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Simultaneous Polynomial Approximations of the Lerch Function

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Abstract. We construct bivariate polynomial approximations of the Lerch function that for certain specialisations of the variables and parameters turn out to be Hermite–Padé approximants either of the polylogarithms or of Hurwitz zeta functions. In the former case, we recover known results, while in the latter the results are new and generalise some recent works of Beukers and Prévost. Finally, we make a detailed comparison of our work with Beukers'. Such constructions are useful in the arithmetical study of the values of the Riemann zeta function at integer points and of the Kubota–Leopold p-adic zeta function.

1 Introduction

In this article, we consider polynomial approximations for the Lerch function, defined to be the multivariate series

$$\Phi_s(x,z) = \sum_{n=1}^{\infty} \frac{z^n}{(n+x)^s}.$$

Here, s is a positive integer and z, x are complex numbers such that $|z| \le 1$, x is not a negative integer, and $(s,z) \neq (1,1)$. These conditions ensure the convergence of the above series, which can be analytically continued in z for any fixed s, x. The Lerch function admits as special cases the Hurwitz function[1] $\zeta(s,x) = \Phi_s(x,1)$, the polylogarithms $\mathrm{Li}_s(z) = \Phi_s(0,z)$, and the Riemann zeta function $\zeta(s) = \Phi_s(0,1)$.

The Hermite–Padé approximants of polylogarithms have been extensively studied, with numerous applications to the diophantine theory of $\zeta(s)$; see for example [4, 5, 9–11, 15] and the references therein. On the other hand, the Hermite–Padé approximants of Hurwitz functions seem to have received much less attention; see for example Beukers [7], Prévost [14] (in which he uses Wilson's orthogonal polynomials [20]) and Rivoal [17]. Our aim is to prove simple formulae for certain diagonal Hermite–Padé type approximants of the Lerch function that provide a unifying approach to these problems.

We remind the reader of the definition of the n-th diagonal Hermite–Padé problem (whose solutions are Hermite–Padé approximants) at $z = z_0 \in \mathbb{C} \cup \{\infty\}$ of a

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[1] In the literature, [1, eq. (1.3.1)], the Hurwitz zeta function is sometimes defined as $\Phi_s(x-1,1)$.

family of formal powers series[2]

$$F_k(z) = \sum_{j=0}^{\infty} f_{j,k}(z - z_0)^k \in \mathbb{C}[[z]],\ k = 1, \dots, K.$$

The problem is to find K polynomials $P_1(z), \dots, P_K(z) \in \mathbb{C}[z]$ of degree at most n such that the order at $z = z_0$ of the power series

$$R(z) = P_1(z)F_1(z) + P_2(z)F_2(z) + \cdots + P_K(z)F_K(z)$$

is at least $K(n+1) - 1$ if $z_0 \in \mathbb{C}$ and at least $(K-1)(n+1)$ if $z_0 = \infty$, which is theoretically the best possible order. Uniqueness (up to a multiplicative constant) is not always ensured. It is sometimes useful to have an order Ω that is less than $K(n+1)-1$ (or $(K-1)(n+1)$); we will say that we have obtained diagonal Hermite–Padé *type* approximants. In diophantine approximation, these approximants are useful when Ω is an increasing function of n, typically $\Omega \approx c_K n$ for some constant $c_K > 0$; see [4, 15] for an example showing that the best possible value for Ω does not necessarily yield the best number-theoretical results. In all cases, we use the notation $R(z) = \mathcal{O}\big((z - z_0)^{\Omega}\big)$.

The functions $\Phi_s(x, z)$ being convergent Taylor series in z, we can consider Hermite–Padé type approximants at $z = 0$ for the family

$$(1, \Phi_1(x, z), \Phi_2(x, z), \dots, \Phi_A(x, z))$$

for any fixed integer $A \geq 1$ and any fixed x. Explicit formulae in the diagonal case have been known for a long time for $x = 0$, and the general case is little different. Surprisingly, it has so far not been mentioned in the literature that these formulae contain more. They also provide, with almost no change, diagonal Hermite–Padé type approximants at $x = +\infty$ for the family $(1, \Phi_1(x, z), \Phi_2(x, z), \dots, \Phi_A(x, z))$ for any fixed integer $A \geq 1$ and any fixed z. Indeed, let us consider the following problem: given any integers $A \geq 2$, $n \geq 0$, $r \geq 0$ such that $A(n+1) \geq r + 2$, find $A + 1$ polynomials $P_0(x, z), P_1(x, z), \dots, P_A(x, z)$ in $\mathbb{Q}[z, x]$, of degree at most r in x and at most n in z, and $\widehat{P}_0(x, z) \in \mathbb{Q}(x)[z]$ of degree at most n in z, such that

$$\begin{aligned}
\mathscr{R}_{A,n,r}(x, z) &= P_0(x, z) + \sum_{j=1}^{A} P_j(x, z)\Phi_j(x, 1/z) \\
&= \mathcal{O}(x^{-A(n+1)+r+1}) \text{ at } x = +\infty, \\
\mathscr{S}_{A,n,r}(x, z) &= \widehat{P}_0(x, z) + \sum_{j=1}^{A} P_j(x + r, z)\Phi_j(x, 1/z) \\
&= \mathcal{O}(z^{-r-1}) \text{ at } z = \infty.
\end{aligned} \tag{1.1}$$

[2] If $z_0 = \infty$, we replace every occurrence of $z - z_0$ by $1/z$. Hermite–Padé approximants are also known as Padé approximants of the first type, not to be confused with our use of the word "type" in this article.

When we fix x, resp. z, the first, resp. second, equation is a diagonal Hermite–Padé type problem in z, resp. in x. Before going further, we warn the reader that the functions $\Phi_s(x,z)$ are not holomorphic nor even real analytic at $x=\infty$. Therefore, the expression "Hermite–Padé type approximants at $x=+\infty$" in Problem (1.1) is *a priori* an abuse of language. For the moment, it is enough to consider that in (1.1) we ask that $\lim_{x\to+\infty} x^{A(n+1)-r-1}\mathscr{R}_{A,n,r}(x,z)$ be finite. In fact, this is not a real abuse of language, because the Lerch function $\Phi_s(x,z)$ admits an asymptotic expansion $\widehat{\Phi}_s(x,z)$ in powers of $1/x$ and the polynomials $P_j(x,z)$ in Problem (1.1) provide a solution to the Hermite–Padé problem for $\big(1,\widehat{\Phi}_1(x,z),\widehat{\Phi}_2(x,z),\dots,\widehat{\Phi}_A(x,z)\big)$ at $x=\infty$. This is a well-known generalisation of Hermite–Padé approximants; see [12, p. 66]. However, since this distinction will be useful, we give more details in Section 2.

