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FILTER MONADS, CONTINUOUS LATTICES AND CLOSURE SYSTEMS

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1. Introduction. The notion of a monad (triple) has become increasingly important as an extension of the classical universal algebraic approach to "algebraic" categories. Indeed the categories of algebras arising from a monad seem to be the most natural generalization of Birkhoff's equational classes. Moreover in [**2**], Barr's concept of a relational model of a monad also coincides nicely with both the concepts of partial algebras (when suitably restricted) and (Moore) closure systems.

In this paper, we wish to examine two particular monads determined by filters. The first is the filter monad $\mathbf{F} = (\mathrm{F}, \eta, \mu)$ over Sets where FX is the set of all (not necessarily proper) filters on X. The second is the open filter monad $\mathbf{\Phi} = (\Phi, \eta, \mu)$ over Top_0, the category of T_0-spaces and continuous functions, where ΦX is the filter space of the lattice of open sets in X as described in Banaschewski [**1**]. We determine the algebras of both these monads and find these two categories to be naturally isomorphic, in fact both of them are in essence the category of continuous lattices as defined in Scott [**5**], together with directed (= non-empty directed) join (= sup) and arbitrary meet (= inf) preserving functions as morphisms. Also, following [**2**], we find that a naturally defined reflective subcategory of the category of all relational $\mathbf{F}$-algebras is naturally isomorphic to Clos, the category of all closure systems and their continuous (inverse image of a closed subset is closed) maps.

2. Preliminaries. For the basic notions of category theory and in particular the definitions and essential facts on monads, see MacLane [**4**]. Following Barr [**2**], a relational $\mathbf{T}$-algebra, for a monad $\mathbf{T} = (T, \eta, \mu)$ over Sets, is a pair (X, r) where X is a set and $r : TX \rightharpoondown X$ is a relation from TX to X satisfying $1_X \subseteq r \circ X$ and $r \circ Tr \subseteq r \circ X$. A morphism $f : (X, r) \to (Y, s)$ is a function $f : X \to Y$ with $f \circ r \subseteq s \circ Tf$. We will identify a relation $r : TX \rightharpoondown X$ with its graph $r \subseteq TX \times X$ to ease notational complexity. Rel ($\mathbf{T}$) is the category of relational T-algebras and their morphisms.

A closure operator on a set X is a map $\Gamma : PX \to PY$ that is extensive ($M \subseteq \Gamma M$), monotone ($M \subseteq N$ implies $\Gamma M \subseteq \Gamma N$) and idempotent ($\Gamma(\Gamma M) = \Gamma M$). A closure system on X is a system of subsets $C \subseteq P(X)$ closed under arbitrary intersections. It is well-known that there exists for any set X a bijective correspondence between the closure operators and the

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closure systems on X. Let Clos be the category of all closure spaces (pairs (X, Γ), Γ a closure operator on X) and continuous maps $f : (X, \Gamma) \to (Y, \Delta)$ ($\equiv x \in \Gamma M$ implies $f(x) \in \Delta(f[M])$). Equivalently; the inverse image under f of any Δ-closed set in Y is Γ-closed in X. The main result in [**2**] then is:

THEOREM (Barr [**2**]). *For any monad* $\mathbf{T} = (T, \eta, \mu)$ *over* Sets, *there exists a faithful functor* $\mathscr{T}$: Rel ($\mathbf{T}$) $\to$ Clos.

A closure operator $\Gamma : PX \to PX$ is called topological (algebraic) if $\Gamma\phi = \phi$ and $\Gamma(M \cup N) = \Gamma M \cup \Gamma N$ (respectively $\Gamma M = \bigcup \{\Gamma F : \text{finite } F \subseteq M\}$). Top (Alg) is the full subcategory of all topological (respectively algebraic) closure spaces.

Finally, for a lattice $(L, \leqq)$, the Scott topology on L (see Scott [**5**]) is given by the directed-open ends of L. A subset $U \subseteq L$ is an end if $x \in U$ and $x \leqq y$ imply $y \in U$. U is directed open if $\bigvee_L D \in U$ implies $D \cap U \neq \phi$ for every (non-empty) directed subset of L. A lattice $(L, \leqq)$ is called continuous if L is complete and for all $x \in L$

$$x = \bigvee\{\bigwedge U : x \in U,\ U \text{ open in the Scott topology}\}.$$

3. The filter monad over sets. A filter on a set X is a system $f \subseteq PX$ that satisfies

(F1) $X \in f$,

(F2) $M, N \in f$ imply $M \cap N \in f$, and

(F3) $M \in f$ and $M \subseteq N (\subseteq X)$ imply $N \in f$.

