BULL. AUSTRAL. MATH. SOC. 35J70, 35J25
VOL. 36 (1987) 425-434

ESTIMATES FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN A LIMIT CASE

Ester Giarrusso and Guido Trombetti

Let u be a weak solution of the homogeneous Dirichlet problem for a second order elliptic equation of divergence form, in a bounded open subset of $\mathbb{R}^n$. We prove, that if the right hand side of the equation is an element of $H^{-1,n}(\Omega)$, then u belongs to the Orlicz space L_Φ where $\Phi(t) = \exp(|t|^{n/(n-1)}) - 1$. We employ the properties of the Schwartz symmetrization thus obtaining the "best" constant of the estimate.

1. Introduction

Denote by Ω a bounded open subset of $\mathbb{R}^n (n \geq 2)$ and consider the Dirichlet problem

$$(1.1)\qquad \begin{cases} -(a_{ij}(x)u_{x_i})_{x_j} = -(f_j)_{x_j} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Received 27 November 1986. Supported in part by M.P.I. (40%).

https://doi.org/10.1017/S0004972700003713 Published online by Cambridge University Press

Here the coefficients a_{ij} are bounded and measurable, the f_j are in $L^p(\Omega)$, $p \geq 2$, and

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 \qquad x \in \Omega , \quad \xi \in \mathbb{R}^n .$$

Here and below the summation convention is employed. Put

$$f = (\sum_{i=1}^{n} |f_i|^2)^{1/2} .$$

It is very well known ([16]) that

$$\|u\|_{p^*} \leq K(p,n) \|f\|_p \quad p^* = \frac{np}{n-p} , \quad \text{if} \quad 2 \leq p < n , \tag{1.2}$$

and that

$$\|u\|_\infty \leq K(p,n) |\Omega|^{\frac{1}{n}-\frac{1}{p}} \|f\|_p \qquad \text{if} \quad p > n . \tag{1.3}$$

Actually, this is the Sobolev imbedding theorem if we replace f by $|Du|$.

For the limit case $p = n$, Stampacchia [17] showed that the solution u of (1.1) is in the Orlicz space L_Φ defined by the function $\Phi(t) = e^{|t|} - 1$. (See [1] and [11] for the exact definition and the properties of Orlicz spaces). On the other hand, Trudinger proved later that $\overset{\circ}{W}{}^{1,n}(\Omega)$ is continuously imbedded in the Orlicz space L_Φ defined by $\Phi(t) = \exp(|t|^{n/(n-1)}) - 1$, ([20], Theorem 2, p.478). Hence one can expect that Stampacchia's result can be sharpened and that the solution u of (1.1) is in the same Orlicz space as that found by Trudinger (that is with $\Phi(t) = \exp(|t|^{n/(n-1)}) - 1)$.

The aim of this paper is to prove the following estimate

$$\|u\|_\Phi \leq C(n, |\Omega|) \|f\|_n \tag{1.4}$$

where

$$\|u\|_\Phi = \inf\{k > 0 : \int_\Omega \Phi(\frac{u}{k})\, dx \leq 1\} \tag{1.5}$$

and

$$\Phi(t) = \exp(|t|^{n/(n-1)}) - 1 .$$

https://doi.org/10.1017/S0004972700003713 Published online by Cambridge University Press

We actually get a slightly better result. We show that

(1.6) $$[u]_{1-1/n} \le (nC_n)^{-1/n} \|f\|_n ,$$

where

C_n = the measure of the unit ball of $\mathbb{R}^n$. Put

$\Omega^\#$ = the ball of $\mathbb{R}^n$ centred at the origin and with the same measure as Ω,

$r = (|\Omega|/C_n)^{1/n}$ = the radius of $\Omega^\#$,

$u^\#(x)$ = the spherically decreasing symmetric rearrangement of u (see Section 2).

Moreover, the value of the constant on the right-hand side of (1.6) is the best possible. The quantity

(1.7) $$[u]_{1-1/n} = \sup_{\Omega^\#} u^\#(x)/(\log\frac{r}{|x|})^{1-1/n}$$

has been introduced in [2], where a "sharp" version of Trudinger's imbedding theorem is given. A paper also related to this topic is [13].