We cannot expect to have a unique solution for Problem (1.1) (even up to multiplicative constant). The following result provides a possible solution.

Theorem 1 *When $r\geq n$, Problem* (1.1) *admits the following explicit solution:*

$$\mathscr{R}_{A,n,r}(x,z)=\frac{n!^A}{r!}\sum_{k=1}^{\infty}\frac{(k)_r}{(k+x)_{n+1}^A}z^{-k}, \tag{1.2}$$

$$\mathscr{S}_{A,n,r}(x,z)=\frac{n!^A}{r!}\sum_{k=1}^{\infty}\frac{(k)_r}{(k+x+r)_{n+1}^A}z^{-k-r}, \tag{1.3}$$

which are holomorphic functions in the variables x,z in the domain defined by $|z|\geq 1$ (including $z=\infty$) and $x\notin\{-1,-2,-3,\dots\}$. We also have

$$\begin{aligned}
P_s(x,z)&=\sum_{j=0}^{n}\frac{(-1)^{jA+r}z^j}{(A-s)!}\Big(\frac{\mathrm{d}}{\mathrm{d}j}\Big)^{A-s}\\
&\quad\times\left(\binom{n}{j}^A\binom{x+j}{r}\right)\in\mathbb{Q}[x,z]\quad \textit{for } s\geq 1,\\
\widehat{P}_0(x,z)&=\sum_{s=1}^{A}\sum_{j=1}^{n}\sum_{\ell=1}^{j}\frac{(-1)^{jA+r+1}z^{j-\ell}}{(\ell+x)^s(A-s)!}\Big(\frac{\mathrm{d}}{\mathrm{d}j}\Big)^{A-s}\\
&\quad\times\left(\binom{n}{j}^A\binom{x+j}{r}\right)\in\mathbb{Q}(x)[z],\\
P_0(x,z)&=\frac{n!^A}{(A-1)!r!}\sum_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}\ell}\right)^{A-1}\\
&\quad\times\left(\frac{(-\ell-x)_r(\ell-j)^A}{(-\ell)_{n+1}^A}\sum_{k=0}^{j-1}\frac{z^k}{\ell+x-k}\right)_{|\ell=j}\in\mathbb{Q}[x,z].
\end{aligned}\tag{1.4}$$

Remarks. (a) The explicit expression for the polynomials P_s is obtained by partial fraction expansion with respect to k of the rational function $(k)_{rn}/(k+x)_{n+1}^A$. For P_s

($s = 1, \dots, A-1$ and $\widehat{P}_0$), the symbol $(\mathrm{d}/\mathrm{d}j)^{A-s}$ is a formal derivative that must be handled carefully: we refer to the proof for the exact meaning of this statement. On the other hand, the derivative $(\mathrm{d}/\mathrm{d}\ell)^{A-1}$ in P_0 is well defined.

(b) Equations (1.2) and (1.3) enable us to control explicitly the error terms in the approximations, which is not the case of the formal $\mathcal{O}\big(z^{-r-1}\big)$ and $\mathcal{O}\big(x^{-A(n+1)+r+1}\big)$ we initially asked for. This could be useful for arithmetical applications of our results although, as Beukers pointed out to the author, the series for $\mathscr{R}_{A,n,r}(x,z)$ does not give its asymptotic expansion in $1/x$ and such an expansion is an important tool in [7].

(c) The most difficult part of the theorem is proving that $P_0(x,z)$ is a polynomial in x, which is done by transformation of a complicated formula for $P_0(x,z)$ into (1.4) (Incidentally, the degrees of $P_0(x,z)$ in x and z are only $r-1$ and $n-1$ respectively). On the other hand, $\widehat{P}_0(x;z)$ is a polynomial in z but not in x; it would be a polynomial in both variables if we replaced the numerator $(k)_r$ of the series in (1.2) and (1.3) by $(k)_{2r}$. This would be a completely different approximation problem: in particular, Corollary 2(ii) below would no longer hold, because r would have to be essentially $\leq An/2$.

Theorem 1 has a number of consequences; we mention two of these in the following corollary.

Corollary 2 (i) *When $r = n$ and $z = 1$, the series $\mathscr{R}_{A,n,r}(x,1)$ in* (1.2) *is a solution of the n-th diagonal Hermite–Padé problem at $x = +\infty$ for the A functions* $(1, \zeta(2,x), \zeta(3,x), \dots, \zeta(A,x))$.

(ii) *When $r = A(n+1) - 1$ and for any fixed x, the series $\mathscr{S}_{A,n,r}(x,z)$ in* (1.3) *is a solution of the n-th diagonal Hermite–Padé problem at $z = \infty$ for the $A+1$ functions* $(1, \Phi_1(x,1/z), \Phi_3(x,1/z), \dots, \Phi_A(x,1/z))$.

Remarks. (a) Assertion (ii) is well known to experts and we mention it for completeness. When $x = 0$, it is part of a theorem of Nikishin [11] (which was generalised in [16] to arbitrary values of r).

(b) The case $A = 2$ and $r = n$ in (i) was obtained by Prévost [14] and Beukers [7] independently in different forms and settings (see Section 4.1 for a comparison of Beukers' solution and ours); Beukers even mentions that his solution is implicit in Stieltjes' classical work on continued fractions. On the other hand, the second case, which deals with Hurwitz functions, seems to be new when $A \geq 3$.

(c) For any fixed $z \neq 1$, there is no value of r such that the series $\mathscr{R}_{A,n,r}(x,z)$ is a solution of the n-th diagonal Hermite-Padé problem at $x = +\infty$ for the $A+1$ functions $(1, \Phi_1(x,1/z), \Phi_2(x,1/z), \dots, \Phi_A(x,1/z))$. This is due to the presence of the function $\Phi_1(x,z)$, whose polynomial coefficient "unexpectedly" vanishes identically when $z = 1$ (yielding (i)).