If FX is the set of all filters on X then we can define an endofunctor F : Sets $\to$ Sets by defining for $\phi : X \to Y$, $F\phi : FX \to FY$ by

$$f \mapsto \{N \in PY : \phi[M] \subseteq N \text{ for some } M \in f\}.$$

Equivalently: $N \in F\phi(f)$ if and only if $\phi^{-1}[N] \in f$. If f_x is the principal (ultra) filter generated by $x \in X$ then the map $x \mapsto f_x$ $(x \in X)$ defines a natural transformation $\eta : 1_{\text{Sets}} \to F$. Also we can define $\mu : F^2 \to F$ by

$$\mu X(\mathscr{F}) = \bigcup \{\bigcap \mathscr{M} : \mathscr{M} \in \mathscr{F}\} \quad (\mathscr{F} \in F^2X).$$

That is, $M \in \mu X(\mathscr{F})$ if and only if $\pi X(M) \in \mathscr{F}$ $(M \subseteq X)$ where $\pi X : PX \to P(FX)$ by $\pi X(M) = \{f \in FX : M \in f\}$. The computations to show that F is an endofunctor and that η and μ are natural transformation are straightforward. We will also use the notation

$$f_M = \{N \subseteq X : M \subseteq N\} \text{ (where } f_{\{x\}} = f_x).$$

(3.1) THEOREM. $\mathbf{F} = (F, \eta, \mu)$ *is a monad over* Sets.

Proof. Let us first note the following obvious properties given $M \subseteq X$:

(1) $(\mu X)^{-1}[\pi X(M)] = \pi FX(\pi X(M))$,

(2) $M = (\eta X)^{-1}[\pi X(M)]$.

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Now for $f \in FX$ and $M \subseteq X$, and $\mathscr{F} \in F^2X$:

(i) $M \in (\mu X \circ \eta FX)(f) \Leftrightarrow \pi X(M) \in \eta FX(f) \Leftrightarrow f \in \pi X(M) \Leftrightarrow M \in f$. Therefore $\mu X \circ \eta FX = 1_{FX}$ for any set X.

(ii) $M \in (\mu X \circ F\eta X)(f) \Leftrightarrow \pi X(M) \in F\eta X(f) \Leftrightarrow M = (\eta X)^{-1}[\pi X(M)] \in f$ by (2). Therefore $\mu X \circ F\eta X = 1_{FX}$ for any set X.

(iii)
$$\begin{aligned} M \in (\mu X \circ \mu FX)(\mathscr{F}) &\Leftrightarrow \pi X(M) \in \mu FX(\mathscr{F}) \\ &\Leftrightarrow \pi FX(\pi X(M)) \in \mathscr{F} \\ &\Leftrightarrow (\mu X)^{-1}(\pi X(M)) \in \mathscr{F} \text{ by (1)} \\ &\Leftrightarrow \pi X(M) \in F\mu X(\mathscr{F}) \\ &\Leftrightarrow M \in (\mu X \circ F\mu X)(\mathscr{F}). \end{aligned}$$

Therefore $\mu \circ \mu F = \mu \circ F\mu$

We will also need the following properties.

(3.2) PROPOSITION.

(1) *$\mu X : F^2X \to FX$ preserves directed joins and arbitrary meets (= intersections)*

(2) *For any $\phi : X \to Y$, $F\phi : FX \to FX$ preserves directed joins and arbitrary meets.*

Proof. (i) Let $\{\mathscr{F}_d : d \in D\}$ be a directed subset of F^2X. Since μX is clearly order-preserving and the join of a directed family of filters is its union, we have:

$$\begin{aligned} M \in \mu X(\bigcup \{\mathscr{F}_d : d \in D\}) &\Leftrightarrow \pi X(M) \in \bigcup \{\mathscr{F}_d : d \in D\} \\ &\Leftrightarrow \pi X(M) \in \mathscr{F}_d \text{ for some } d \in D \\ &\Leftrightarrow M \in \mu X(\mathscr{F}_d) \text{ for some } d \in D \\ &\Leftrightarrow M \in \bigcup \{\mu X(\mathscr{F}_d) : d \in D\} \\ &\qquad = \bigvee \{\mu X(\mathscr{F}_d) : d \in D\}. \end{aligned}$$

Since $\bigwedge \phi = P(FX)$ in F^2X,

$$\begin{aligned} M \in \mu X(P(FX)) &\Leftrightarrow \pi X(M) \in P(F(X)) \\ &\Leftrightarrow M \in P(X), \end{aligned}$$

i.e., $\mu X(\bigwedge \phi) = P(X) = \bigwedge \phi$ in FX.