In [2] it is proved that $\forall\, u \in \mathring{W}^{1,n}(\Omega)$

(1.8) $$[u]_{1-1/n} \le (nC_n)^{-1/n} \|Du\|_n$$

and the constant is the best one. The connection between (1.5) and (1.7) is given by the following inequality ([2])

(1.9) $$\|u\|_\Phi \le (\frac{1+|\Omega|}{n})^{1-1/n} [u]_{1-1/n}$$

We finally point out that more generally our result applies to elliptic operators of degenerate type like those in [14] and in [21]. For this kind of operator the estimate (1.4) has already been obtained by Trudinger for the case $n=2$, ([21, Theorem 4.1]).

2. Hypotheses and preliminaries

Throughout this paper we assume

(i) $$a_{ij}(x)\xi_i\xi_j \ge m(x)|\xi|^2 \quad \text{a.e. in } \Omega,\ \xi \in \mathbb{R}^n$$
$$m(x) \ge 0 \quad \text{in } \overline{\Omega}$$

(ii) $$m(x) \in L^s(\Omega),\ m^{-1}(x) \in L^q(\Omega),\ \frac{1}{s}+\frac{1}{q} \le \frac{2}{n}$$
$$a_{ij}m^{-1} \in L^\infty,\ f_j m^{-1/p} \in L^p(\Omega),\ p \ge 2 .$$

We also use the following notation.

Given a measurable real valued function f in Ω, the distribution function of f is

$$\mu(t) = \text{meas}\{x \in \Omega : \; |f| > t\} \;;$$

the decreasing rearrangement of f is

$$f^*(s) = \inf\{t \geq 0 : \; \mu(t) < s\} \;;$$

and the spherically symmetric rearrangement of f is

$$f^\#(x) = f^*(C_n|x|^n) \;.$$

Finally, for $f \in L^p(\Omega)$ we denote by

$$K(f) = \{F \in L^p(0,|\Omega|) \text{ for which } \exists\{f_k\} \subset L^p(0,|\Omega|) :$$

$$f_k^* = f^* \text{ and } f_k \rightharpoonup F \text{ in } L^p(0,|\Omega|)\} \;.$$

Now we state a lemma of weak approximation, which is one of the main tools for the proof of Theorem 3.1.

Consider the distribution function $\mu(t)$ of the solution $u(x)$ of (1.1) and for any $s \in [0,|\Omega|]$ a subset $D(s)$ of Ω such that

$$\text{meas}\; D(s) = s$$

$$s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$$

$$D(s) = \{x \in \Omega : \; |u| > t\} \quad \text{if} \quad s = \mu(t) \;.$$

Hence, given a function $f \in L^p(\Omega)$, there exists a function $F(t)$ such that

$$\int_{D(s)} f(x)dx = \int_0^s F(t)dt \;. \tag{2.1}$$

LEMMA 2.1. *The function $F(t)$ given by (2.1) belongs to $K(f)$.*

See the proof in [4]. See also [12] and [15] for a similar topic.

3. The estimate for the limit case

To get our result we use a symmetrization technique ([4], [5]. [10], [18]). First we prove the following comparison lemma.

LEMMA 3.1. *Let (i) - (ii) be satisfied and let u be the solution of (1.1). Then*

$$u^\#(x) \leq v(x)$$

where $v(x)$ *is a solution of the problem*

$$\begin{cases} -(\underline{m}(C_n|x|^n)v_{x_i})_{x_i} = (F(C_n|x|^n)\underline{m}^{1/2}(C_n|x|^n)\, x_i/|x|)_{x_i} & \text{in } \Omega^* \\ v = 0 \quad \text{on } \partial\Omega^*, & \end{cases}$$

$F^2(s) \in K(f^2/m)$ *and* $\underline{m}^{-1}(s) \in K(m^{-1})$.