We now consider another Hermite–Padé type problem. Given any integers $A \geq 2$, $n \geq 0$, and $r \geq 0$ such that $A(n+1) \geq 2r+3$, find $A-1$ polynomials $Q_0(x), Q_2(x), \dots, Q_{A-1}(x)$ in $\mathbb{Q}[x]$, of degree at most $2r+1$, such that

$$\begin{aligned}\mathscr{T}_{A,n,r}(x) &= Q_0(x) + \sum_{\substack{j=2,\dots,A-1\\ j\equiv A-1 \pmod 2}} Q_j(x)\Phi_j(x;(-1)^A)\\ &= \mathcal{O}(x^{-A(n+1)+2r+2}) \ \text{ at } x = +\infty.\end{aligned} \tag{1.5}$$

When A is even and $r = n$, Problem (1.5) is by definition the $(2n+1)$-th diagonal Hermite–Padé problem for $\big(1, \zeta(3,x), \zeta(5,x), \dots, \zeta(A-1,x)\big)$. When $r = n$, the case $A = 4$ was solved by Prévost [14] and Beukers [7] in different forms; their solutions also have a different form from the one proposed here, which has the advantage of being generalisable to larger values of A and r. See Section 4.2 for a comparison with Beukers' solution for $A = 4, r = n$.

Theorem 3 *When $r \geq n$, Problem* (1.5) *admits the following solution:*

$$\mathscr{T}_{A,n,r}(x) = \frac{n!^A}{r!^2} \sum_{k=1}^{\infty} (-1)^{kA} \Big(k + x + \frac{n}{2}\Big) \frac{(k)_r (k+2x+n-r+1)_r}{(k+x)_{n+1}^A},$$

which converges for all $x \in \mathbb{C} \setminus \{-1,-2,-3,\dots\}$. We also have that

$$Q_s(x) = \frac{(-1)^r}{(A-s)!} \sum_{j=0}^{n} \Big(\frac{\mathrm{d}}{\mathrm{d}j}\Big)^{A-s} \left(\Big(\frac{n}{2} - j\Big) \binom{n}{j}^A \binom{x+j}{r} \binom{x+n-j}{r} \right) \in \mathbb{Q}[x]$$

for $s \geq 2$ and

$$Q_0(x) = \frac{(-1)^r n!^A}{(A-1)! r!^2} \sum_{j=1}^{n} \Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{A-1}$$

$$\times \left(\Big(\frac{n}{2} - \ell\Big) \frac{(-\ell-x)_r (\ell-x-n)_r (\ell-j)^A}{(-\ell)_{n+1}^A} \sum_{k=0}^{j-1} \frac{1}{\ell+x-k} \right)_{|\ell=j} \in \mathbb{Q}[x].$$

Remark. When A is odd, there is no choice of r such that Problem (1.5) becomes the $(2r+1)$-th diagonal Hermite–Padé for $\big(1, \widetilde{\zeta}(2,x), \widetilde{\zeta}(4,x), \dots, \widetilde{\zeta}(A-1,x)\big)$, where $\widetilde{\zeta}(s,x) = \Phi_s(x,-1)$.

The rest of the article is organised as follows. In Section 2, we compare Hermite–Padé approximants with the similar approximations obtained by replacing formal series by asymptotic expansions; this result is stated as Proposition 4. We prove Theorem 1, Corollary 2, and Theorem 3 in that order in Section 3. Finally, in Section 4, we compare our results with the results of Beukers quoted above, which present interesting differences.

2 Asymptotic and Formal Hermite–Padé Approximants

For fixed s, z, the function $\Phi_s(x,z)$ is not defined as a formal power series in x and, as indicated in the introduction, it is not holomorphic or real analytic at $x = \infty$ (and hence cannot be expanded as a convergent power series in $1/x$). Therefore, strictly speaking, we cannot seek Hermite–Padé type approximants at $x = \infty$ for it. Nevertheless, this problem can be fixed by a classical extension of the notion of Hermite–Padé approximants; see [12, p. 66] for more details and references.

Indeed, the Lerch function admits an asymptotic expansion

$$\Phi_s(x,z) \sim \sum_{k=1}^{\infty} \phi_k(s,z)x^{-k}$$

in Poincaré's sense: for all $N \geq 0$, we have that

$$\Phi_s(x,z) = \sum_{k=0}^{N} \phi_k(s,z)x^{-k-s+1} + \mathcal{O}\big(x^{-s-N}\big),$$

uniformly in the half-plane $\Re(x) > 0$, where

$$\phi_k(s,z) = \begin{cases} \binom{k+s-2}{s-2}\dfrac{B_k}{s-1} & \text{if } z = 1, s \geq 2, \\ (-1)^k \binom{k+s-1}{s-1} \vartheta^k\Big(\dfrac{z}{1-z}\Big) & \text{if } z \neq 1, s \geq 1. \end{cases}$$

Here, B_k is the k-th Bernoulli number and $\vartheta = z\mathrm{d}/\mathrm{d}z$. Since a function admits at most one asymptotic expansion in powers of $1/x$, it is natural to use the following notion of "order at infinity": given a function g defined in a ray $S = (e^{i\vartheta}A, e^{i\vartheta}\infty)$ (for a certain $A \in \mathbb{R}$) and which admits an asymptotic expansion as $x \to e^{i\vartheta}\infty$, we say that g is of order at least K at $x = e^{i\vartheta}\infty$ if for every integer k with $0 \leq k \leq K-1$, $\lim_{x\to e^{i\vartheta}\infty, x\in S} x^j f(x) = 0$. In this case, we write $g(x) = \mathcal{O}\big(x^{-K}\big)$. It may happen that the asymptotic expansion holds in a much larger domain than a ray, typically in an open angular sector: the definition extends accordingly. For example, $\Phi_s(x,z)$ has order $s-1$ in x at infinity in the half-plane $\Re(x) > 0$.

