Can. J. Math., Vol. XXV, No. 2, 1973, pp. 370–385

TESTING CATEGORIES AND STRONG UNIVERSALITY

J. SICHLER

1. Introduction. A category **A** is *binding* (or *universal*) if any full category of algebras is isomorphic to a full subcategory of **A**. There are many binding categories: the category of all commutative rings with unit and all unit-preserving homomorphisms [**1**], the category of bounded lattices [**2**], the category of semigroups [**3**], the category **A**(1, 1) of all algebras with two unary fundamental operations and the category of directed graphs [**4**], the category of all commutative groupoids [**11**] and many others. A considerable number of known universal categories are equational classes of algebras so that a question of certain importance arises: how the universal equational classes can be distinguished among all equational classes of algebras? The present paper offers one possible criterion for equational classes of unary algebras; it takes on the following form:

There exists a finite category **C** *such that the full embeddability of* **C** *into an equational class* **A** *of unary algebras is equivalent to the universality of* **A**.

We may say that the finite category **C** tests the universality of an equational class of unary algebras. The present paper is primarily concerned with the proof of the existence of such a testing category; it will be explicitly described here. The finiteness of **C** is the property to be emphasized for there are many testing categories which are not small: a category **K** is binding if and only if some binding category can be fully embedded into it. On the other hand, there is no small category testing universality of an arbitrary category for all the universal categories are large. These obvious facts force us to narrow the range of categories to be tested considerably; the admissible categories have to satisfy the conditions (0)–(6) listed below.

Let **A** be a binding category with a small left adequate subcategory **L** and let $\mathbf{F} : \mathbf{L} \rightarrow \mathbf{B}$ be a full embedding to a cocomplete category **B**. The small category **L** appears to be a good candidate for a testing category because the functor **F** can be naturally extended by colimits of suitable diagrams to a functor $\mathbf{F}^* : \mathbf{A} \rightarrow \mathbf{B}$. There is, however, no reason for the functor $\mathbf{F}^*$ to be one-to-one let alone full and there are indeed numerous examples of its failure to possess either of the two properties even if **B** was an equational class of unary algebras. The next question to ask is whether a small left adequate subcategory **L** of some particular binding category could be enlarged to a

Received January 20, 1972 and in revised form, May 9, 1972. This research was partially supported by NRC grant No. A-7690.

subcategory **C** of **A** whose representability in **B** would guarantee both desired properties of the extended functor. Fortunately enough, this can be done: the finite testing category **C** is found among full subcategories of a binding subcategory of the category of all undirected graphs and all their compatible mappings.

The categories tested for universality by this small category are not just equational classes of unary algebras. The criterion applies to any category **K** satisfying the conditions (0)–(6).

(0) **K** is a concrete category with a faithful functor $| \ | : \mathbf{K} \to \mathbf{Set}$. **K** and $| \ |$ satisfy (1)–(6) below:

(1) there are class **E** of epimorphisms and a class **M** of monomorphisms of **K** such that **K** is a bicategory in the sense of Isbell with respect to these two classes (cf. [**7**]),

(2) $|m|$ is a one-to-one mapping for every $m \in \mathbf{M}$,

(3) for every object A of **K** and for every bijection $b : |A| \to X$ there is an isomorphism i of **K** such that $b = |i|$,

(4) **K** has and $| \ |$ preserves equalizers,

(5) **K** is a cocomplete category,

(6) if $\mathbf{D} : \mathbf{S} \to \mathbf{K}$ is a diagram and $(A, r) = \mathrm{colim}_{\mathbf{K}}(\mathbf{D})$ (where A is an object of **K** and $r : \mathbf{D} \to \mathbf{const}_A$ is the colimiting cone), then

$$|A| = \bigcup_{s \in \mathrm{Obj}(\mathbf{S})} |r_s|(|\mathbf{D}(s)|).$$

All our considerations will involve a concrete category **K** together with one of its forgetful functors $| \ |$. To emphasize the fact that a particular faithful functor $| \ |$ has been chosen we will often write $(\mathbf{K}, | \ |)$ instead of **K**.

Recall that the condition (1) means that

(7) both **E** and **M** are subcategories of **K**,

(8) $\mathbf{E} \cap \mathbf{M} = \mathrm{Iso}(\mathbf{K})$ (the class of all isomorphisms of **K**),

(9) every morphism k of **K** has a decomposition $k = m \circ e$ for some $m \in \mathbf{M}$ and $e \in \mathbf{E}$,

(10) if $f \circ e = m \circ g$, $m \in \mathbf{M}$, $e \in \mathbf{E}$, then there is $h \in \mathbf{K}$ such that $h \circ e = g$ (and, consequently, $m \circ h = f$).

The condition (2) is certainly a natural one; (3) says that every object of **K** has enough isomorphic copies. All the conditions are naturally satisfied by any equational class of unary algebras; let us point out that the condition (6) is the most restrictive one – it eliminates equational classes of algebras which are not unary. There are, however, some other categories for which (0)–(6) do hold. Let us mention at least two different types of these.

The functor categories. Let $(\mathbf{K}, | \ |)$ be a category satisfying (0)–(6) and let **c** be a small category. Let $\mathbf{K}^{\mathbf{c}}$ denote the category whose objects are all functors $\mathbf{F} : \mathbf{c} \to \mathbf{K}$ and whose morphisms are all natural transformations of these

https://doi.org/10.4153/CJM-1973-038-3 Published online by Cambridge University Press

functors. Define a functor $|\ \ |^* : \mathbf{K}^{\mathbf{c}} \to \mathbf{Set}$ as follows. For an object $\mathbf{F} : \mathbf{c} \to \mathbf{K}$ put

$$|\mathbf{F}|^* = \bigcup_{c \in \mathrm{Obj}(\mathbf{c})} |\mathbf{F}(c)| \times \{c\};$$

for a natural transformation $n : \mathbf{F} \to \mathbf{G}$, $n = (n_c : c \in \mathrm{Obj}(\mathbf{c}))$, define $|n|^*$ by $|n|^*(x, c) = (|n_c|(x), c)$ for all $c \in \mathrm{Obj}(\mathbf{c})$ and all $x \in |\mathbf{F}(c)|$. The functor $|\ \ |^*$ is clearly faithful and the properties (0)–(6) of $(\mathbf{K}^{\mathbf{c}}, |\ \ |^*)$ are extensions of the corresponding properties of $(\mathbf{K}, |\ \ |)$. The special case of $\mathbf{K} = \mathbf{Set}$ is referred to in [**3**], where the concept of a rich small category is defined. A small category $\mathbf{c}$ is called *rich* if the functor category $\mathbf{Set}^{\mathbf{c}}$ is binding. Observe that every equational class of algebras is isomorphic to some full subcategory of $\mathbf{Set}^{\mathbf{c}}$ with a suitable $\mathbf{c}$.

The categories $S(\mathbf{F})$. Let $\mathbf{F} : \mathbf{Set} \to \mathbf{Set}$ be a covariant functor. The objects of the category $S(\mathbf{F})$ are all pairs (X, R) where X is a set and R is a subset of $\mathbf{F}(X)$. A mapping $f : X \to X'$ is a morphism of $S(\mathbf{F})$ from (X, R) to (X', R') if $\mathbf{F}(f)(R) \subseteq R'$. The natural functor $|\ \ |$ defined by $|(X, R)| = X$ and by $|f| = f$ is faithful. Let $\mathbf{E}$ be the class of all morphisms $e : (X, R) \to (X', R')$ with $e(X) = X'$ and $\mathbf{F}(e)(R) = R'$; let $\mathbf{M}$ be the class of all one-to-one morphisms. It is easy to see that $S(\mathbf{F})$ becomes a bicategory which is both complete and cocomplete. $|\ \ |$ preserves all limits and colimits so that all the properties (0)–(6) are possessed by $(S(\mathbf{F}), |\ \ |)$.