Proof. Since $u \in H_0^1(m)$ (where $H_0^1(m)$ is (see [14]) the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^1(m)} = \left(\int_\Omega m(x)|Du|^2 dx\right)^{1/2} \Big)$$

is a weak solution of (1.1), then

$$\int_\Omega a_{ij}\, u_{x_i} \psi_{x_j}\, dx = \int_\Omega f_j \psi_{x_j}\, dx \qquad \forall \psi \in H_0^1(m) .$$

Now choose in the above

$$\psi(x) = \begin{cases} (|u|-t)\,\text{sgn}\, u & \text{if} \quad |u| > t \\ 0 & \text{if} \quad |u| \le t \end{cases}$$

to obtain

$$\int_{|u|>t} a_{ij} u_{x_i} u_{x_j} = \int_{|u|>t} f_j u_{x_j}\, dx \quad , \quad t > 0 .$$

Recall (i) and perform some standard calculations ([4], [18]) to get

$$(3.1) \qquad -\frac{d}{dt}\int_{|u|>t} m(x)|Du|^2 dx \le -\frac{d}{dt}\int_{|u|>t} \frac{f^2(x)}{m(x)}\, dx .$$

Now, with the same technique as that used in [4], [18], by the De Giorgi isoperimetric inequality ([6]) and a Fleming-Rishel formula ([7]), we have

$$(3.2) \qquad nC_n^{1/n}\mu(t)^{1-1/n} \le -\frac{d}{dt}\int_{|u|>t} |Du|\, dx .$$

Use Hölder's inequality on the right-hand side of (3.2) and recall (3.1) to get

$$(3.3) \qquad (nC_n\mu(t)^{1-1/n})^2 \le \left(-\frac{d}{dt}\int_{|u|>t}\frac{f^2(x)}{m(x)}\, dx\right)\left(-\frac{d}{dt}\int_{|u|>t}\frac{1}{m(x)}\, dx\right) .$$

https://doi.org/10.1017/S0004972700003713 Published online by Cambridge University Press

According to Lemma 2.1 consider a function $F^2(s)$ and a function $1/\underline{m}(s)$ such that

$$\int_{|u|>t} \frac{f^2(x)}{m(x)}\,dx = \int_0^{\mu(t)} F^2(s)\,ds$$

and

$$\int_{|u|>t} \frac{1}{m(x)}\,dx = \int_0^{\mu(t)} \frac{1}{\underline{m}(s)}\,ds .$$

Then use (3.3) to see that

$$1 \le \frac{1}{nC_n^{1/n}} \frac{F(\mu(t))}{\underline{m}^{1/2}(\mu(t))} \mu(t)^{1/n-1}(-\mu'(t)) .$$

Integrate the above between 0 and t and recall the definition of $u^*(s)$ to obtain

$$u^*(C_n|x|^n) \le \frac{1}{nC_n^{1/n}} \int_{C_n|x|^n}^{|\Omega|} \frac{F(r)}{\underline{m}^{1/2}(r)} r^{-1+1/n}\,dr = v(x) . \qquad \Box$$

THEOREM 3.1. *Let (i), (ii) be satisfied,* $q > n$ *and* $p = n(q-1)/(q-n)$ *, then*

$$[u]_{1-1/n} \le (nC_n)^{-1/n} \|m^{-1}\|_q^{1-1/p} \|fm^{-1/p}\|_p .$$

Proof. By definition of $[u]_{1-1/n}$ (see (1.7)) and by Lemma 3.1 we have

$$[u]_{1-1/n} \le [v]_{1-1/n} . \tag{3.4}$$

On the other hand

$$\begin{aligned} (3.5)\quad [v]^n_{1-1/n} &\le (nC_n)^{-1} \|Dv\|^n_n && \text{by (1.8)} \\ &= (nC_n)^{-1} \|F(C_n|x|^n)/\underline{m}^{1/2}(C_n|x|^n)\|^n_n && \text{by definition of } v(x) \\ &= (nC_n)^{-1} \int_0^{|\Omega|} \left|\frac{F^2(s)}{\underline{m}(s)}\right|^{n/2} ds . \end{aligned}$$