Let us now consider K complex valued functions $F_1(x), F_2(x), \dots, F_K(x)$ defined on an interval $(A, +\infty)$ ($A \in \mathbb{R}$). Let us suppose that all these functions have an asymptotic expansion to all orders when $x = +\infty$, *i.e.,*

$$F_k(x) = \sum_{j=0}^{N-1} f_{j,k}x^{-j} + \mathcal{O}(x^{-N})$$

as $x \to +\infty$ for every integer $N \geq 0$. For each $k = 1, \dots, K$, we denote by $\widehat{F}_k(x) = \sum_{j=0}^{\infty} f_{j,k}x^{-j}$ the formal power series associated with F_k (this series may be not convergent or may converge to a function different from $F_k(x)$). Let us also suppose that for a given integer $n \geq 0$, we can find K polynomials $P_1(x), \dots, P_K(x) \in \mathbb{C}[x]$ of degree at most n such that

$$\lim_{x\to+\infty} x^{(K-1)(n+1)-1} \sum_{k=1}^{K} P_k(x)F_k(x) = 0. \tag{2.1}$$

In this situation, we have the following easy result, which will be used in what follows. We prove it for completeness.

Proposition 4 *The set of polynomials* $\big(P_1(x), P_2(x), \dots, P_K(x)\big)$ *is a solution of the n-th diagonal Hermite–Padé problem at* $x = \infty$ *for* $(\widehat{F}_1(x), \widehat{F}_2(x), \dots, \widehat{F}_K(x))$.

Remarks. (a) Obviously, by a change of variable, we obtain similar results when the asymptotic is taken along a ray direction $x \to e^{i\vartheta}\infty$ or in the case of an asymptotic expansion in a neighborhood of a complex x_0, where

$$F_k(x) = \sum_{j=0}^{N-1} f_{j,k}(x - x_0)^j + \mathcal{O}((x - x_0)^N).$$

(b) This theorem shows that it is consistent with the usual definition of Hermite–Padé approximants of formal series to consider P's to be solutions of the "n-th Hermite–Padé (type) problem at $x = +\infty$ of $F_1(x), F_2(x), \dots, F_K(x)$". To emphasize the difference between these notions, we could say that we compute the "n-th asymptotic Hermite–Padé (type) approximants at $x = +\infty$" for these functions. In Theorems 1 and 3 the convergence is even stronger than just $x \to +\infty$ since, in those cases, the asymptotic expansion $\Phi_s(x, z) \sim \sum_{k=1}^{\infty} \phi_k(s, z)x^{-k}$ holds uniformly in the half-plane $\Re(x) > 0$.

Proof It is easier to work at $x = 0$. Changing the variable x to $1/x$, we suppose given functions $G_1(x), \dots, G_K(x)$ (obtained from the F's by the formula $G_j(x) = F_j(1/x)$) which all admit an asymptotic expansion at $x = 0$ in an interval $[0, A]$. We then have $G_k(x) = \sum_{j=0}^{N-1} f_{j,k}x^j + \mathcal{O}(x^N$ for each $N \geq 0$. By hypothesis (2.1), there exist K polynomials $Q_1(x), \dots, Q_K(x) \in \mathbb{C}[x]$ of degree at most n (obtained from the P's by the formula $Q_j(x) = x^n P_j(1/x)$) such that

$$\lim_{x \to 0+} x^{-K(n+1)+2} \sum_{k=1}^{K} Q_k(x)G_k(x) = 0. \tag{2.2}$$

Set $\widehat{G}_k(x) = \sum_{j=0}^{\infty} f_{j,k}x^j \in \mathbb{C}[[x]]$ and $Q_k(x) = \sum_{j=0}^{n} q_{j,k}x^j$. We have that

$$\sum_{k=1}^{K} Q_k(x)G_k(x) = \sum_{k=1}^{K}\Big(\sum_{j=0}^{n} q_{j,k}x^j\Big)\Big(\sum_{j=0}^{K(n+1)-2} f_{j,k}x^j\Big) + \mathcal{O}(x^{K(n+1)-1}),$$

and condition (2.2) ensures that the polynomial

$$\sum_{k=1}^{K}\Big(\sum_{j=0}^{n} q_{j,k}x^j\Big)\Big(\sum_{j=0}^{K(n+1)-2} f_{j,k}x^j\Big)$$

is identically 0. But this means exactly that $Q_1(x), \dots, Q_K(x)$ are solutions of the n-th diagonal Hermite–Padé problem for the formal series $\widehat{G}_1(x), \dots, \widehat{G}_K(x)$. ■

3 Proofs of the Results

3.1 Proof of Theorem 1

Under the conditions of Problem 1.1, let us define the function

$$\widehat{\mathscr{S}}_{A,n,r}(x,z) = \frac{n!^A}{r!} \sum_{k=1}^{\infty} \frac{(k)_r}{(k+x+r)_{n+1}^A} z^{-k-r},$$

which is clearly holomorphic in z in $|z| \geq 1$ (including $z = \infty$) and meromorphic in x in $\mathbb{C} \setminus \{-1-r, -2-r, -3-r, \dots\}$. Furthermore, we obviously have $\widehat{\mathscr{S}}_{A,n,r}(x,z) = \mathcal{O}(z^{-r-1})$. With $k = \ell - r$, we have that

$$\widehat{\mathscr{S}}_{A,n,r}(x,z) = \frac{n!^A}{r!} \sum_{\ell=r+1}^{\infty} \frac{(\ell-r)_r}{(\ell+x)_{n+1}^A} z^{-\ell} = \frac{n!^A}{r!} \sum_{\ell=1}^{\infty} \frac{(\ell-r)_r}{(\ell+x)_{n+1}^A} z^{-\ell},$$

where the second equality holds because $(\ell - r)_r = 0$ for all $\ell \in \{1, 2, \dots, r\}$. By partial fraction expansion, we have

$$\frac{n!^A}{r!} \frac{(\ell-r)_r}{(\ell+x)_{n+1}^A} = \sum_{s=1}^{A} \sum_{j=0}^{n} \frac{C_{j,s}(x)}{(\ell+x+j)^s}$$

with

$$C_{j,s}(x) = \frac{1}{(A-j)!} \Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{A-j} \Big(\frac{n!^A}{r!} \frac{(\ell-r)_r}{(\ell+x)_{n+1}^A} (\ell+x+j)^A\Big)_{|\ell=-j-x}.$$

Note that we have used here the property that $A(n+1) \geq r+2$, which ensures that there is no polynomial part in the expansion. Hence, we have $\widehat{\mathscr{S}}_{A,n,r}(x,z) = \widehat{P}_0(x,z) + \sum_{s=1}^{A} \widehat{P}_j(x,z)\Phi_s(x,z)$ with

$$\widehat{P}_0(x,z) = -\sum_{s=1}^{A} \sum_{j=1}^{n} C_{j,s}(x) \sum_{\ell=1}^{j} \frac{z^{j-\ell}}{(\ell+x)^s}$$

and, for $s \geq 1$, $\widehat{P}_s(x,z) = \sum_{j=0}^{n} C_{j,s}(x) z^j$.

