathrm{d}t < \lambda_1\|y\|_p^p \quad \left(\text{since } \eta(t) \leqslant \frac{1}{p}\vartheta_2(t) \text{ a.e. on } T\right),$$

which contradicts (3.12).

From the analysis of the three distinct cases, it follows that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\text{per}}(T)$ is bounded. Thus by passing to a suitable subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{\text{per}}(T)$ and $x_n \to x$ in $C_{\text{per}}(T)$. By virtue of hypothesis H(j) (iii), we have that the sequence $\{u_n\}_{n\geqslant 1} \subseteq L^1(T)$ is bounded and so $\int_0^b u_n(x_n - x)\,\mathrm{d}t \to 0$ as $n \to \infty$. Because

$$\left|\langle A(x_n), x_n - x\rangle - \int_0^b u_n(t)(x_n - x)(t)\,\mathrm{d}t\right| \leqslant \varepsilon_n\|x_n - x\|,$$

we obtain that

$$\lim_{n\to\infty}\langle A(x_n), x_n - x\rangle = 0.$$

Exploiting the generalized pseudomonotonicity of A and the Kadec–Klee property of $L^p(T)$ as before, we conclude that $x_n \to x$ in $W^{1,p}_{\text{per}}(T)$. □

Consider the symmetric, pointed, closed and convex cone $C \subseteq W^{1,p}_{\text{per}}(T)$, defined by

$$C = \left\{x \in W^{1,p}_{\text{per}}(T) : \int_0^b |x(t)|^{p-2}x(t)\,\mathrm{d}t = 0\right\}.$$

Proposition 3.3. ***If*** *hypotheses H(j) hold,* ***then*** *$\varphi|_C$ is coercive (i.e. if $\|x\| \to \infty$, then $\varphi(x) \to +\infty$).*

Proof. Suppose that the result of the proposition is not true. Then we can find $\{x_n\}_{n\geqslant 1} \subseteq C$ and $M_2 > 0$ such that

$$\varphi(x_n) \leqslant M_2 \quad \text{for all } n \geqslant 1 \text{ and } \|x_n\| \to \infty.$$

Let $y_n = x_n/\|x_n\|$, $n \geqslant 1$. We may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\text{per}}(T) \quad \text{and} \quad y_n \to y \quad \text{in } C_{\text{per}}(T), \quad \text{with } y \in C.$$

We have

$$\frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}\,\mathrm{d}t \leqslant \frac{M_2}{\|x_n\|^p}. \tag{3.13}$$

As in the proof of Proposition 3.2, we can show that $j(\cdot,x_n(\cdot))/\|x_n\|^p \xrightarrow{w} g$ in $L^1(T)$ with $0 \leqslant g(t) \leqslant (\lambda_1/p)|y(t)|^p$ a.e. on T. So if we pass to the limit as $n \to \infty$ in (3.13), we obtain

$$\frac{1}{p}\|y'\|_p^p \leqslant \frac{\lambda_1}{p}\|y\|_p^p.$$

Because $y \in C$, it follows that (see [**15**])

$$\|y'\|_p^p = \lambda_1 \|y\|_p^p. \tag{3.14}$$

If $y \equiv 0$, then from (3.13) we see that $\|y_n'\|_p \to 0$ and so $y_n \to 0$ in $W^{1,p}_{\mathrm{per}}(T)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geqslant 1$. So $y \neq 0$ and it is an eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$, hence $y(t) \neq 0$ a.e. on T. Moreover, as in the proof of Proposition 3.2, we can check that $g(t) = \eta(t)|y(t)|^p$ with $\vartheta_1(t) \leqslant p\eta(t) \leqslant \vartheta_2(t)$ a.e. on T. Therefore, we have

$$\|y'\|_p^p = \int_0^b \eta(t)|y(t)|^p \, \mathrm{d}t < \lambda_1 \|y\|_p^p,$$

a contradiction to (3.14). This proves the proposition. □

Proposition 3.4. ***If*** *hypotheses* $H(j)$ *hold,* ***then*** $\varphi|_{\mathbb{R}}$ *is anticoercive (i.e.* $\varphi(c) \to -\infty$ *as* $|c| \to \infty$, $c \in \mathbb{R}$*).*

Proof. This is a direct consequence of hypothesis H(j) (v). □

Now we have all the necessary tools to apply Theorem 2.2 and produce a solution for the problem (1.1).

Theorem 3.5. ***If*** *hypotheses* $H(j)$ *hold,* ***then*** *problem (1.1) has a solution* $x \in C^1_{\mathrm{per}}(T)$ *with* $|x'|^{p-2}x' \in W^{1,p}_{\mathrm{per}}(T)$.