Let $(\mathbf{K}, |\ \ |)$ and $(\mathbf{L}, |\ \ |^*)$ be concrete categories. A full embedding $\mathbf{G} : \mathbf{K} \to \mathbf{L}$ is called *strong* if there exists a functor $\mathbf{F} : \mathbf{Set} \to \mathbf{Set}$ such that $|\ \ |^* \circ \mathbf{G} = \mathbf{F} \circ |\ \ |$. A category $(\mathbf{L}, |\ \ |^*)$ is *strongly binding* or *strongly universal* if any category of algebras (equipped by the standard underlying-set functor as the forgetful one) can be strongly embedded into $(\mathbf{L}, |\ \ |^*)$. The embeddings constructed here will enable us to show that every category satisfying (0)–(6) which is binding is also strongly binding; in particular, universality and strong universality are properties equivalent for every equational class of unary algebras.

In the second section a small category $\mathbf{C}$ (which will turn out to be our testing category) is described and its representation in a category $\mathbf{K}$ satisfying (0)–(6) modified to a form more suitable for the construction of the colimit construction which is discussed in Section 3. Let us point out that the purpose of the second section is to eliminate undesired properties of a representation one starts with; every modifying step is necessary as can be shown by a series of examples. The fourth section shows the equivalence of universality and strong universality for the type of category investigated here.

2. The testing category. We will be working within the framework of the Gödel-Bernays set theory; an ordinal will be the set of all the smaller ordinals and cardinals will be identified with initial ordinals. The symbol $\vee$ will denote the disjoint union.

Let $\mathbf{U}$ be the category of all undirected graphs without diagonal and all their compatible mappings; i.e., the objects of $\mathbf{U}$ are all pairs (X, R) where X

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is a set and R is a set of two-element subsets of X. A mapping $f : X \to X'$ is a morphism of $\mathbf{U}$ from (X, R) to (X', R') if $\{x_1, x_2\} \in R$ implies $\{f(x_1), f(x_2)\} \in R'$. As a concrete category $\mathbf{U}$ will always be equipped by the natural functor $|\ \ |$ defined by $|(X, R)| = X$, $|f| = f$.

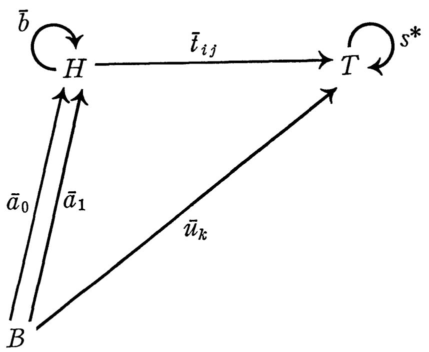
Let $\mathbf{C}$ be the abstract category isomorphic to the full subcategory of $\mathbf{U}$ determined by the graphs $\mathbf{1} = (\{e\}, \emptyset)$, $\mathbf{2} = (\{0, 1\}, \{\{0, 1\}\})$ and by the full graph $\mathbf{4}$ with four vertices. Let us assume that $\mathbf{B}$ is already represented as a full subcategory of a category $(\mathbf{K}, |\ \ |)$ satisfying (0)–(6); let B, H, and T be the three objects of $\mathbf{K}$ representing $\mathbf{1}$, $\mathbf{2}$, and $\mathbf{4}$, respectively. Let S_4 be the group of all permutations of the set $4 = \{0, 1, 2, 3\}$. The following is a description of the full representation of $\mathbf{C}$ in $\mathbf{K}$; it is easy to see that this is indeed a representation of the full subcategory of $\mathbf{U}$ determined by the objects $\mathbf{1}$, $\mathbf{2}$, and $\mathbf{4}$ of $\mathbf{U}$:

$$\mathbf{K}(B, B) = \{1_B\},\ \mathbf{K}(H, H) = \{1_H, \bar{b}\},\ \mathbf{K}(T, T) = \{s^* : s \in S_4\},$$

$$\mathbf{K}(B, H) = \{\bar{a}_0, \bar{a}_1\},\ \mathbf{K}(B, T) = \{\bar{u}_i : i \in 4\},$$

$$\mathbf{K}(H, T) = \{\bar{t}_{ij} : i \neq j,\ i, j \in 4\},\ \text{and no other morphisms.}$$

(11)



The composition is defined by

(12) $\bar{b} \circ \bar{a}_i = \bar{a}_{1-i}$ for $i = 0, 1$; $\bar{b} \circ \bar{b} = 1_H$,

(13) $\bar{t}_{ij} \circ \bar{b} = \bar{t}_{ji}$ for all $\bar{t}_{ij}$,

(14) $\bar{t}_{ij} \circ \bar{a}_0 = \bar{u}_i$ and $\bar{t}_{ij} \circ \bar{a}_1 = \bar{u}_j$ for all $\bar{t}_{ij}$,

(15) $s^* \circ \bar{u}_k = \bar{u}_{s(k)}$ for all $s \in S_4$ and all $\bar{u}_k$,

(16) $s^* \circ \bar{t}_{ij} = \bar{t}_{s(i)s(j)}$ for all $s \in S_4$ and all $\bar{t}_{ij}$.

Observe that $\mathbf{C}$ is a finite category. Using the above representation of $\mathbf{C}$ in $(\mathbf{K}, |\ \ |)$ we will produce a new five-object subcategory of $\mathbf{K}$ with all the properties needed for the construction discussed in the next section.

Let $\bar{t}_{01} = t_{01} \circ d$ be a decomposition of $\bar{t}_{01}$ provided for by (9), let $d : H \to A$, $t_{01} : A \to T$; (10) yields the uniqueness of A up to isomorphism. Let $i \neq j$ and let $s \in S_4$ be such that $s(0) = i$ and $s(1) = j$. Put $t_{ij} = s^* \circ t_{01}$. Since s^*

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is an isomorphism and $t_{01} \in \mathbf{M}$ we may conclude with the use of (8) and (9) that every t_{ij} is in the class $\mathbf{M}$. Also

(17) $s^* \circ t_{ij} = t_{s(i)s(j)}$ and $\bar{t}_{ij} = t_{ij} \circ d$ for all t_{ij}.

As d is an epimorphism of $\mathbf{K}$ all t_{ij} are pairwise different; using the fact that $\mathbf{K}(H, T) = \{\bar{t}_{ij} : i \neq j, i, j \in 4\}$ together with (17) we obtain

(18) $\mathbf{K}(A, T) = \{t_{ij} : i \neq j, i, j \in 4\}$.

Similarly, there is a morphism $b : A \to A$ such that

(19) $t_{ij} \circ b = t_{ji}$ for all t_{ij}, $b \circ b = 1_A$, $b \circ d = d \circ \bar{b}$.