Let us now define

$$\widehat{\mathscr{R}}_{A,n,r}(x,z) = z^r \widehat{\mathscr{S}}_{A,n,r}(x-r,z) = \frac{n!^A}{r!} \sum_{k=1}^{\infty} \frac{(k)_r}{(k+x)_{n+1}^A} z^{-k},$$

which is holomorphic in z for $|z| \geq 1$ (including $z = \infty$) and meromorphic in x in $\mathbb{C} \setminus \{-1, -2, -3, \dots\}$. By partial fraction expansion, we have

$$\frac{n!^A}{r!} \frac{(k)_r}{(k+x)_{n+1}^A} = \sum_{s=1}^{A} \sum_{j=0}^{n} \frac{D_{j,s}(x)}{(k+x+j)^s}$$

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with

$$D_{j,s}(x) = \frac{1}{(A-j)!}\Big(\frac{\mathrm{d}}{\mathrm{d}k}\Big)^{A-j}\Big(\frac{n!^A}{r!}\frac{(k)_r}{(k+x)_{n+1}^A}(k+x+j)^A\Big)_{|k=-j-x}. \tag{3.1}$$

Hence, we have $\widehat{\mathscr{R}}_{A,n,r}(x,z) = P_0(x,z) + \sum_{s=1}^{A} P_j(x,z)\Phi_s(x,z)$ with

$$P_0(x,z) = -\sum_{j=1}^{n}\sum_{s=1}^{A} D_{j,s}(x)\sum_{k=1}^{j}\frac{z^{j-k}}{(k+x)^s}$$

and, for $s \geq 1$, $P_s(x,z) = \sum_{j=0}^{n} D_{j,s}(x)z^j$. We have $C_{j,s}(x) = D_{j,s}(x+r)$ and therefore for all $s \geq 1$, $\widehat{P}_s(x,z) = P_s(x+r,z)$.

The right-hand side of equation (3.1) shows clearly that, for $s \geq 1$, $P_s(x,z)$ and $Q_s(x,z)$ are both polynomials in z of degree (at most) n and polynomials in x of degree (at most) r. We have the following symbolic expression for P_s:

$$P_s(x,z) = \frac{1}{(A-s)!}\sum_{j=0}^{n}(-1)^{jA+r}\Big(\frac{\mathrm{d}}{\mathrm{d}j}\Big)^{A-s}\Big(\binom{n}{j}^A\binom{x+j}{r}\Big)z^j.$$

We also have that $P_0(x,z)$ is a polynomial in z of degree at most $n-1$, but it is much less obvious that $P_0(x,z)$ is also a polynomial in x. To prove this, we first note that

$$D_j(x) = \frac{(-1)^{A-s}}{(A-s)!}\Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{A-s}\Big(\frac{n!^A}{r!}\frac{(-\ell-x)_r}{(-\ell)_{n+1}^A}(j-\ell)^A\Big)_{|\ell=j},$$

and

$$\sum_{k=1}^{j}\frac{z^{j-k}}{(k+x)^s} = \frac{(-1)^{s-1}}{(s-1)!}\Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{s-1}\Big(\sum_{k=0}^{j-1}\frac{z^k}{\ell+x-k}\Big)_{|\ell=j}.$$

Hence, by Leibniz' formula,

$$\begin{aligned}
&\sum_{s=1}^{A} D_{j,s}(x)\sum_{k=1}^{j}\frac{z^{j-k}}{(k+x)^s}\\
&\quad= \frac{(-1)^{A-1}}{(A-1)!}\sum_{s=1}^{A}\binom{A-1}{s-1}\Big(\frac{n!^A}{r!}\frac{(-\ell-x)_r}{(-\ell)_{n+1}^A}(j-\ell)^A\Big)^{(A-s)}_{|\ell=j}\Big(\sum_{k=0}^{j-1}\frac{z^k}{\ell+x-k}\Big)^{(s-1)}_{|\ell=j}\\
&\quad= \frac{(-1)^{A-1}}{(A-1)!}\Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{A-1}\Big(\frac{(-\ell-x)_r(j-\ell)^A}{(-\ell)_{n+1}^A}\sum_{k=0}^{j-1}\frac{z^k}{\ell+x-k}\Big)_{|\ell=j}.
\end{aligned}$$

We have not yet used the hypothesis that $r \geq n$, and this enables us to use the following trivial but crucial fact; for any $j \in \{1,\dots,n\}$ and any $k \in \{1,\dots,j-1\}$, the polynomial $x-k$ is a factor of the polynomial $(-x)_r$ for any $r \geq n$. Hence, the rational function in x, ℓ,

$$F(x,\ell)=\frac{(-x)_r(j-\ell)^A}{(-\ell)_{n+1}^A}\sum_{k=0}^{j-1}\frac{z^k}{x-k}$$

is actually a polynomial in x of degree at most $r-1$ and a rational function of ℓ; the same is true of the function

$$\frac{\partial^{A-1}}{\partial \ell^{A-1}}F(x+\ell,\ell)=\Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{A-1}\Big(\frac{(-\ell-x)_r(j-\ell)^A}{(-\ell)_{n+1}^A}\sum_{k=0}^{j-1}\frac{z^k}{\ell+x-k}\Big).$$

Finally, the formula

$$P_0(x,z)=\frac{n!^A}{(A-1)!r!}\sum_{j=1}^{n}\Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{A-1}\Big(\frac{(-\ell-x)_r(\ell-j)^A}{(-\ell)_{n+1}^A}\sum_{k=0}^{j-1}\frac{z^k}{\ell+x-k}\Big)_{|\ell=j}$$

shows that $P_0(x,z)$ is a polynomial in x of degree at most $r-1$.