For non-empty meets, the calculations are straightforward.

(ii) Now consider $\phi : X \to Y$ and directed family $\{f^d : d \in D\}$ in FX. Again $F\phi$ is clearly order preserving hence the image of a directed family is again directed. For $N \subseteq Y$ then:

$$\begin{aligned} N \in F\phi(\bigcup \{f^d : d \in D\}) &\Leftrightarrow \phi^{-1}[N] \in \bigcup \{f^d : d \in D\} \\ &\Leftrightarrow \phi^{-1}[N] \in f^d \text{ for some } d \in D \\ &\Leftrightarrow N \in F_\phi(f^d) \text{ for some } d \in D \\ &\Leftrightarrow N \in \bigcup \{F_\phi(f^d) : d \in D\} \\ &\qquad = \bigvee \{F_\phi(f^d) : d \in D\}. \end{aligned}$$

For arbitrary meets, the calculations are obvious.

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We now wish to describe the F-algebras in a more natural way. A D-lattice is a complete lattice L which for any family $(D_i : i \in I)$ of directed subsets:

(D) $\bigwedge\{\bigvee D_i : i \in I\} = \bigvee\{\bigwedge(d_i : i \in I) : d \in \prod(D_i : i \in I)\}.$

A morphism $\phi : L \to M$ between D-lattices is a directed join and arbitrary meet preserving function. D is the category of all D-lattices and their morphisms. Let us note that all algebraic lattices (in particular all FX) are in D and by (3.2), all $F\phi : FX \to FY$ and all $\mu X : F^2X \to FX$ are morphisms in D.

(3.3) THEOREM. *D is precisely monadic over sets by* $\mathbf{F}$.

Proof. For D-lattice L, define $\epsilon L : FL \to L$ by

$$\epsilon L(f) = \bigvee\{\bigwedge M : M \in f\}.$$

We claim that ϵL is a D-morphism.

Since $\epsilon L(f_M) = \bigwedge M$ for all $M \subseteq L$ and $\bigwedge_{FL}\phi = f_\phi$ we have $\epsilon L(\bigwedge_{FL}\phi) = \epsilon L(f_\phi) = \bigwedge_L \phi$. For nonempty family $(g_i : i \in I)$ in FL, each set $\{\bigwedge M_i : M_i \in g_i\}$ is directed and $\bigwedge(g_i : i \in I) = \{\bigcup(M_i : i \in I) : M \in \prod(g_i : i \in I)\}$. Therefore,

$$\begin{aligned}\bigwedge(\epsilon L(g_i) : i \in I) &= \bigwedge\{\bigvee\{\bigwedge M_i : M_i \in g_i\} : i \in I\}\\ &= \bigvee\{\bigwedge\{\bigwedge M_i : i \in I\} : M \in \prod(g_i : i \in I)\} \text{ by } (D)\\ &= \bigvee\{\bigwedge \bigcup(M_i : i \in I) : M \in \prod(g_i : i \in I)\}\\ &= \epsilon L(\bigwedge(g_i : i \in I)).\end{aligned}$$

For directed subset $\{g_d : d \in D\}$, $\bigvee(g_d : d \in D) = \bigcup(g_d : d \in D)$. Therefore

$$\begin{aligned}\bigvee(\epsilon L(g_d) : d \in D) &= \bigvee\{\bigvee\{\bigwedge M : M \in g_d\} : d \in D\}\\ &= \bigvee\{\bigwedge M : M \in g_d \text{ for some } d \in D\}\\ &= \epsilon L(\bigvee(g_d : d \in D)).\end{aligned}$$

By letting $\bar{F} : \text{Sets} \to D$ be F, lifted to D, with the natural D-lattice structure imposed, we have $\epsilon : \bar{F}U \to 1_D$ and $\eta : 1_{\text{Sets}} \to F = U\bar{F}$ providing the back and front adjunctions, where $U : D \to \text{Sets}$ is the natural forgetful functor.

To check the monadic part, let θ be a congruence relation on a D-lattice L; that is, θ is an equivalence relation L that is also a subalgebra of $L \times L$ $(i : \theta \rightarrowtail L \times L$ is a D-map). Define on $L/\theta : x/\theta \leqq y/\theta$ if and only if $x\, \theta\, x \wedge y$. Now $u\, \theta\, x$, and $v\, \theta\, y$ imply

$$u\, \theta\, x\, \theta\, x \wedge y\, \theta\, u \wedge v$$

hence $\leqq$ is well defined on L/θ. Moreover it is clear that $\leqq$ is a partial order and that the canonical $\kappa : L \to L/\theta$ is order preserving. We need that κ also preserves directed joins and arbitrary meets and that L/θ is a D-lattice.