Let f and g be morphisms of $\mathbf{K}$ such that $t_{01} \circ f = t_{02} \circ g$. Let $s_1 \in S_4$ be a permutation interchanging 1 with 3 and leaving 0 and 2 fixed; let s_2 be defined by $s_2(0) = 0$, $s_2(1) = 2$, $s_2(2) = 1$, and $s_2(3) = 3$. Now, $t_{03} \circ f = s_1^* \circ t_{01} \circ f = s_1^* \circ t_{02} \circ g = t_{02} \circ g$ and also $t_{01} \circ g = s_2^* \circ t_{02} \circ g = s_2^* \circ t_{03} \circ f = t_{03} \circ f$; hence $t_{01} \circ f = t_{02} \circ g = t_{03} \circ f = t_{01} \circ g$. But $t_{01} \in \mathbf{M}$, so that $f = g$. Let $(E, a_0) = \text{Equalizer}(t_{01}, t_{02})$, put $a_1 = b \circ a_0$. Using the transitivity of S_4^* on $\mathbf{K}(A, T)$ we easily conclude that

(20) for every i, j, k, $j \neq k$, $(E, a_0) = \text{Eq}(t_{ij}, t_{ik})$ and $(E, a_1) = \text{Eq}(t_{ji}, t_{ki})$.

Another easily drawn conclusion is

(21) $|t_{ij}|(a) = |t_{ik}|(a')$ implies $a = a'$; if $j \neq k$ then the equality holds if and only if $a = a' = |a_0|(e)$ for some $e \in |E|$.

An analogous statement is valid also for $|t_{ji}|$, $|t_{ki}|$, and $|a_1|$.

Since $t_{01} \circ d \circ \bar{a}_0 = \bar{t}_{01} \circ \bar{a}_0 = \bar{t}_{02} \circ \bar{a}_0 = t_{02} \circ d \circ \bar{a}_0$, (20) yields the existence of a unique $e : B \to E$ with $a_0 \circ e = d \circ \bar{a}_0$; let $e = m \circ n$ be its decomposition according to (9) and let P be the domain of m. We have also $a_1 \circ e = b \circ a_0 \circ e = b \circ d \circ \bar{a}_0 = d \circ \bar{b} \circ \bar{a}_0 = d \circ \bar{a}_1$; if $a_0 \circ m = a_1 \circ m$, then $\bar{t}_{01} \circ \bar{a}_0 = t_{01} \circ d \circ \bar{a}_0 = t_{01} \circ a_0 \circ m \circ n = t_{01} \circ a_1 \circ m \circ n = \bar{t}_{01} \circ \bar{a}_1$, a contradiction. Put $v_i = a_i \circ m$ for $i = 0, 1$ and observe that $|v_0|$ and $|v_1|$ are one-to-one mappings. For every $k \in 4$ define $w_k = t_{k0} \circ v_0$. It is easy to see that

(22) $w_k = t_{ki} \circ v_0 = t_{ik} \circ v_1$ for all t_{ik}.

Because n is an epimorphism and all w_k are pairwise different,

(23) $\mathbf{K}(P, T) = \{w_k : k \in 4\}$.

Now let $v : P \to A$ be a morphism of $\mathbf{K}$. From (23) it follows that $t_{01} \circ v = w_k$ for some $k \in 4$. If $k \neq 0, 1$, then there is an involution s such that $s(i) = i$ for $i = 0, 1$ and $s(k) \neq k$. Hence $w_k = t_{01} \circ v = s^* \circ t_{01} \circ v = s^* \circ w_k = w_{s(k)}$, a contradiction. If $k = 0$, then $t_{01} \circ v = t_{01} \circ v_0$ and $v = v_0$ as $t_{01} \in \mathbf{M}$. If $k = 1$, $v = v_1$. Thus

(24) $\mathbf{K}(P, A) = \{v_0, v_1\}$.

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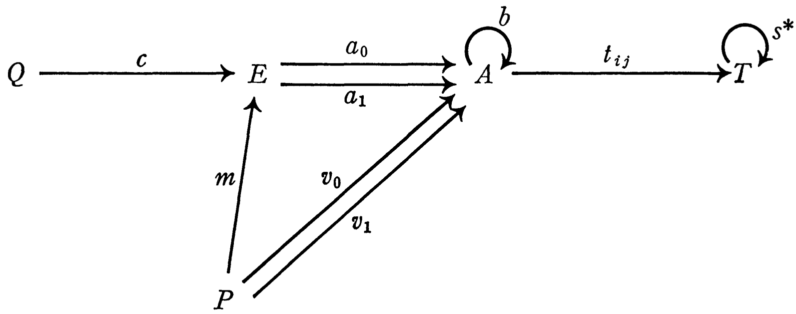
Suppose that $a_0 \circ f = a_1 \circ g$ for some $f, g \in \mathbf{K}$; then $t_{01} \circ a_0 \circ f = t_{01} \circ a_1 \circ g = t_{21} \circ a_1 \circ g = t_{21} \circ a_0 \circ f = t_{20} \circ a_0 \circ f = t_{20} \circ a_1 \circ g$. But $t_{01} \circ a_0 = t_{10} \circ b \circ a_0 = t_{10} \circ a_1 = t_{20} \circ a_1$ is a monomorphism so that $f = g$. Put $(Q, c) = \mathrm{Eq}(a_0, a_1)$; using (2) and (4) we conclude that

(25) $|a_0|(x) = |a_1|(y)$ if and only if $x = y = |c|(q)$ for some $q \in |Q|$.

The modified five-object category is generated by a_0, m, c, b, t_{01}, and all s^* (cf. (26)).

(26)



The above category will be used for the construction of a full embedding in the next section. Note that all its morphisms are monomorphisms; by (2), their underlying mappings are all one-to-one.

3. The construction. An undirected graph $G = (X, R)$ is 3-*colourable* if there is a compatible mapping φ of G into the complete graph **3** with three vertices. G is *connected* if for every two distinct vertices x, y there is a finite sequence $x = x_0, \ldots, x_{n-1}, x_n = y$ such that

$$\{x_i, x_{i+1}\} \in R \qquad \text{for} \qquad i = 0, \ldots, n-1.$$

Let **G** be the category of all connected 3-colourable graphs and all compatible mappings between them. **G** is a binding category; this can be proved easily using [**5**; **4**], and [**11**] as the sources of suitable full embeddings.

A full embedding of the category **G** into a category $(\mathbf{K}, |\ \ |)$ satisfying (0)–(6) will be constructed in this section as a colimit functor extending a representation of the modified small category of the previous section.

To every object $G = (X, R)$ of **G** a small category $\mathbf{S}_G$ will be assigned as follows:

$\mathrm{Obj}(\mathbf{S}_G) = (O_x : x \in X) \cup (O_{xy} : (x, y) \in X^2$ and $\{x, y\} \in R)$; the morphisms of $\mathbf{S}_G$ beside the units are:

$\beta_{xy} : O_{xy} \to O_{yx}$ for each O_{xy},

$\alpha_{0xy} : O_x \to O_{xy}$ for each pair O_x, O_{xy},

$\alpha_{1xy} : O_y \to O_{xy}$ for each pair O_y, O_{xy};

their composition is defined by

$$\beta_{xy} \circ \alpha_{0xy} = \alpha_{1yx},\ \beta_{xy} \circ \alpha_{1xy} = \alpha_{0yx},\ \beta_{xy} \circ \beta_{yx} = 1_{0_{yx}}.$$

The category $\mathbf{S}_G$ is the scheme of the diagram $\mathbf{D}_G : \mathbf{S}_G \to \mathbf{K}$ defined by $\mathbf{D}_G(O_x) = E$, $\mathbf{D}_G(O_{xy}) = A$, $\mathbf{D}_G(\beta_{xy}) = b$, $\mathbf{D}_G(\alpha_{0xy}) = a_0$, and $\mathbf{D}_G(\alpha_{1xy}) = a_1$ for all x and all (x, y), respectively. It is clear that $\mathbf{D}_G$ is a well-defined functor. The category $\mathbf{K}$ is cocomplete; set $(\mathbf{L}(G), r) = \operatorname{colim}_{\mathbf{K}}(\mathbf{D}_G)$, where $\mathbf{L}(G)$ is an object of $\mathbf{K}$ and $r : \mathbf{D}_G \to \mathbf{const}_{\mathbf{L}(G)}$ is the colimiting cone. Writing r_x for r_{O_x} and r_{xy} for $r_{O_{xy}}$ one has $r = (r_s : O_s \in \operatorname{Obj}(\mathbf{S}_G))$. Let us note explicitly that $r_{xy} \circ b = r_{yx}$, $r_{xy} \circ a_0 = r_x$, and $r_{xy} \circ a_1 = r_y$ for all x and (x, y), respectively.