To complete the proof of the theorem, it remains to show that

$$\lim_{x\to+\infty}x^{A(n+1)-r-2}\widehat{\mathscr{R}}_{A,n,r}(x,z)=0.$$

But for $\Re(x)>0$ we have that

$$(3.2)\qquad |x^{A(n+1)-r-2}\widehat{\mathscr{R}}_{A,n,r}(x,z)|\leq\sum_{k=1}^{\infty}\frac{(k)_r|x|^{A(n+1)-r-2}}{|k+x|^{A(n+1)}}$$
$$\leq\sum_{k=1}^{\infty}\frac{(k)_r|x|^{A(n+1)-r-2}}{|x|^{A(n+1)-r-2}|k+x|^{r+2}}=\sum_{k=1}^{\infty}\frac{(k)_r}{|k+x|^{r+2}}.$$

Obviously this last series tends to 0 as $|x|\to\infty$ uniformly in the half-plane $\Re(x)>0$. Since $\widehat{\mathscr{R}}_{A,n,r}(x,z)$ admits an asymptotic expansion as $x\to+\infty$, we therefore have that $\widehat{\mathscr{R}}_{A,n,r}(x,z)=\mathcal{O}\big(x^{-A(n+1)+r+1}\big)$. Hence, the series $\widehat{\mathscr{R}}_{A,n,r}(x,z)$ and $\widehat{\mathscr{S}_{A,n,r}}(x,z)$ are a solution to the approximation Problem 1.1.

3.2 Proof of Corollary 2

Case (ii) does not need a proof because this is the definition of Hermite–Padé approximants at $z=\infty$.

This is almost true for case (i), except that we must explain how the divergent series $\zeta(1,x)$ disappears. This follows from a general fact. Under the conditions of Theorem 1, for all $r\geq n$, the polynomial $P_1(x,z)$ is identically zero for $z=1$. This can be proved as follows: the series $\mathscr{R}_{A,n,r}(x,z)$ in equation (1.2) is convergent for any z such that $|z|\geq 1$ and $x\notin\{-1,-2,-3,\dots\}$. In particular $\mathscr{R}_{A,n,r}(x,1)$ is convergent, and the only potentially divergent term amongst the $P_s(x,1)\Phi_s(x,1)$ is $s=1$. Abel's continuity theorem implies that $\lim_{z\to 1}P_s(x,z)\Phi_s(x,z)$ exists and is finite. Therefore, for all $x\in\mathbb{C}$, $P_1(x,1)=0$ and $\mathscr{R}_{A,n,r}(x,1)=P_0(x,1)+\sum_{s=2}^{A}P_s(x,1)\zeta(s,x)$.

3.3 Proof of Theorem 3

The proof is similar to that of Theorem 1. We first define the series

$$\mathscr{T}_{A,n,r}(x,z) = \frac{n!^A}{r!^2} \sum_{k=1}^{\infty} \Big(k + x + \frac{n}{2}\Big) \frac{(k)_r (k+2x+n-r+1)_r}{(k+x)_{n+1}^A} z^{-k},$$

which, under the conditions of the theorem, converges for

$$|z| \geq 1 \quad \text{and} \quad x \notin \{-1,-2,-3,\dots\}.$$

We also define the rational fraction of k

$$R(k) = \Big(k + x + \frac{n}{2}\Big) \frac{(k)_r (k+2x+n-r+1)_r}{(k+x)_{n+1}^A},$$

which satisfies the functional equation $R(k) = (-1)^{A(n+1)+1} R(-k-2x-n)$. By partial fraction expansion, we have

$$R(k) = \sum_{s=1}^{A} \sum_{j=0}^{n} \frac{E_{j,s}(x)}{(k+x+j)^s}$$

where

$$E_{j,s}(x) = \frac{1}{(A-j)!} \Big(\frac{\mathrm{d}}{\mathrm{d}k}\Big)^{A-j} \big(R(k)\,(k+x+j)^A\big)_{|k=-j-x}.$$

By uniqueness of this expansion, the functional equation for $R(k)$ implies that

$$E_{n-j,s}(x) = (-1)^{s+A(n+1)+1} E_{j,s}(x). \tag{3.3}$$

By the same process as in Section 3.1, we deduce that

$$\mathscr{T}_{A,n,r}(x,z) = Q_0(x,z) + \sum_{s=1}^{A} Q_s(x,z)\Phi_s(x,z),$$

where $Q_s(x,z) = \sum_{j=0}^{n} E_{j,s}(x) z^j$ for $s \geq 1$ and

$$Q_0(x,z) = \frac{(-1)^r n!^A}{(A-1)! r!^2} \sum_{j=1}^{n} \Big(\frac{\mathrm{d}}{\mathrm{d}\ell}\Big)^{A-1}$$
$$\times \Big(\Big(\frac{n}{2} - \ell\Big) \frac{(-\ell-x)_r (\ell-x-n)_r (\ell-j)^A}{(-\ell)_{n+1}^A} \sum_{k=0}^{j-1} \frac{z^k}{\ell+x-k} \Big)_{|\ell=j}.$$

Clearly, for $s \geq 1$, the $Q_s(x,z)$ are polynomials of degree (at most) n in z and $2r$ in x, while $Q_0(s,z)$ is of degree (at most) $n-1$ in z and $2r-1$ in x.

Equation (3.3) implies that $z^nQ_s(x,1/z) = (-1)^{A(n+1)+s+1}Q_s(x,1/z)$ and, in particular, that

$$Q_s(x,(-1)^A) = (-1)^{A+s+1}Q_s(x,(-1)^A). \tag{3.4}$$

Therefore, $Q_s(x,(-1)^A) = 0$ if $s \equiv A \pmod 2$, and we have

$$\mathscr{T}_{A,n,r}(x,(-1)^A) = Q_0(x,(-1)^A) + \sum_{\substack{1\le s\le A\\ s\equiv A+1 \pmod 2}} Q_s\big(x,(-1)\big)\,\Phi_s\big(x;(-1)^A\big). \tag{3.5}$$

The term $Q_1(x,(-1)^A)\Phi_1(x;(-1)^A)$ appears only when A is even, in which case the series $\Phi_1(x;(-1)^A)$ is divergent. By the same argument as in Section 3.2, this implies that $Q_1(x,(-1)^A) = 0$ identically. Furthermore, we also have $Q_A(x,(-1)^A) = 0$ identically for any $A \geq 3$ by the functional equation (3.4). Therefore, the sum in (3.5) starts at $j = 2$ and stops at $j = A-1$. Finally, the order at $x =+ \infty$ of $\mathscr{T}_{A,n,r}(x,(-1)^A)$ is obtained by upper bounds similar to (3.2), and Problem (1.5) is completely solved.