Now because of Lemma 2.1 there exists a sequence $\{\psi_k^2(s)\}$ of functions equidistributed with f^2/m and such that

$$\psi_k^2(s) \longrightarrow F^2(s)$$

https://doi.org/10.1017/S0004972700003713 Published online by Cambridge University Press

in $L^{\alpha}(0,|\Omega|)$, with

$$\frac{1}{\alpha} = \frac{2}{p} + \frac{1}{q}\,\frac{p-2}{p} .$$

Hence

$$\int_0^{|\Omega|} |F^2(s)|^{\alpha}\, ds \le \lim_k \int_0^{|\Omega|} |\psi_k^2(s)|^{\alpha} ds \tag{3.6}$$

$$= \int_0^{|\Omega|} (\, (\psi_k^2(s))^*)^{\alpha} ds = \int_{\Omega} (\frac{f^2(x)}{m(x)}\,)^{\alpha} dx ,$$

by using the well known result (see [5], [10], [18])

$$\int_{\Omega} u(x)dx = \int_0^{|\Omega|} u^*(s)ds .$$

Analogously by Lemma 2.1 there exists a sequence $\{\zeta_k(s)\}$ equidistributed with $m^{-1}(x)$ and such that

$$\zeta_k(s) \longrightarrow \frac{1}{\underline{m}(s)}$$

in $L^q(0,|\Omega|)$. Thus

$$\int_0^{|\Omega|} \frac{1}{\underline{m}^q(s)}\, ds \le \lim_k \int_0^{|\Omega|} |\zeta_k(s)|^q ds = \int_{\Omega} \frac{1}{m^q(x)}\, dx . \tag{3.7}$$

Now, $\frac{n}{2\alpha} + \frac{n}{2q} = 1$ since $p = n(q-1)/(q-n)$, and hence by Hölder's inequality and by (3.6) and (3.7)

$$\int_0^{|\Omega|} (\frac{F^2(s)}{\underline{m}(s)})^{n/2} ds$$

$$\le \left\{ \int_{\Omega} (\frac{f^2(x)}{m(x)})^{\alpha} dx\right\}^{n/2\alpha} \left\{ \int_{\Omega} \frac{1}{m^q(x)}\, dx\right\}^{n/2q}$$

$$\le \| f\, m^{-1/p}\|_p^n \;.\; \|m^{-1}\|_q^{n(p-1)/p}$$

(again by Holder's inequality since $\frac{2\alpha}{p} + \frac{\alpha}{q}\,\frac{p-2}{p} = 1$) .

Now the above inequality with (3.4) and (3.5) gives the required estimate. □

https://doi.org/10.1017/S0004972700003713 Published online by Cambridge University Press

Remarks. (1) At least for the uniformly elliptic case $(m=1)$, the constant in Theorem 3.1 is the best possible.

(2) An obvious consequence of Theorem 3.1 and (1.9) is that

$$\|u\|_{\Phi} \le \frac{(1+|\Omega|)^{(n-1)/n}}{nC_n^{1/n}} \|m^{-1}\|_q^{1-1/p} \|fm^{-1/p}\|_p$$

with $\Phi(t) = \exp(|t|^{n/(n-1)}) - 1$.

(3) Theorem 3.1 improves the estimate (b) of Theorem 7.2 in [14] and th estimate obtained in Theorem 4.1 of [21] for case II.

(4) Using the Comparison Lemma 3.1 we can derive the numerical value of the constants in "sharp" estimates for a solution u of (1.1) by known methods ([4], [18], [8]).

For example we have for $m=1$ and $p>n$:

$$\|u\|_\infty \le K_1(n,p)\, |\Omega|^{\frac{1}{n}-\frac{1}{p}} \|f\|_p$$

where

$$K_1(n,p) = \frac{1}{nC_n^{1/n}} \left[\frac{n(p-1)}{p-n}\right]^{1-1/p}$$

This constant is the best one as already known (see [22]).