(i) For $M \subseteq L$, $(\bigwedge M)/\theta = \bigwedge\{m/\theta : m \in M\}$: For $M = \phi$ this is clear as $x/\theta \leqq 1/\theta$ for all $x \in L$. If $M \neq \phi$, then $\bigwedge M \leqq m$ for all $m \in M$ gives

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$(\bigwedge M)/\theta$ as a lower bound for this set. Now if $u/\theta \leqq m/\theta$ for all $m \in M$, we have $(u, u \wedge m) \in \theta$ for all $m \in M$ which gives $\bigwedge \{(u, u \wedge m) : m \in M\} = (u, u \wedge \bigwedge M) \in \theta$. Therefore $(\bigwedge M)/\theta = \bigwedge\{m/\theta : m \in M\}$.

(ii) For directed $D \subseteq L$, $(\bigvee D)/\theta = \bigvee\{d/\theta : d \in D\}$: Again $(\bigvee D)/\theta$ is clearly an upper bound of the right hand set. If $d/\theta \leqq u/\theta$ for all $d \in D$, we have

$$\bigvee\{(d, u \wedge d) : d \in D\} = (\bigvee D, \bigvee (u \wedge D)) = (\bigvee D, u \wedge \bigvee D) \in \theta$$

as $\{(d, u \wedge d) : d \in D\}$ is directed and L is a D-lattice.

(iii) L/θ is a D-lattice:

Let $(\{x/\theta : x \in X_i\} : i \in I)$ be a family of directed subsets of L/θ. Then for each $i \in I$ define $D_i \subseteq L$ by $D_i = \{d_x : x \in X_i\}$ where $d_x = \bigvee\{y \in X_i : y/\theta \leqq x/\theta\}$. Clearly each D_i is directed and for each $x \in X_i$ $d_x/\theta = x/\theta$. Since

$$\bigwedge\{\bigvee D_i : i \in I\} = \bigvee\{\bigwedge (d_i : i \in I) : d \in \Pi(D_i : i \in I)\}$$

we may apply κ and obtain our result.

Since the congruence relations on a D-lattice obviously form a closure system, the conditions of Linton [**3**, Proposition 3, p. 88] hold. Since $U : D \to$ Sets clearly creates isomorphisms, the theorem is proven.

4. The open filter monad over Top_0**.** From Banaschewski [**1**], one can form the filter space of any partially-ordered set. One obtains the filter space of a T_0-space X by considering the filter space of $\mathbf{O}X$, its lattice of open sets. That is, ΦX is the set of all filters in $\mathbf{O}X$ together with the topology generated by the standard open sets $\phi X(U) = \{u \in |\Phi X| : U \in u\}$ for every $U \in \mathbf{O}X$. To obtain an endofunctor of Top_0 one defines for continuous $f : X \to Y$, $\Phi f : \Phi X \to \Phi Y$ by $V \in \Phi f(u)$ if and only if $f^{-1}[V] \in u$ $(V \in \mathbf{O}Y, u \in |\Phi X|)$. The map $x \mapsto \mathbf{O}X(x)$, the open neighbourhoods of $x \in |X|$ is then an embedding of X into ΦX. To ease notation we will let $v_x(v_M)$ be the filter of all open neighbourhoods of $x \in |X|$ (respectively $M \subseteq |X|$). This embedding is natural and provides us with a natural transformation $\eta : 1_{\mathrm{Top}_0} \to \Phi$. Moreover we can also define $\mu : \Phi^2 \to \Phi$ as in section 3 by $U \in \mu X(\mathcal{U})$ if and only if $\phi X(U) \in \mathcal{U}$. In fact $\mu X(\mathcal{U})$ is the largest limit point of $\mathcal{U}$ in ΦX (i.e. $u \in \mathrm{Lim}_{\Phi X}\mathcal{U}$ if and only if $u \subseteq \mu X(\mathcal{U})$). We can then show immediately that:

(4.1) THEOREM. $\mathbf{\Phi} = (\Phi, \eta, \mu)$ *is a monad over* Top_0.

(4.2) PROPOSITION. $\mu X : \Phi^2 X \to \Phi X$ *and* $\Phi f : \Phi X \to \Phi Y$ (for $f : X \to Y$) *preserve directed joins and arbitrary meets.*

As noted in [**1**] and [**5**], there is a natural partial order on any T_0-space; viz:

$$x \leqq y \Leftrightarrow v_x \subseteq v_y \quad (x, y \in |X|)$$

and that any continuous function between two T_0-spaces preserves these orders. Moreover every filter space ΦX is a continuous lattice in which the

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induced T_0-space order coincides with the natural order of set-theoretical inclusion of the open filters.