Let $f : G \to G' = (X', R')$ be a morphism in $\mathbf{G}$, denote $(\mathbf{L}(G'), r') = \operatorname{colim}_{\mathbf{K}}(\mathbf{D}_{G'})$. The system $(r_{f(x)}' : x \in X) \cup (r_{f(x)f(y)}' : (x, y) \in X^2, \{x, y\} \in R)$ of morphisms of $\mathbf{K}$ is a natural transformation of $\mathbf{D}_G$ into $\mathbf{const}_{\mathbf{L}(G')}$ so that there is a unique morphism $\mathbf{L}(f) : \mathbf{L}(G) \to \mathbf{L}(G')$ such that

(27) $\mathbf{L}(f) \circ r_x = r_{f(x)}'$ for all $x \in X$, and
$\mathbf{L}(f) \circ r_x = r_{f(x)f(y)}'$ for all $(x, y) \in X^2$ with $\{x, y\} \in R$.

Clearly, $\mathbf{L}$ is a functor.

Now let $\varphi : (X, R) \to \mathbf{4}$ be a colouring of (X, R), i.e., let φ be a compatible mapping of the two graphs. Again, the system of morphisms

$$(t_{\varphi(x)\varphi(y)} \circ a_0 : x \in X) \cup (t_{\varphi(x)\varphi(y)} : (x, y) \in X^2, \{x, y\} \in R)$$

is a natural transformation of $\mathbf{D}_G$ into $\mathbf{const}_T$ which gives a unique morphism φ^* of $\mathbf{K}$ such that

(29) $\varphi^* \circ r_x = t_{\varphi(x)\varphi(y)} \circ a_0$ and $\varphi^* \circ r_{xy} = t_{\varphi(x)\varphi(y)}$ for all $x \in X$ and all $(x, y) \in X^2$ with $\{x, y\} \in R$.

The connectedness of (X, R) was used here.

LEMMA 1. *Let (X, R) be a 3-colourable graph. Then*

(a) *$x = y$ if and only if for every colouring $\varphi : (X, R) \to \mathbf{4}$, $\varphi(x) = \varphi(y)$ holds;*

(b) *$\{x, y\} \in R$ if and only if $\varphi(x) \neq \varphi(y)$ for every colouring $\varphi : (X, R) \to \mathbf{4}$.*

The connectedness of a graph G together with (6) imply that every element l of $|\mathbf{L}(G)|$ is of the form $l = |r_{xy}|(z)$ for some $\{x, y\} \in R$ and some $x \in |A|$.

LEMMA 2. *Let $G = (X, R)$ be an object of $\mathbf{G}$. Let $l = |r_{xy}|(z)$ and $l' = |r_{x'y'}|(z')$ be elements of $|L(G)|$ such that $|\varphi^*|(l) = |\varphi^*|(l')$ for every colouring $\varphi : G \to \mathbf{4}$. Let W be a set and let $k : |\ \ | \circ \mathbf{D}_G \to \mathbf{const}_W$ be a natural transformation, $k = (k_s : O_s \in \operatorname{Obj}(\mathbf{S}_G))$. Then $k_{xy}(z) = k_{x'y'}(z')$.*

Proof. Let φ be a colouring of G by three colours; we may assume that $\varphi(X) \subseteq \{0, 1, 2\}$ and that $\varphi(x) = \varphi(x') = 0$ (all the other cases can be reduced to this one by a suitable choice of the representations of l, l' and by an application of a suitable permutation $s \in S_3$). Define a new colouring ψ of G by $\psi(\xi) = \varphi(\xi)$ for all vertices ξ of G different from y', $\psi(y') = 3$. Let us observe that $|t_{0,\psi(y)}|(z) = |\psi^* \circ r_{xy}|(z) = |\psi^*|(l) = |\psi^*|(l') = |\psi^* \circ r_{x'y'}|(z') = |t_{0,\psi(y')}|(z') = |t_{03}|(z')$. There are several cases to consider now. First of all, assume that $y = y'$; in this case $\psi(y) = 3$ and $z = z'$ because $|t_{03}|$ is a one-to-one mapping. If $y \neq y'$, then $\psi(y) \neq \psi(y')$ and (21) yields that $z = z' = |a_0|(e)$ for some $e \in |E|$. These two observations imply that $k_{xy}(z) = k_{x'y'}(z')$ whenever $x = x'$. If $x \neq x'$ and $y = y'$, an argument similar to the previous one and using a colouring $\rho : G \to \mathbf{4}$ with $\rho(x) = \mathbf{0}$ and $\rho(x') = \mathbf{1}$ proves the equality again. The only remaining case is the case of $x \neq x'$ and $y \neq y'$; we know already that $z = z' = |a_0|(e)$. Simultaneously,

$$|t_{01}|(|a_0|(e)) = |t_{0,\rho(y)} \circ a_0|(e) = |\rho^* \circ r_{xy} \circ a_0|(e) = |\rho^*|(l) = |\rho^*|(l')$$
$$= |\rho^* \circ r_{x'y'} \circ a_0|(e) = |t_{1,\rho(y')} \circ a_0|(e) = |t_{10} \circ a_0|(e) = |t_{01}|(|a_1|(e)).$$

Since $|t_{01}|$ is one-to-one, $|a_0|(e) = |a_1|(e)$; (25) yields the existence of a $q \in |Q|$ such that $|c|(q) = e$. G is a connected graph so that there are vertices $x = x_0, x_1, \ldots, x_n = x'$ with $\{x_i, x_{i+1}\} \in R$. Thus

$$k_{xy}(z) = k_{xy}(|a_0|(e)) = k_{xy} \circ |a_0 \circ c|(q) = k_{x_0x_1} \circ |a_0 \circ c|(q)$$
$$= k_{x_0x_1} \circ |a_1 \circ c|(q) = k_{x_1}(|c|(q)) = \ldots = k_{x_n}(|c|(q)) = k_{x'y'}(z').$$

This finishes the proof.

Note that, in particular, $|\varphi^*|(l) = |\varphi^*|(l')$ holding for all colourings φ implies $l = l'$. Since the functor $|\ \ |$ is faithful we conclude

(30) $\varphi^* \circ f = \varphi^* \circ g$ for all colourings $\varphi : G \to \mathbf{4}$ implies that $f = g$.

Also,

(31) $(|\mathbf{L}(G)|, |r|) \cong \mathrm{colim}_{\mathbf{Set}}(|\ \ | \circ \mathbf{D}_G)$.