Proof. By virtue of Propositions 3.3 and 3.4, we can find $c > 0$ such that

$$\varphi(\pm c) < \inf_C \varphi = m_C.$$

Let

$$E_1 = \{\pm c\}, \quad E = \{x \in W^{1,p}_{\mathrm{per}}(T) : -c \leqslant x(t) \leqslant c \text{ for all } t \in T\} \quad \text{and} \quad D = C.$$

We claim that the sets E_1 and D link in $W^{1,p}_{\mathrm{per}}(T)$. To this end note that $E_1 \cap D = \emptyset$ and let $\gamma \in C(E, W^{1,p}_{\mathrm{per}}(T))$ such that $\gamma|_{E_1} = \mathrm{id}_{E_1}$. So we have $\gamma(\pm c) = \pm c$. Consider the map $\psi : W^{1,p}_{\mathrm{per}}(T) \to \mathbb{R}$ defined by

$$\psi(x) = \int_0^b |x(t)|^{p-2} x(t) \, \mathrm{d}t.$$

Evidently, ψ is continuous. Then $\psi \circ \gamma \in C(E)$ and we have

$$(\psi \circ \gamma)(-c) = \psi(-c) < 0 < \psi(c) = (\psi \circ \gamma)(c).$$

So, by the intermediate-value theorem, we can find $x \in E$ such that

$$\begin{aligned} & \psi(\gamma(x)) = 0, \\ \Rightarrow\quad & \gamma(E) \cap D \neq \emptyset, \\ \Rightarrow\quad & E_1 \text{ and } D \text{ link in } W^{1,p}_{\mathrm{per}}(T). \end{aligned}$$

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Therefore, we can apply Theorem 2.2 and obtain $x \in W^{1,p}_{\mathrm{per}}(T)$ such that

$$0 \in \partial\varphi(x) \quad \text{and} \quad m_C \leqslant \varphi(x).$$

From the inclusion as in the proof of Proposition 3.2, we obtain that $x \in C^1_{\mathrm{per}}(T)$ with $|x'|^{p-2}x' \in W^{1,p}_{\mathrm{per}}(T)$ is a solution of problem (1.1). □

Remark 3.6. As the referee pointed out, it would be interesting to know if we can replace the second condition in hypothesis H(j) (iv) by the weaker conditions

$$\lim_{|c|\to\infty} \int_0^b \left[j(t, cu_1(t)) - \frac{\lambda_1}{p} |cu_1(t)|^p \right] \mathrm{d}t = -\infty \quad \text{and} \quad \lim_{|c|\to\infty} \int_0^b j(t,c)\,\mathrm{d}t = +\infty,$$

where u_1 is an eigenfunction associated with λ_1 (Ahmad–Lazer–Paul-type conditions in terms of the potential function $j(t,x)$).

Example 3.7. Consider the following non-smooth locally Lipschitz function:

$$j_1(x) = \begin{cases} \dfrac{\lambda_1}{2p}|x|^p - |x| & \text{if } x \leqslant 0, \\ \dfrac{\alpha}{p}\lambda_1|x|^p + \dfrac{c}{p}\lambda_1|x|^p \sin(\ln(1+x)) & \text{if } x \geqslant 0. \end{cases}$$

Here $\alpha, c \in (0,1)$, $\alpha > c$ and $\alpha + c\sqrt{1 + (1/p^2)} = 1$. We have

$$\begin{aligned} \lim_{x\to-\infty} \frac{pj_1(x)}{|x|^p} &= \tfrac{1}{2}\lambda_1, \\ \liminf_{x\to+\infty} \frac{pj_1(x)}{|x|^p} &= (\alpha - c)\lambda_1 > 0, \\ \limsup_{x\to+\infty} \frac{pj_1(x)}{|x|^p} &= (\alpha + c)\lambda_1 < \lambda_1. \end{aligned}$$

Also, because $j_1 \in C^1(\mathbb{R}\backslash\{0\})$, we have

$$\begin{aligned} \lim_{x\to-\infty} \frac{j_1'(x)}{|x|^{p-2}x} &= \tfrac{1}{2}\lambda_1, \\ \liminf_{x\to+\infty} \frac{j_1'(x)}{x^{p-1}} &= \left(\alpha + c - \frac{c}{p}\right)\lambda_1 > 0, \\ \limsup_{x\to+\infty} \frac{j_1'(x)}{x^{p-1}} &= \left(\alpha + c\sqrt{1 + \frac{1}{p^2}}\right)\lambda_1 = \lambda_1. \end{aligned}$$

Thus we have resonance at λ_1. Then, for any $\alpha \in L^1(T)$ and $h \in L^\infty(T)$, the function $j(t,x) = j_1(x) + \alpha(t)\max\{|x|, |x|^{1/2}\} + h(t)$ satisfies hypothesis H(j).

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