(4.3) THEOREM. $(X, \alpha) \in |\mathrm{Top}_0{}^\Phi|$ *if and only if* X *is a continuous lattice and* $\alpha : \Phi X \rightarrow X$ *is given by*

$$\alpha(u) = \bigwedge\{\bigwedge U : U \in u\}.$$

Proof. Let (X, α) be a Φ-algebra and consider X with the induced T_0-order. Since α is continuous we have:

(1) $u \subseteq v$ implies $\alpha(u) \leqq \alpha(v)$.

We need a few more properties.

(2) $\alpha(u) \in \mathrm{Lim}_X u$:

For $U \in v_{\alpha(u)}$, there exists a standard open neighbourhood $\phi X(V)$ of $u \in |\Phi X|$ with $\alpha[\phi X(V)] \subseteq U$. Now $u \in \phi X(V)$ implies $V \in u$. Since V is open, $y \in V$ implies $v_y \in \phi X(V)$ whence $y = \alpha(v_y) \in U$. That is $V \subseteq U$ and therefore $U \in u$. Since U was arbitrary, $v_{\alpha(u)} \subseteq u$ and $\alpha(u) \in \mathrm{Lim}_X u$.

(3) $x \in \mathrm{Lim}_X u$ if and only if $x \leqq \alpha(u)$:

$x \leqq \alpha(u)$ implies $v_x \subseteq v_{\alpha(u)} \subseteq u$ by (2). Therefore $x \in \mathrm{Lim}_X u$. Conversely, $x \in \mathrm{Lim}_X u$ means $v_x \subseteq u$ and therefore by (1), $x = \alpha(v_x) \leqq \alpha(u)$.

(4) For any $A \subseteq |X|$, $\alpha(v_A) = \bigwedge A$:

Since $v_A \subseteq v_x$ for all $x \in A$, $\alpha(v_A)$ is a lower bound for A. If $x \leqq a$ for all $a \in A$ then $v_x \subseteq \bigcap \{v_a : a \in A\} = v_A$ and so

$$x = \alpha(v_x) \leqq \alpha(v_A).$$

(5) $(|X|, \leqq)$ is a complete lattice.
(6) α preserves arbitrary meets:

$$\begin{aligned} x \leqq \bigwedge (\alpha(v_i) : i \in I) &\Leftrightarrow x \leqq \alpha(v_i)(i \in I) \\ &\Leftrightarrow x \in \mathrm{Lim}_X v_i (i \in I) \\ &\Leftrightarrow x \in \mathrm{Lim}_X \bigwedge (v_i : i \in I) \\ &\Leftrightarrow x \leqq \alpha(\bigwedge (v_i : i \in I)); \end{aligned}$$

$$\begin{aligned} x \leqq \alpha(\bigwedge_{\Phi X} \phi) &\Leftrightarrow x \leqq \alpha(\mathbf{O}X) \\ &\Leftrightarrow x \in \mathrm{Lim}_X \mathbf{O}X \\ &\Leftrightarrow x \leqq \bigwedge_X \phi. \end{aligned}$$

(7) For $u \in |\Phi X|$, $\alpha(u) = \bigvee \{\bigwedge U : U \in u\}$:
Clearly $\alpha(u)$ is an upper bound for $S = \{\bigwedge U : U \in u\}$. If x is an upper bound for S then $\alpha(V_{U \cup \{x\}}) = \alpha(v_U)$ for all $U \in u$. For each $U \in u$, define

$$U^* = \{v_v \cap v_x : U \subseteq V\}, \quad \text{and } U^+ = \{v_v : U \subseteq V\}.$$

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Then by assumption, $\alpha[U^*] = \alpha[U]$. Therefore

$$\begin{aligned}\alpha(u) \cap \alpha(v_x) = \alpha(u \cap v_x) &= \alpha(\bigcup \{v_U \cap v_x : U \in u\}) \\ &= (\alpha \circ \mu x)(\bigcup \{\mathscr{V}_{U*} : U \in u\}) \\ &= \alpha \circ \Phi\alpha(\bigcup \{\mathscr{V}_{U*} : U \in u\}) \\ &= \alpha(\bigcup \{v_{\alpha[U*]} : U \in u\}) \\ &= \alpha(\bigcup \{v_{\alpha[U+]} : U \in u\}) \\ &= \alpha \circ \Phi\alpha(\bigcup \{\mathscr{V}_{U+} : U \in u\}) \\ &= \alpha \circ \mu X(\bigcup \{\mathscr{V}_{U+} : U \in u\}) \\ &= \alpha(u).\end{aligned}$$

Therefore $\alpha(u) \leqq \alpha(v_x) = x$.