To prove (31), let $(W, k) = \mathrm{colim}_{\mathbf{Set}}(|\ \ | \circ \mathbf{D}_G)$. There is a unique mapping $h : W \to |\mathbf{L}(G)|$ such that for every $\{x, y\}$ in R, $h \circ k_{xy} = |r_{xy}|$. From (6) we conclude that h is an onto mapping. Let $w = k_{xy}(z)$ and $w' = k_{x'y'}(z')$ be two elements of W such that $h(w) = h(w')$. It follows $|r_{xy}|(z) = |r_{x'y'}|(z')$ and Lemma 2 yields $w = w'$.

The next lemma can be proved easily from (31) with the use of (21) and (25); it describes the functor $\mathbf{L}$ more explicitly.

LEMMA 3. *The composite functor* $|\ \ | \circ \mathbf{L}$ *is naturally equivalent to the functor* $\mathbf{H} : \mathbf{G} \to \mathbf{Set}$ *defined by*

$$\mathbf{H}(X, R) = |Q| \vee (X \times I) \vee (R \times J);$$

https://doi.org/10.4153/CJM-1973-038-3 Published online by Cambridge University Press

if $f : (X, R) \to (X', R')$ *is a morphism of* $\mathbf{G}$ *then* $\mathbf{H}(f)$ *has the form*

$$\mathbf{H}(f)(q) = q \text{ for all } q \in |Q|,$$

$$\mathbf{H}(f)(x, i) = (f(x), i) \text{ for all } (x, i) \in X \times I,$$

$$\mathbf{H}(f)(\{x, y\}, j) = (\{f(x), f(y)\}, j) \text{ for all } (\{x, y\}, j) \text{ in } R \times J.$$

$I = |E| \backslash |c|(|Q|)$ *and* $J = |A| \backslash (|a_0|(|E|) \cup |a_1|(|E|))$.

If $I = \emptyset$, then $|c|$ is onto so that c is an epimorphism; hence $a_0 = a_1$ which contradicts (24). Fix an element i of I and define a mapping

$$\mu_{(X,R)} : |(X, R)|^* \to \mathbf{H}(X, R) \qquad \text{by} \qquad \mu_{(X,R)}(x) = (x, i).$$

The system of all mappings μ_G is a natural transformation from the forgetful functor $|\ \ |^*$ of the category $\mathbf{G}$ to the functor $\mathbf{H}$; all the mappings μ_G are one-to-one. The composite functor $|\ \ | \circ \mathbf{L}$ (which is naturally equivalent to $\mathbf{H}$) is faithful and thus $\mathbf{L}$ is faithful itself. The following statement is needed for the proof of its fulness.

LEMMA 4. *Let* $G = (X, R)$ *be an object of* $\mathbf{G}$. *Then every morphism* $h : A \to \mathbf{L}(G)$ *is of the form* $h = r_{xy}$ *for some* $\{x, y\} \in R$.

Proof. The first fact to be shown is that every morphism $f : P \to \mathbf{L}(G)$ is equal to some $r_x \circ m$ $(x \in X)$.

Let f be such a morphism and let $p \in |P|$. According to (6), $|f|(p) = |r_{xy}|(z)$ for some $z \in |A|$; from (23) there it follows that $\varphi^* \circ f = w_{k(\varphi)}$ for any colouring $\varphi : G \to \mathbf{4}$. Thus

$$|w_{k(\varphi)}|(p) = |\varphi^*|(|f|(p)) = |\varphi^*|(|r_{xy}|(z)) = |t_{\varphi(x)\varphi(y)}|(z)$$

and since $\varphi(x) \neq \varphi(y)$, either $k(\varphi) \neq \varphi(x)$ or $k(\varphi) \neq \varphi(y)$. Without loss of generality it may be assumed that $k(\varphi) \neq \varphi(x)$. Now $|t_{\varphi(x)\varphi(y)}|(z) = |w_{k(\varphi)}|(p) = |t_{\varphi(x)k(\varphi)}|(|a_1| \circ |m|(p))$, so that $z = |a_1 \circ m|(p)$ and $|f|(p) = |r_{xy} \circ a_1 \circ m|(p) = |r_y \circ m|(p)$. We conclude that for every $p \in |P|$ there is a $x = x_p$ such that $|f|(p) = |r_x \circ m|(p)$. According to the considerations at the end of the proof of Lemma 2, $|r_x|(|c|(q)) = |r_y|(|c|(q))$ for arbitrary $x, y \in X$. Assume now that $p, p' \in |P|$ and $|f|(p) = |r_x|(|m|(p))$, $|f|(p') = |r_{x'}|(|m|(p'))$ with $x \neq x'$ and both $|m|(p)$ and $|m|(p')$ lying in I. Choose a colouring ψ of G for which $\psi(x) \neq \psi(x')$ and observe that $|w_{k(\psi)}|(p) = |w_{\psi(x)}|(p)$, $|w_{k(\psi)}|(p') = |w_{\psi(x')}|(p')$ hold for an arbitrary colouring ψ (cf. (29) and (22)). It may be assumed without loss of generality that $k(\psi) \neq \psi(x)$; observing that $|t_{\psi(x)k(\psi)}|(|a_0 \circ m|(p)) = |w_{\psi(x)}|(p) = |w_{k(\psi)}|(p) = |t_{\psi(x)k(\psi)}|(|a_1 \circ m|(p))$ we conclude from (25) that $|m|(p) = |c|(q)$ for some $q \in |Q|$, which is a contradiction. Thus $x = x'$ so there is a unique $x \in X$ such that $|f|(p) = |r_x \circ m|(p)$ for all $p \in |P|$; since $|\ \ |$ is a faithful functor, $f = r_x \circ m$.

Let $h : A \to \mathbf{L}(G)$ be a morphism in $\mathbf{K}$. The composite morphisms $h \circ v_0$ and $h \circ v_1$ are equal to some $r_x \circ m$ and $r_{x'} \circ m$, respectively. By (18), $\varphi^* \circ h = t_{i(\varphi)j(\varphi)}$ for an arbitrary colouring φ. Hence $w_{i(\varphi)} = t_{i(\varphi)j(\varphi)} \circ v_0 = \varphi^* \circ h \circ v_0 = \varphi^* \circ r_x \circ m = t_{\varphi(x),k} \circ a_0 \circ m = t_{\varphi(x),k} \circ v_0 = w_{\varphi(x)}$, and, analogously $w_{j(\varphi)} = w_{\varphi(x')}$; (23) yields $i(\varphi) = \varphi(x)$ and $j(\varphi) = \varphi(x')$. In particular, $\varphi(x) \neq \varphi(x')$ for every colouring φ. Lemma 1 gives $\{x, x'\} \in R$; for an arbitrary colouring φ, $\varphi^* \circ h = t_{i(\varphi)j(\varphi)} = t_{\varphi(x)\varphi(x')} = \varphi^* \circ r_{xx'}$ and from (30) we conclude that $h = r_{xx'}$.