4 Comparison with Some Results of Beukers

In this section, we compare our formulas with those obtained by Beukers in certain special cases. The example in Subsection 4.2 is particularly instructive, since it shows the relevance of asymptotic Padé approximants.

4.1 Theorem 1 for $A = 2$ and $r = n$

Let us define the formal series $R(x) = \sum_{k=0}^{\infty} B_k x^{-k-1}$, which is the asymptotic expansion of $\zeta(2,x)$ at $x = +\infty$. Up to change of notation, Beukers [7] showed that

$$\sum_{k=1}^{\infty} \frac{(k-n)_n}{(k)_{n+1}}\frac{(k-1)!}{(x+1)_k} = p_n(x)R(x) + q_n(x) = \mathcal{O}(x^{-n-1}), \tag{4.1}$$

where $p_n(x)$ and $q_n(x)$ are polynomials in $\mathbb{C}[x]$ of degree n. It is important to note that these equalities are equalities of formal series; this is a solution to the n-th diagonal Padé problem for $(1,R(x))$. Our Corollary 2 (case (ii) with $A = 2$, $r = n$) produces a solution to the n-th Padé problem for $(1,\zeta(2,x))$

$$\mathscr{R}_{2,n,n}(x,1) = n!\sum_{k=1}^{\infty}\frac{(k)_n}{(k+x)_{n+1}^2} = P_{2,n}(x)\zeta(2,x) + P_{0,n}(x) = \mathcal{O}(x^{-n-1}),$$

where the equalities have an analytical meaning. Since $R(x)$ is the asymptotic expansion of $\zeta(2,x)$, Proposition 4 shows that $(P_{2,n}(x), P_{0,n}(x))$ is also a solution of the formal Padé problem for $(1,R(x))$.

Let us call $R_n(x)$ the series on the left-hand side of (4.1). It is not just a formal series: it converges uniformly on $\Re(x) > -1$, where it defines a holomorphic function. Since $p_0(x) = 1$ and $q_0(x) = 0$, $R_n(x)$ can be viewed as a solution to the n-th (asymptotic) Padé problem at $x = +\infty$ for $(1, R_0(x))$. It is natural to compare the series $R_n(x)$ and $\mathscr{R}_{2,n,n}(x,1)$, because $R_0(x)$ and $\zeta(2,x)$ have the same asymptotic expansion. We will prove that they are actually equal[3] on $\Re(x) > -1$. For this, we first express them as hypergeometric functions:

$$R_n(x) = \frac{\Gamma(n+1)^3\Gamma(x+1)}{\Gamma(2n+2)\Gamma(x+n+2)}\,{}_3F_2\left[\begin{matrix} n+1,\ n+1,\ n+1 \\ x+n+2,\ 2n+2 \end{matrix};1\right],$$

$$\mathscr{R}_{2,n,n}(x,1) = \frac{\Gamma(n+1)^2\Gamma(x+1)^2}{\Gamma(x+n+2)^2}\,{}_3F_2\left[\begin{matrix} n+1,\ x+1,\ x+1 \\ x+n+2,\ x+n+2 \end{matrix};1\right].$$

We can now apply Thomae's ${}_3F_2$ relation [19, (4.3.1)], and we obtain that

$$\mathscr{R}_{2,n,n}(x,1) = R_n(x)$$

on $\Re(x) > -1$.

In particular, for $n = 0$, the equality $\mathscr{R}_{2,0,0}(x,1) = R_0(x)$ reads

$$\zeta(2,x) = \sum_{k=1}^{\infty} \frac{k!}{k^2(x+1)_k}, \tag{4.2}$$

and we remark that the right-hand side is the expansion of $\zeta(2,x)$ in a *série de facultés*, (see [13, Ch. VI]). Equation (4.2) can also be proved as follows:

$$\begin{aligned}\zeta(2,x) &= \int_0^{\infty} \frac{t}{1-e^{-t}}\,e^{-t(x+1)}\,\mathrm{d}t = -\int_0^1 \frac{\log(1-u)}{u}\,(1-u)^x\,\mathrm{d}u \\ &= \sum_{k=1}^{\infty} \frac{1}{k}\int_0^1 u^{k-1}(1-u)^x\,\mathrm{d}u = \sum_{k=1}^{\infty} \frac{k!}{k^2(x+1)_k},\end{aligned} \tag{4.3}$$

where we have used the change of variable $t = -\log(1-u)$. All steps are justified by the absolute convergence on $\Re(x) \geq \delta > -1$. Nörlund [13, pp. 213–214] mentions that the functions admitting an expansion in a *série de facultés* are exactly those which admit an asymptotic expansion of the form considered by Borel, *i.e.,* coming from a suitable Laplace transform, which is the case for $\zeta(2,x)$ by the first equality of (4.3).

4.2 Theorem 3 for $A = 4$ and $r = n$

Let us define the formal series

$$T(x) = -\frac{1}{2}\,R'(x) = \frac{1}{2}\sum_{k=0}^{\infty}(k+1)B_k x^{-k-2},$$

which is the asymptotic expansion of $\zeta(3,x)$ at $x = +\infty$. Beukers [7] also showed that

$$\frac{(-1)^{n+1}}{2}\sum_{k=1}^{\infty}\frac{(k-n)_n}{(k)_{n+1}}\,\frac{(k-1)!^2}{(x+1)_k(-x)_k} = u_n(x)T(x) + v_n(x) = \mathcal{O}(x^{-2n-2}), \tag{4.4}$$

[3] This is by no means obvious; see Section 4.2 for a counterexample in an apparently similar situation.

where $u_n(x)$ and $v_n(n)$ are polynomials in $\mathbb{C}[x]$ of degree $2n$. Again, these equalities are equalities of formal series. This is a solution to the $(2n+1)$-th diagonal Padé problem for $(1, T(x))$. In Theorem 3 (case (ii) with $A = 4$, $r = n$), we have also produced an analytic solution to the $(2n+1)$-th diagonal Padé problem for $(1, \zeta(3, x))$, and we now proceed to compare the two approaches.