(8) X is a continuous lattice:

For any $x \in |X|$, $x = \alpha(v_x) = \bigvee \{\bigwedge V : V \in v_x\}$. Therefore by the argument in [**5**, p. 67–78], X is a continuous lattice with α as required (Note: To know that X is a continuous lattice since it is a retract of ΦX is not sufficient as there could be many possible retractions $\alpha : \phi X \to X$.)

Conversely let X be a continuous lattice and define $\alpha : \Phi X \to X$ by $\alpha(u) = \bigvee\{\bigwedge U : U \in u\}$. By [**5**, 2.4], $\alpha \circ \eta X = 1_X$. Also, $\alpha(u) \in V \in \mathbf{O}X$, the Scott topology, implies $\bigwedge U \in V$ for some $U \in u$. Now $w \in \phi X(U)$ implies $U \leqq \alpha(w)$ hence $\alpha(w) \in V$. Therefore $\alpha[\phi X(U)] \subseteq V$ for this U and α is continuous. As α is a map between continuous lattices, it therefore preserves directed joins. Also by [**5**, 2.1], $x \in \mathrm{Lim}_X u$ if and only if $x \leqq \alpha(u)$ and therefore by the reverse argument of point (6) in the first half of this proof, α preserves arbitrary meets. Therefore to establish (X, α) as a Φ-algebra, we need only consider open filters of the form $\mathscr{V}_u \in |\Phi^2 X|$. But

$$(\alpha \circ \Phi\alpha)(\mathscr{V}_u) = \alpha(v_{\alpha(u)} = \alpha(u) = (\alpha \circ \mu X)(\mathscr{V}_u).$$

Therefore (X, α) is a Φ-algebra.

(4.4) THEOREM. *The category of continuous lattices with directed join (= continuous) and arbitrary meet preserving maps is monadic over* Top_0.

Proof. We need only check out the Φ-algebra morphisms. For (X, α) and $(Y, \beta) \in |\mathrm{Top}_0{}^{\Phi}|$, $f : X \to Y$ is a Φ-algebra morphism if and only if $f \circ \alpha = \beta \circ \Phi f$. But for arbitrary $A \subseteq X$

$$f(\bigwedge A) = f \circ \alpha(v_A) \text{ and } \beta(\bigwedge f[A]) = (\beta \circ \Phi f)(v_A)$$

Since any filter is the direct union of its principal subfilters, the result follows.

(4.5) THEOREM. *D is precisley the category of continuous lattices with directed join and arbitrary meet preserving maps.*

Proof. Let L be a continuous lattice, $(D_i : i \in I)$ a family of directed sets, $x = \bigwedge(\bigvee D_i : i \in I)$ and U a (Scott) open set with $x \in U$. Then for every

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$i \in I$, $D_i \cap U \neq \phi$ and therefore there exists a $d^U \in \prod(D_i : i \in I)$ with $d_i^U \in U$. This forces $\bigwedge U \leqq \bigwedge (d_i^U : i \in I)$ and

$$x = \bigvee \{\bigwedge U : U \in v_x\} \leqq \bigvee\{\bigwedge (d_i : i \in I) : d \in \prod(D_i : i \in I)\}.$$

Therefore L is a D-lattice.

Conversely let X be a D-lattice. By (3.3) $\alpha : FX \to X$ by $\alpha(f) = \bigvee\{\bigwedge M : M \in f\}$ makes (X, α) an $\mathbf{F}$-algebra with α directed join and arbitrary meet preserving. Now since FX is an algebraic lattice, it is a continuous lattice; moreover with the Scott topology on both FX and X, α is continuous.

Define $\beta : X \to FX$ by

$$\beta(x) = \bigwedge \{f \in FX : x \leqq \alpha(f)\}.$$

β then satisfies (1) $\beta(x) \leqq f \Leftrightarrow x \leqq \alpha(f)$; (2) $x \leqq y \Leftrightarrow \beta(x) \subseteq \beta(y)$; (3) $\beta(\bigvee M) =$ is $\bigvee \beta[M]$; and (4) $\alpha \circ \beta = 1_X$. We need also that β continuous.

Now the basic open sets of FX are the $\pi x(M)$, $M \subseteq X$. Moreover $x \in \beta^{-1}[\pi X(M)]$ if and only if $M \in \beta(x)$. Therefore $x \in \beta^{-1}[\pi X(M)]$ and $x \leqq y$ implies $M \in \beta(x) \subseteq \beta(y)$ and $y \in \beta^{-1}[\pi X(M)]$.