To prove the fulness of $\mathbf{L}$, let $k : \mathbf{L}(X, R) \to \mathbf{L}(X', R')$ be a morphism in $\mathbf{K}$; by Lemma 4, $k \circ r_{xy} = r_{x'y'}'$ for any $\{x, y\} \in R$. If $r_{x'y'}' = r_{x''y''}'$, then $t_{\varphi(x')\varphi(y')} = \varphi^* \circ r_{x'y'}' = \varphi^* \circ r_{x''y''}' = t_{\varphi(x'')\varphi(y'')}$; (18) and Lemma 1 yield $x' = x''$, $y' = y''$. Define a mapping $f : X \to X'$ by $k \circ r_x = k \circ r_{xy} \circ a_0 = r_{x'y'}' \circ a_0 = r_{f(x)}'$; since G is connected, f is a well-defined mapping. Now $k \circ r_{xy} = r_{f(x)f(y)}' = \mathbf{L}(f) \circ r_{xy}$ for all $\{x, y\} \in R$; therefore $f \in \mathbf{G}$ and $k = \mathbf{L}(f)$.

The functor $\mathbf{L}$ is full and faithful, but we need a functor which is also one-to-one on the class of all objects of $\mathbf{G}$. The existence of a full embedding is guaranteed by the following modification of Lemma 5 of [**9**].

LEMMA 5 (A. Pultr). *Let* $\mathbf{G}$ *be equipped by the standard forgetful functor* $|\ \ |^*$ *and let* $(\mathbf{K}, |\ \ |)$ *satisfy* (3). *If* $\mathbf{L} : \mathbf{G} \to \mathbf{K}$ *is a full functor and if there is a monotransformation* $\rho : |\ \ |^* \to |\ \ | \circ \mathbf{L}$, *then there is a full embedding* $\mathbf{L}^* : \mathbf{G} \to \mathbf{K}$ *naturally equivalent to* $\mathbf{L}$.

The existence of a monotransformation ρ follows from the existence of the monotransformation $\mu : |\ \ |^* \to \mathbf{H}$ described earlier. The colimit-defined functor $\mathbf{L}$ determines a full embedding $\mathbf{L}^*$ up to a natural equivalence.

It is shown in [**4**] that every small category can be fully embedded into a binding category; this enables us to state the main result of the present paper.

THEOREM 1. *The following two conditions are equivalent for any category* $(\mathbf{K}, |\ \ |)$ *satisfying* (0)–(6):

(i) *The finite category* $\mathbf{C}$ *is isomorphic to a full subcategory of* $\mathbf{K}$.

(ii) *Every full category of algebras is isomorphic to a full subcategory of* $\mathbf{K}$.

In other words, the universality of such a category $(\mathbf{K}, |\ \ |)$ is tested by the full representability of $\mathbf{C}$ in $\mathbf{K}$.

4. Binding and strongly binding categories. Many binding equational classes of algebras are also strongly binding. Among the strongly binding classes are the class of commutative groupoids [**11**], the class of semigroups [**10**] and [**13**], and many equational classes of unary algebras. A class $\mathbf{A}(\Delta)$ of all algebras of the type Δ is binding if and only if it is strongly binding. In this section we will show that "binding" has the same meaning as "strongly binding" for any category $(\mathbf{K}, |\ \ |)$ satisfying (0)–(6); in particular, these two concepts are equivalent for any equational class of unary algebras. The general problem of equational classes of algebras of an arbitrary type remains unsolved; there are binding equational classes (e.g. the equational class of bounded lattices [**2**] and the class of commutative rings with unit [**1**]) for which no proof of their strong universality is known. On the other hand, no equational class is known to be universal but not strongly so.

To prove the equivalence of these two concepts for our type of category we will construct a special embedding of a strongly binding category of algebras with two unary fundamental operations into the category **G** and follow it by the full embedding $\mathbf{L}^*$ of **G** into a binding category **K**; the composite functor will turn out to be a strong embedding of a very special type (cf. Theorem 3 below). This will prove the equivalence since the converse implication is trivial.

The category **A** of all connected algebras with two unary fundamental operations is strongly binding (for the proof, cf. [**12**]). A unary algebra is *connected* if the union of the graphs of its fundamental operations is a connected directed graph, that is a graph whose symmetrization is a connected undirected graph according to the definition in the previous section. A *directed graph* is a pair (X, R) in which X is a set and $R \subseteq X^2$. A mapping $f : X \to X'$ is a morphism of the category **D** of all irreflexive connected directed graphs from (X, R) to (X', R') if $(f(x), f(y)) \in R'$ whenever $(x, y) \in R$.

Given a set S, define functors $\mathbf{V}_S$, $\mathbf{K}_S$, $\mathbf{Q}_S$ from the category **Set** of all sets and mappings into itself by $\mathbf{V}_S(X) = X \vee S$, $\mathbf{V}_S(f)(s) = s$ for $s \in S$ and $\mathbf{V}_S(f)(x) = f(x)$; $\mathbf{K}_S(X) = X \times S$, $\mathbf{K}_S(f)(x, s) = (f(x), s)$; $\mathbf{Q}_S$ $(S \neq \emptyset)$ is the covariant Hom-functor $\mathbf{Hom}(S, -)$.

First, a full embedding $\Phi : \mathbf{A} \to \mathbf{D}$ will be constructed.

Let $(X; a, b)$ be a connected algebra with two unary fundamental operations $a, b : X \to X$. Set $\Phi(X; a, b) = (X \times 5, R)$, where

$$R = \{((x, i), (x, i + 1)) : i \in 4, x \in X\} \cup \{((x, 0), (a(x), 2)) : x \in X\} \cup \{((x, 2), (b(x), 4)) : x \in X\}.$$

If $f : (X; a, b) \to (X'; a', b')$ is a homomorphism of the two algebras, define $\Phi(f)(x, i) = (f(x), i)$ for all $i \in 4$. It is easy to see that Φ is a one-to-one functor from **A** to **D**. To prove its fulness, consider an arbitrary morphism $h : \Phi(X; a, b) \to \Phi(X'; a', b')$ in **D**. The only vertices z for which both (z, z') and (z'', z) belong to R for some z', z'' are of the form (x, i) with $i = 1, 2, 3$; thus $h(X \times \{1, 2, 3\})$ is contained in $X' \times \{1, 2, 3\}$. If $h(x, 2) = (x', 3)$, then $h(x, 3) = (x', 4)$ which is impossible; similarly $h(x, 2) \neq (x', 1)$. Hence $h(x, 2) = (x', 2)$ and this, in turn, implies that $h(x, i) = (x', i)$ for all $i \in 5$. There is a mapping $f : X \to X'$ such that $(f(x), i) = h(x, i)$ for all these i and for all $x \in X$. $((x, 0), (a(x), 2)) \in R$ for every $x \in X$; since h is a morphism in D, $(h(x, 0), h(a(x), 2)) \in R'$. But $h(x, 0) = (f(x), 0)$ and $h(a(x), 2) = (f(a(x)), 2)$ so that $f(a(x)) = a(f(x))$. The compatibility of f with the second fundamental operation is established in a similar way; $h = \Phi(f)$ and f is a morphism of **A**.

Observe that R is naturally bijective to $X \times 6$ with respect to all morphisms of the graphs of the form $\Phi(X; a, b)$.

The full embedding of the category of all directed irreflexive graphs into the category of all 3-colourable undirected graphs which is constructed in [**5**] carries connected graphs to connected undirected graphs without diagonal; hence its restriction Ψ to **D** is a full embedding of **D** into **G**. The set Z of vertices of a graph $(Z, T) = \Psi(Y, S)$ is naturally bijective to the set $Y \vee (S \times 11)$ and the set T of its edges is bijective naturally to the set $S \times 15$. The naturality is again with respect to all morphisms between the graphs in question. Consider the composite functor $\Psi \circ \Phi : \mathbf{A} \to \mathbf{G}$ now; the undirected graph $(Z, T) = (\Psi \circ \Phi)(X; a, b)$ has Z bijective naturally to the set $X \times 71 \cong (X \times 5) \vee (X \times 66)$ and T is naturally bijective to $X \times 90 \cong (X \times 6) \times 15$.