Let us call $T_n(x)$ the series on the left-hand side of (4.4). It is convergent and holomorphic for any $x \in \mathbb{C} \setminus \mathbb{Z}$. Since $\mathscr{T}_{4,n,n}(x)$ is holomorphic at any

$$x \in \mathbb{C} \setminus \{-1, -2, -3, \dots\},$$

both series cannot be equal for all complex non-integral x; for example, numerically, $T_0(1/2) \approx 1.4704$, while $\mathscr{T}_{4,0,0}(x)(1/2) = \zeta(3, 1/2) = 7\zeta(3) - 8 \approx 0.4143$. Therefore, it seems paradoxical that both series produce solutions of the same Padé problem for the formal series $T(x)$.

There is no paradox at all for the following simple reason: $T_n(x)$ can also be viewed as a solution of the $(2n+1)$-th diagonal Padé problem for the *convergent* series $T_0(x)$ (note that $u_0(x) = 1$ and $v_0(x) = 0$), and it happens that in any sector that does not contain the real axis, $T_0(x)$ and $\zeta(3, x)$ have the same asymptotic expansion (although they do not coincide analytically). Hence, Proposition 4 implies that $T_n(x)$ and $\mathscr{T}_{4,n,n}(x)$ enable us to solve the same formal Padé problem for $\big(1, T(x)\big)$.

It is an interesting problem to identify the difference between $T_n(x)$ and $\mathscr{T}_{A,n,n}(x)$. To do this, we first express them as hypergeometric functions:

$$T_n(x) = \frac{1}{2}\frac{\Gamma(n+1)^4\Gamma(x-n)}{\Gamma(x+n+2)\Gamma(2n+2)}\;{}_4F_3\left[\begin{matrix} n+1,\, n+1,\, n+1,\, n+1 \\ x+n+2, n-x+1, 2n+2\end{matrix};1\right],$$

$$\mathscr{T}_{4,n,n}(x) = \frac{1}{2}\frac{\Gamma(n+1)^3\Gamma(x+1)^4\Gamma(2x+n+3)}{\Gamma(2x+2)\Gamma(x+n+2)^4} \times {}_7F_6\left[\begin{matrix} n+2x+2,\, x+2+\frac{n}{2},\, n+1,\, x+1,\, x+1,\, x+1,\, x+1 \\ x+1+\frac{n}{2}, 2x+2, x+n+2, x+n+2, x+n+2, x+n+2\end{matrix};1\right].$$

Luckily, we can now apply Bailey's identity [19, (2.4.4.3)], which relates two Saalschutzian ${}_4F_3$ (like $T_n(x)$) and one very well-poised ${}_7F_6$ (like $\mathscr{T}_{4,n,n}(x)$), and we obtain the desired expression:

(4.5)
$$\mathscr{T}_{4,n,n}(x) = T_n(x) + \frac{1}{2}\frac{\Gamma(n-x)\Gamma(x+1)^4}{\Gamma(2x+2)\Gamma(x+n+2)}\;{}_4F_3\left[\begin{matrix} x+1,\, x+1,\, x+1,\, x+1 \\ x+n+2, x-n+1, 2x+2\end{matrix};1\right],$$

valid for $x \in \mathbb{C} \setminus \mathbb{Z}$.

The second term $\Gamma[\,\cdot\,]_4F_3[\,\cdot\,]$ on the right-hand side of (4.5) cancels the poles of $T_n(x)$ at $x = 0, 1, 2, \dots$, and it decreases exponentially to 0 as $|x| \to +\infty$ uniformly in any sector that does not contain the real axis. Its asymptotic expansion in powers of $1/x$ is thus necessarily $\sum_{k=0}^{\infty} 0 \cdot x^{-k}$. This confirms that $T_n(x)$ and $\mathscr{T}_{4,n,n}(x)$ have the same asymptotic expansion in such a sector.

After some transformations, the case $n = 0$ of (4.5) gives the identity, valid for $x \in \mathbb{C} \setminus \mathbb{Z}$:

$$\begin{aligned}(4.6)\quad \zeta(3,x) = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{k!^2}{(x+1)_k(-x)_k k^3} \\ - \frac{\pi}{2\sin(\pi x)} \frac{1}{\binom{2x}{x}} \sum_{k=1}^{\infty} \frac{(x+1)_k^2}{(k-1)!(2x+1)_k(k+x)^3}.\end{aligned}$$

We remark that (4.6) is not the expansion of $\zeta(3,x)$ in a *série de facultés*, which is (for $\Re(x) > -1$)

$$\zeta(3,x) = \frac{1}{2}\sum_{k=1}^{\infty}\Big[\sum_{j=1}^{k} \frac{1}{j(k-j+1)}\Big] \frac{k!}{(x+1)_{k+1}},$$

by formula [13, (77)]. Hence, it is not at all clear to us how to find Hermite–Padé approximants for $\zeta(A,x)$ when $A \geq 4$ in the spirit of Beukers' approach for $A = 2$ and $A = 3$. This adds to the interest of the more flexible method developed in the present article.

5 Arithmetical Applications

Calegari [8] recently addressed the problem of the irrationality of certain values of the p-adic zeta function of Kubota–Leopold. In particular, he showed that $\zeta_p(3) \notin \mathbb{Q}$ when $p = 2$ and $p = 3$, by means of the sophisticated machinery of overconvergent p-adic modular forms. This method can be viewed as the p-adic analogue of the "modular proof" of Apéry's theorem [2] proposed by Beukers [6]. As a matter of fact, Beukers's main purpose [7] was to obtain new proofs of Calegari's theorems by simpler methods, *i.e.*, by using Padé approximants in the special cases of the Hurwitz zeta function. In the archimedean case, polylogarithms seemed unavoidable until recently [5, 10], but see [18] for a new approach. The present article provides a generalisation of the results of [7] in the hope that it could lead to results analogous to those in [4, 11, 15, 16] for the p-adic zeta function or related functions.

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