For directed $D \subseteq X$, $M \in \beta(\bigvee D) = \bigvee \beta[D] = \bigcup \{\beta(d) : d \in D\}$ implies $M \in \beta(d)$ for some $d \in D$. Therefore β is continuous and X as a retract of a continuous lattice is continuous by [**5**, 2.10].

5. Relational F-algebras and closure systems. As mentioned in the preliminaries, Barr's result can be generalized for an arbitrary monad over sets. In our particular case, the result becomes:

(5.1) THEOREM. *There exists a functor* C : Rel $(\mathbf{F}) \to$ Clos *given by:*

$$C(A, \alpha) = (A, \Gamma_\alpha)$$
$$C(f : (A, \alpha) \to (B, \beta)) = f : (A, \Gamma_\alpha) \to (B, \Gamma_\beta)$$

where $x \in \Gamma_\alpha(M)$ *if and only if there exists an* $f \in FX$ *with* $M \in f$ *and* $(f, x) \in \alpha$.

Now using the usual topological results, for any closure system (X, Γ) and $f \in FX$ we may define the adherence or accumulation points of f by: $\mathrm{Acc}\, f = \bigcap \{\Gamma(M) : M \in f\}$. This allows us to define a functor J : Clos $\to$ Rel $(\mathbf{F})$ by:

(a) $J(X, \Gamma) = (X, r_\Gamma)$ where $(f, x) \in r_\Gamma$ if and only if $x \in \mathrm{Acc}\, f$;

(b) for $f : (X, \Gamma) \to (Y, \Delta)$, $Jf = f : (X, r_\Gamma) \to (Y, r_\Delta)$.

(5.2) THEOREM. *The functors*

$$\mathrm{Rel}\ (\mathbf{F}) \underset{J}{\overset{C}{\rightleftarrows}} \mathrm{Clos}$$

define Clos *as a reflective subcategory of* Rel $(\mathbf{F})$.

Proof. First we must show that J is indeed a functor. For $(X, \Gamma) \in |\mathrm{Clos}|$, $1_X \subseteq r_\Gamma \circ \eta X$ holds trivially. To show that $r_\Gamma \circ F r_\Gamma \subseteq r_\Gamma \circ \mu X$ follows the proof of [**2**, Proposition 3.2]. Namely for $(\mathscr{F}, f) \in F r_\Gamma$ and $(f, x) \in r_\Gamma$ we must

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show $(\mu X(\mathscr{F}), x) \in r_\Gamma$. Now $M \in \mu X(\mathscr{F})$ implies $\pi X(M) \in \mathscr{F}$ which gives

$$\begin{aligned}\Gamma(M) &= \{y \in X : y \in \Gamma M\} \\ &= \{y \in X : (g, y) \in r_\Gamma \text{ for some } g \in \pi X(M)\} \\ &= Fcr_\Gamma \circ Fdr_\Gamma^{-1}[\pi X(M)] \in f.\end{aligned}$$

Therefore $x \in \Gamma(\Gamma(M)) = \Gamma M$ for any $M \in \mu X(\mathscr{F})$. Therefore $J(X, \Gamma) \in |\mathrm{Rel}(\mathbf{F})|$. The fact that $f : (X, r_\Gamma) \to (Y, r_\Delta)$ is a relational $\mathbf{F}$-algebra morphism is precisely [**2**, 3.3] and this gives in fact that J and C are both full and faithful.

Now for $(X, \Gamma) \in |\mathrm{Clos}|$ and $M \subseteq X$,

$$\begin{aligned}x \in \Gamma_{r\Gamma}(M) &\Leftrightarrow \text{there exists } f \in \pi x(M) \text{ with } (f, x) \in r_\Gamma \\ &\Leftrightarrow x \in \operatorname{Acc} f \subseteq \Gamma M.\end{aligned}$$

However $x \in \Gamma M$ implies $(f_M, x) \in r_\Gamma$ whence $x \in \Gamma_{r\Gamma}(M)$. Therefore $CJ = 1_{\mathrm{Clos}}$.