If the category $(\mathbf{K}, |\ \ |)$ is binding, then the small category **C** described in the first section is fully representable in **K**; let (26) be its modified representation and let $\mathbf{L}^*$ be the full embedding constructed above. The composite functor $\mathbf{L}^* \circ \Psi \circ \Phi$ is a full embedding of **A** into **K**. The functor $|\ \ | \circ \mathbf{L}^*$ is naturally equivalent to **H** (see Lemma 3 and Lemma 5). Let **U** be the standard underlying-set functor of the category **A** of unary algebras. Taking into account the naturality properties of Φ and Ψ we conclude that the functor $|\ \ | \circ \mathbf{L}^* \circ \Psi \circ \Phi$ is naturally equivalent to the functor $\mathbf{V}_{|Q|} \circ \mathbf{K}_B \circ \mathbf{U}$ with $B = (I \times 71) \vee (J \times 90)$; I, J, and $|Q|$ are the same as in Lemma 3. Consequently, there is a functor $\mathbf{F} : \mathbf{Set} \to \mathbf{Set}$ satisfying $\mathbf{F} \circ \mathbf{U} = |\ \ | \circ \mathbf{L}^* \circ \Psi \circ \Phi$; **F** is naturally equivalent to $\mathbf{V}_{|Q|} \circ \mathbf{K}_B$. The full embedding $\mathbf{L}^* \circ \Psi \circ \Phi$ is strong; since **A** is a strongly binding category, so is $(\mathbf{K}, |\ \ |)$. This concludes the proof of our second theorem.

THEOREM 2. *A category* $(\mathbf{K}, |\ \ |)$ *satisfying* (0)–(6) *is binding if and only if it is strongly binding.*

A natural question to ask now is how complicated are the set-functors carrying the strong embeddings of full categories of algebras into equational classes of unary algebras or, how simple these functors can really be. The next theorem shows that a very narrow class of functors is completely sufficient.

THEOREM 3. *Let* $(\mathbf{K}, |\ \ |)$ *be a binding category satisfying the conditions* (0)–(6). *Then*

(I) *for every full category* **N** *of algebras there are sets* A, B, *and* C *and a strong embedding of* **N** *into* $(\mathbf{K}, |\ \ |)$ *over a set-functor naturally equivalent to* $\mathbf{V}_A \circ \mathbf{K}_B \circ \mathbf{Q}_C$;

(II) *in the case of a full category* **N** *of unary algebras the strong embedding is carried by a functor naturally equivalent to some* $\mathbf{V}_A \circ \mathbf{K}_B$.

Theorem 3 will be proved in several steps: first of all, a strong embedding of the category $\mathbf{A}(\Delta)$ of all algebras of the type Δ into a suitable category of algebras with many unary operations will be constructed over a set-functor $\mathbf{Q}_C$ with sufficiently large C. This construction is a very standard one and it is included here for the sake of completeness only. The second step is a strong embedding of this category into the category $\mathbf{A}(1, 1, 1)$ of all algebras with three fundamental unary operations over a functor $\mathbf{K}_{B'}$. $\mathbf{A}(1, 1, 1)$ is in turn being embedded strongly into $\mathbf{A}(1, 1)$ and this strong embedding is carried by the functor $\mathbf{K}_5$; Theorem 1 of [**12**] describes a strong embedding $\mathbf{R}$ of $\mathbf{A}(1, 1)$ into the category $\mathbf{A}$ of all connected algebras with two fundamental unary operations whose carrying functor is $\mathbf{V}_2 \circ \mathbf{K}_7$. Composing all these embeddings one obtains a strong embedding $\Sigma_\Delta : \mathbf{A}(\Delta) \to \mathbf{A}$ carried by a set-functor naturally equivalent to some $\mathbf{V}_{A^*} \circ \mathbf{K}_{B^*} \circ \mathbf{Q}_C$, since

$$\mathbf{V}_D \circ \mathbf{K}_C \circ \mathbf{V}_B \circ \mathbf{K}_A$$

is naturally equivalent to $\mathbf{V}_{(B\times C)\vee D} \circ \mathbf{K}_{A\times C}$. Applying the strong embedding of $\mathbf{A}$ into $\mathbf{K}$ we obtain the desired result (recall that the last embedding is again carried by a functor of the form $\mathbf{V}_{A'} \circ \mathbf{K}_{B'}$ and apply the above natural equivalence).

A *type* Δ is a sequence of ordinal numbers indexed by an ordinal number, i.e., $\Delta = (k_a : a \in b)$, where a, b, and k_a are ordinal numbers. The objects of the category $\mathbf{A}(\Delta)$ are all pairs $(X, (o_a : a \in b))$ in which X is a set and $o_a : X^{k_a} \to X$ is a mapping for every $a \in b$. Morphisms are all mappings $f : X \to X'$ such that $o_a'(f \circ \varphi) = f(o_a(\varphi))$ for all $\varphi \in X^{k_a}$. $\mathbf{A}(\Delta)$ is equipped with the standard underlying-set functor as the forgetful one; every full category of algebras is a full subcategory of some $\mathbf{A}(\Delta)$.

The following three lemmas describe the strong embeddings required for the proof of Theorem 3. The symbol $+$ means the ordinal sum.

LEMMA 6. *Let $\Delta = (k_a : a \in b)$ be a type and let $k = \sup\{k_a : a \in b\}$. Set $\Delta' = (1_c : c \in b + k)$. There is a strong embedding*

$$\mathbf{G} : \mathbf{A}(\Delta) \to \mathbf{A}(\Delta')$$

carried by the functor $\mathbf{Q}_k$.

Proof. Observe that the replacement of every nullary operation o_a by a constant unary operation with the value o_a defines a strong embedding of $\mathbf{A}(\Delta)$ into a category of algebras without nullary operations; the embedding is carried by the identity functor. Thus we may assume that $k_a > 0$ for all $a \in b$. For an algebra $(X, (o_a : a \in b)) \in \mathbf{A}(\Delta)$ set $G(X, (o_a : a \in b)) = (X^k, (o_c^* : c \in b + k))$ where the unary operations $o_c^* : X^k \to X^k$ are defined as

$o_c^*(\varphi)(m) = o_c(\psi)$ for every $m \in k$, $c \in b$; ψ is the restriction of φ to k_c,
$o_c(\varphi)(m) = \varphi(c)$ for every $m \in k$ and $c \notin b$.

The unary operations of the second group are all the projections of X^k onto its diagonal; it is easy to see that the mappings compatible with all of them are exactly all the mappings of the form $\mathbf{Q}_k(f)$; $\mathbf{Q}_k(f)$ is compatible with all the operations of the first group if and only if f is a homomorphism in $\mathbf{A}(\Delta)$; this shows that $\mathbf{G}$ is a full embedding.

LEMMA 7. *Let* $\Delta = (1_a : a \in k)$ *be a type. For every set* B *with* $\text{card}(B) \geqq \text{card}(k)$ *there is a strong embedding* $\mathbf{S} : \mathbf{A}(\Delta) \to \mathbf{A}(1, 1, 1)$ *carried by the functor* $\mathbf{K}_B$.