For $m=1$ and $2\le p\le n$, via the Bliss inequality (see [18])

$$\int_0^\infty \left(\int_r^\infty \psi(s)ds\right)^q dr \le B\left(\int_0^\infty \psi(r)^p r^{-1+p+p/q}\, dr\right)^{q/p} \quad q>p>1$$

where

$$B = \frac{B}{q(p-1)} \left\{\frac{\Gamma(pq/(q-p))}{\Gamma(q/(q-p))\ \Gamma(p(q-1)/(q-p))}\right\}^{(q/p)-1} (q(1-1/p))^{q-(q/p)+1}$$

we get:

$$\|u\|_{p^*} \le K_2(n,p)\, \|f\|_p$$

where

$$K_2(n,p) = \frac{1}{nC_n^{1/n}} \left\{\frac{\Gamma(n)}{\Gamma(n/p)\ \Gamma(1+n-n/p)}\right\}^{1/n} \left\{\frac{n(p-1)}{n-p}\right\}^{1-1/p}$$

Also this constant is the best possible one ([19]) .

References

[1] R.A. Adams, *Sobolev Spaces*, (Academic Press, New York, San Francisco, London, 1975).

[2] A. Alvino, "Un caso limite della diseguaglianza di Sobolev in spazi di Lorentz", *Rend. Accad. Sci. Fis. Mat. Napoli* XLIV (1977), 105-112.

[3] A. Alvino, P.L. Lions and G. Trombetti, "On optimization problems with prescribed rearrangements", (to appear).

[4] A. Alvino and G. Trombetti, "Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri", *Ricerche Mat.*, XXVII (1978), 413-428.

[5] C. Bandle, *Isoperimetric Inequalities and Applications*, (Pitman, London 1980).

[6] E. De Giorgi, "Su una teoria generale della misura $(r-1)$ dimensionale in uno spazio ad r dimensioni", *Ann. Mat. Pura Appl.*, 36 (1954), 197-213.

[7] W. Fleming and R. Rishel, "An integral formula for total gradient variation", *Arch Math.*, 11 (1960).

[8] E. Giarrusso and D. Nunziante, "Symmetrization in a class of first-order Hamilton-Jacobi equations", *Nonlinear Anal.* 8 (1984), 289-299.

[9] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983).

[10] B. Kawohl, *Rearrangements and Convexity of Level Sets in P.D.E.*, (Lecture Notes in Mathematics, 1150, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985).

[11] M.A. Krasnosel'skii and Y.A. Rutickii, *Convex Functions and Orlicz Spaces*, (P. Noordhoff Ltd., Groningen, 1961).

[12] L. Migliaccio, "Sur une condition de Hardy-Littlewood-Polya", *C.R. Acad. Sci. Paris, Sér. I. Math.* 297 (1983), 25-28.

[13] J. Moser, "A sharp form of an inequality by N. Trudinger", *Indiana Univ. Math. J.*, 20 (1971), 1077-1092.

[14] M.K.V. Murthy and G. Stampacchia, "Boundary value problems for some degenerate elliptic operators", *Ann. Math. Pura Appl.*, 90 (1971), 1-122.

[15] J. Ryff, "Majorized functions and measure", *Indag. Math.* 30 (1968), 431-437.

[16] G. Stampacchia, "Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus", *Ann. Inst. Fourier (Grenoble)* 15 (1965), 189-258.

[17] G. Stampacchia, "Some limit cases of L^p-estimates for solutions of second order elliptic equations", *Comm. Pure Appl. Math.* XVI (1963), 505-510.

[18] G. Talenti, "Elliptic equations and rearrangements", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 3 (1976), 697-718.

[19] G. Talenti, "Best constant in Sobolev inequality", *Ann. Mat. Pura Appl.* 110 (1976), 353-372.

[20] N.S. Trudinger, "On imbeddings into Orlicz spaces and some applications", *J. Math. Mech.*, 17 (1967), 473-484.

[21] N.S. Trudinger, "Linear elliptic operators with measurable coefficients", *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat.*, 27 (1973), 265-308.

[22] H.F. Weinberger, "Symmetrization in uniformly elliptic problems", *Studies in Math. Analysis and related topics*, (Stanford Univ. Press, Stanford, California, 1962), 424-428.

Dipartimento di Matematica ed Applicazioni
"R. Caccioppoli", Universita di Napoli
Via Mezzocannone, 8; 80134 Napoli
Italia