For any set A, let ρA be the natural partial order on FA. $(A, \alpha) \in |\mathrm{Rel}(\mathbf{F})|$ is called ρ-closed if $\alpha \circ \rho A \subseteq \alpha$. We claim that $JC(A, \alpha) = (A \circ \rho A)$ for any $(A, \alpha) \in |\mathrm{Rel}\ (\mathbf{F})|$. Firstly, if $(f, x) \in \alpha$ and $g \subseteq f$ then since $\operatorname{Acc} f \subseteq \operatorname{Acc} g$ we have $(g, x) \in r_{\Gamma\alpha}$. Now $(g, x) \in r_{\Gamma\alpha}$ if and only if for all $M \in g$, $x \in \Gamma_\alpha M$. That is: for all $M \in g$ there exists $g_M \in \pi X(M)$ with $(g_M, x) \in \alpha$. We define as in [**2**, 3.7] a filter Φ on $F\alpha$ (here the filters on the graph of α) by the filter base

$$\begin{aligned}B &= \{(h, x) : (h, x) \in \alpha\} \\ B_M &= \{(h, y) : h \in \pi X(M) \text{ and } (h, y) \in \alpha\} \quad (M \in g).\end{aligned}$$

Since $Fd\alpha(\Phi) = f_x$ by the inclusion in Φ of B and $c\alpha[B_M] \subseteq \pi X(M)$ for all $M \in g$ we have as (A, α) is a relational $\mathbf{F}$-algebra

$$(\mu X \circ Fc\alpha(\Phi), x) \in \alpha \quad \text{and} \quad g \subseteq \mu X Fc\alpha(\Phi).$$

Therefore $(g, x) \in r \circ \rho A$.

To complete the proof let $\mathrm{Rel}_\rho(\mathbf{F})$ be the full sub-category of all ρ-closed relational $\mathbf{F}$-algebras where (A, α) is ρ-closed if $\alpha \circ \rho A \subseteq \alpha$. It is clear that $\mathrm{Rel}_\rho(\mathbf{F})$ is isomorphic to Clos. To show the reflection then take $f : (A, \alpha) \to (B, \beta)$ with $(B, \beta) \in \mathrm{Rel}_\rho(\mathbf{F})$. Then

$$f \circ \alpha \circ \rho A \subseteq \beta \circ Ff \circ \rho A \subseteq \beta \circ \rho B \circ Ff \subseteq \beta \circ Ff$$

with the second inclusion occurring since $Ff \circ \rho A \subseteq \rho B \circ Ff$ from 3.4(1).

(5.3) COROLLARY. Clos *is naturally isomorphic to the reflective subcategory of all ρ-closed relational* $\mathbf{F}$*-algebras.*

We should note at this time that the relation ρA on FA has a natural definition for any monad $\mathbf{T} = (T, \eta, \mu)$ over sets, viz: for $p, q \in TX$, $(p, q) \in \rho^{\mathbf{T}} X$ if and only if all $e : M \mapsto X$ ($p \in Te[TM]$ implies $q \in Te[TM]$). $\rho^{\mathbf{T}} X$ is always a pre-order (= reflexive and transitive) and a $\rho^{\mathbf{T}}$-closed relational T-algebra can be defined analogously. In order to make $\mathrm{Rel}_\rho(\mathbf{T})$ a reflective subcategory

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one also needs at least that for any $f : X \to Y$, $Tf \circ \rho X \subseteq \rho Y \circ Tf$. These properties hold also for the following monads

(a) $\mathbf{\Omega}$, the ultrafilter or compact Hausdorff monad: $(u, v) \in \rho^{\mathbf{\Omega}} X$ if and only if $u = v$.

(b) $\mathbf{P}$, the power set monad: $(M, N) \in \rho^{\mathbf{P}} T$ if and only if $M \supseteq N$ (M, $N \in P(X)$).

(c) $\mathbf{P}_\omega$, the finite subset monad: $(M, N) \in \rho^{\mathbf{P}\omega} X$ if and only if $M \supseteq N$.

(d) $\mathbf{I}$, the identity monad: $(x, y) \in \rho^{\mathbf{I}} X$ if and only if $x = y$.

In cases (a) and (d), all relational $\mathbf{T}$-algebras are ρ-closed and we obtain a natural isomorphism with Top. In (b), $\mathrm{Rel}_\rho(\mathbf{P})$ is naturally isomorphic with Clos ($x \in \Gamma_\alpha(M)$ if and only if $(M, X) \in \alpha$) and in (c) $\mathrm{Rel}_\rho(\mathbf{P}_\omega) \simeq$ Alg($x \in \Gamma_\alpha(M)$ if and only if there exists finite $F \subseteq M$ with $(F, x) \in \alpha$). We do not know if Reg, the category of all regular closure systems

$$(\Gamma(\Gamma(M) \cap N) = \Gamma(M) \cap \Gamma(N))$$

can be so obtained. Also we have not been able to find the precise conditions on a monad $\mathbf{T}$ to make its category of ρ-closed relational T-algebras naturally isomorphic to a full subcategory of Clos.

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