Proof. Combining results of [**14**; **4**; **12**] it is easy to show that there is a rigid connected algebra with two fundamental unary operations of any infinite cardinality (an object R of a category $\mathbf{K}$ is called *rigid* if $\mathbf{Hom}_{\mathbf{K}}(R, R) = \{1_R\}$). It is not difficult to construct such an algebra on a finite set: the symmetric group S_n ($n \geqq 3$) has two generators and no nontrivial mapping commutes with all the permutations on an n-element set.

Let $B^* = (B; \bar{c}, \bar{d})$ be a rigid connected algebra with unary operations $\bar{c}, \bar{d}$. There is a one-to-one mapping of k into the set B so that the operational symbols of the class $\mathbf{A}(\Delta)$ can be indexed by the elements of B. Let $(X, (o_b : b \in B))$ be an algebra from $\mathbf{A}(\Delta)$. Define

$$\mathbf{S}(X, (o_b : b \in B)) = (X \times B; c, d, e)$$

where

$$c(x, b) = (x, \bar{c}(b)),\ d(x, b) = (x, \bar{d}(b)),\ e(x, b) = (o_b(x), b)$$
$$\text{for all } (x, b) \in X \times B.$$

Obviously, $\mathbf{S}$ is a well-defined functor which is one-to-one both on objects and on morphisms ($\mathbf{S}(f) = \mathbf{K}_B(f)$ for every morphism f of $\mathbf{A}(\Delta)$). Now let h be a homomorphism from $\mathbf{S}(X, (o_b : b \in B))$ into $\mathbf{S}(X', (o_b' : b \in B))$. Note that $(\{x\} \times B; c_x, d_x)$ (where c_x and d_x are the restrictions of c and d, respectively, to $\{x\} \times B$) is an algebra isomorphic to B^*. Any homomorphism sends a connected part of the algebra $(X \times B; c, d)$ into a connected part of $(X' \times B; c', d')$; the maximal connected parts of these algebras are, however, isomorphic to the rigid algebra B^* and we conclude from this the existence of a mapping $f : X \to X'$ such that $\mathbf{K}_B(f) = h$. The compatibility of h with the third operation implies that f is a morphism in $\mathbf{A}(\Delta)$. This shows the fulness of $\mathbf{S}$.

LEMMA 8. $\mathbf{A}(1, 1, 1)$ *can be strongly embedded into* $\mathbf{A}(1, 1)$ *over the set-functor* $\mathbf{K}_5$.

Proof. For an algebra $(X; c, d, e)$ in $\mathbf{A}(1, 1, 1)$ set $\mathbf{T}(X; c, d, e) = (X \times 5; a, b)$ with a, b described by the formulae:

$a(x, i) = (x, i + 1)$ for all $x \in X$ and $i \in 5$; the addition is modulo 5,
$b(x, i) = (x, 1 - i)$ for all $x \in X$ and $i = 0, 1$,
$b(x, 2) = (c(x), 4)$, $b(x, 3) = (d(x), 2)$, $b(x, 4) = (e(x), 3)$ for all $x \in X$.

Define $\mathbf{T}(f) = \mathbf{K}_5(f)$ for every morphism f of $\mathbf{A}(1, 1, 1)$; $\mathbf{T}$ is a one-to-one functor. Let $h : \mathbf{T}(X; c, d, e) \to \mathbf{T}(X'; c', d', e')$ be a morphism of $\mathbf{A}(1, 1)$. Since $a(z) = b(z)$ if and only if $z \in X \times \{0\}$, $a'(h(x, 0)) = h(a(x, 0)) = h(b(x, 0)) = b'(h(x, 0))$ implies the existence of a mapping $f : X \to X'$ such that $h(x, 0) = (f(x), 0)$. Furthermore,

$$h(x, i) = h(a^i(x, 0)) = (a')^i(f(x), 0) = (f(x), i)$$

for all i; hence $h = \mathbf{K}_5(f)$. Now, $(fc(x), 4) = h(c(x), 4) = h(b(x, 2)) = b'(h(x, 2)) = b'(f(x), 2) = (c'f(x), 4)$; we establish similarly that f is compatible with the other two operations.

To finish the proof of Theorem 3, recall that the strong embedding $\Theta = \mathbf{L}^* \circ \Psi \circ \Phi : \mathbf{A} \to (\mathbf{K}, |\ \ |)$ is carried by a set-functor $\mathbf{V}_{A'} \circ \mathbf{K}_{B'}$ (with A', B' dependent on the representation of the testing category used to construct $\mathbf{L}^*$) and define

$$\Sigma_\Delta = \mathbf{R} \circ \mathbf{T} \circ \mathbf{S} \circ \mathbf{G} : \mathbf{A}(\Delta) \to \mathbf{A};$$

Σ_Δ is a strong embedding over the set-functor $(\mathbf{V}_2 \circ \mathbf{K}_5) \circ \mathbf{K}_{B(\Delta)} \circ \mathbf{Q}_{\sup(\Delta)}$. $\Theta \circ \Sigma_\Delta$ is the desired strong embedding.

5. A problem. Let us emphasize once more that the condition (6) imposes a strong restriction on the range of the categories admissible for testing by the presented method; all the equational classes not equivalent to equational classes of unary algebras are excluded. Is there a small category testing the universality of an arbitrary equational class of (finitary) algebras?

References

1. E. Fried and J. Sichler, *Homomorphisms of commutative rings with unit element* (to appear).
2. G. Grätzer and J. Sichler, *On the endomorphism semigroup (and category) of bounded lattices*, Pacific J. Math. *35* (1970), 639–647.
3. Z. Hedrlín and J. Lambek, *How comprehensive is the category of semigroups*, J. Algebra *11* (1969), 195–212.
4. Z. Hedrlín and A. Pultr, *On full embeddings of categories of algebras*, Illinois J. Math. *10* (1966), 392–406.
5. ——— *Relations (graphs) with given infinite semigroups*, Monatsh. Math. *68* (1964), 421–425.
6. Z. Hedrlín and J. Sichler, *Any boundable binding category contains a proper class of mutually disjoint copies of itself*, Algebra Universalis *1* (1971), 97–103.
7. J. R. Isbell, *Algebras of uniformly continuous functions*, Ann. of Math. *68* (1958), 96–125.
8. A. Pultr, *Eine Bemerkung über volle Einbettungen von Kategorien von Algebren*, Math. Ann. *178* (1968), 78–82.
9. ——— *On full embeddings of concrete categories with respect to forgetful functors*, Comment. Math. Univ. Carolinae *9* (1968), 281–304.
10. J. Sichler, *A(1, 1) can be strongly embedded into the category of semigroups*, Comment. Math. Univ. Carolinae *9* (1968), 257–262.
11. ——— *The category of commutative groupoids is binding*, Comment. Math. Univ. Carolinae *8* (1967), 753–755.

12. ——— *Concerning minimal primitive classes of algebras containing any category of algebras as a full subcategory*, Comment. Math. Univ. Carolinae *9* (1968), 627–635.

13. V. Trnková, *Strong embedding of category of all groupoids into category of semigroups*, Comment. Math. Univ. Carolinae *9* (1968), 251–256.

14. P. Vopěnka, A. Pultr, and Z. Hedrlín, *A rigid relation exists on any set*, Comment. Math. Univ. Carolinae *6* (1965), 149–155.

University of Manitoba,
Winnipeg, Manitoba

https://doi.org/10.4153/CJM-1973-038-3 Published online by Cambridge University Press