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GYROSCOPICALLY STABILIZED SYSTEMS: A CLASS OF QUADRATIC EIGENVALUE PROBLEMS WITH REAL SPECTRUM

Dedicated to the memory of M. G. Krein.

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ABSTRACT. Eigenvalue problems for selfadjoint quadratic operator polynomials $L(\lambda) = I\lambda^2 + B\lambda + C$ on a Hilbert space H are considered where $B, C \in \mathcal{L}(H)$, $C > 0$, and $|B| \geq kI + k^{-1}C$ for some $k > 0$. It is shown that the spectrum of $L(\lambda)$ is real. The distribution of eigenvalues on the real line and other spectral properties are also discussed. The arguments rely on the well-known theory of (weakly) hyperbolic operator polynomials.

1. **Introduction.** Let H be a Hilbert space with inner product $(\cdot,\cdot)$, and let $\mathcal{L}(H)$ be the algebra of bounded linear operators on H. Let $A, B, C \in \mathcal{L}(H)$ and be selfadjoint with $A \gg 0$ (i.e. A is uniformly positive; $A \geq \delta I$ for some $\delta > 0$ and I the identity in $\mathcal{L}(H)$). Here and subsequently it is convenient to use several notations, terminologies and results from the monograph [12].

Define the quadratic operator polynomial $L(\lambda) = A\lambda^2 + B\lambda + C$ and note that when $\lambda \in \mathbb{R}$, $L(\lambda)^* = L(\lambda)$. There will be no loss of generality if it is assumed from the beginning that $L(\lambda)$ is monic (i.e. that $A = I$) and take

$$L(\lambda) = I\lambda^2 + B\lambda + C. \tag{1.1}$$

It will also be assumed that $C > 0$. If also $B > 0$ or $B < 0$, and is sufficiently large, in the sense that, for all nonzero $x \in H$

$$(Bx, x)^2 > 4(x, x)(Cx, x), \tag{1.2}$$

then the spectrum of $L(\lambda)$, say $\sigma\big(L(\lambda)\big)$ (see Section 2 for definitions), is real and all eigenvalues (if any) are semisimple. In this case the system is said to be *hyperbolic* (see p. 169 and Lemma 31.23 of [12]). Such systems originated in the work of Duffin in 1955, [3], on finite-dimensional problems and he called them *overdamped.* Problems of this kind in a general Hilbert space setting were studied by Krein and Langer in the early

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1960's ([8] and [9]) and their results led to several generalizations, some of which appear in [12] (see also Chapter 11 of [4]).

The purpose of this paper is to investigate pencils of the form (1.1) with $C > 0$ and B invertible and *indefinite* (see Section 2). It will be seen that, once more, if B is large compared to I and C in a suitable sense, then $\sigma\big(L(\lambda)\big) \subset \mathbb{R}$ and all eigenvalues are semisimple. In this case we require

$$|B| > kI + k^{-1}C \tag{1.3}$$

for some $k > 0$, where $|B|$ denotes the positive square root of B^2.

The motivation for this work came from problems arising in engineering for which $B = iG$ and $G^* = -G$. In this case the spectral properties of $L(\lambda)$ determine the motions of a conservative, time-invariant, linear system oscillating about a position of *unstable* equilibrium under the action of gyroscopic forces. Indeed, corresponding homogeneous equations of motion have the form

$$\ddot{u}(t) + G\dot{u}(t) - Cu(t) = 0. \tag{1.4}$$

Making the substitution $u(t) = x\exp(\mu t)$, with x independent of t, and then the rotation of parameter $\lambda = -i\mu$ leads to the eigenvalue problem $L(\lambda)x = 0$ where $L(\lambda) = I\lambda^2 + (iG)\lambda + C$ and, of course, $B = iG$ is indefinite.

The origins of the problem are discussed, and several results announced in the paper [1]. Here we give the proofs and elaborate the theory in some interesting ways. In view of the physical origins of our problem $L(\lambda)$ of (1.1) will be said to be *gyroscopically stabilized*, or a *GS system* if $C > 0$, B is invertible and indefinite, and (1.3) is satisfied.

At first sight conditions (1.2) and (1.3) concerning the magnitudes of B relative to I and C seem to have little in common. The connection between them is revealed, however, when it is observed that $L(\lambda)$ is hyperbolic and $B > 0$ if and only if $L(-k) > 0$ for some $k > 0$ (see Section 31 of [12]). Using (1.1) this is equivalent to

$$B > kI + k^{-1}C. \tag{1.5}$$

Comparison of (1.3) with (1.5) reveals the essential differences between the coefficients of hyperbolic and GS systems.

For the stability of (1.4) it is necessary (and sufficient when $\dim H < \infty$) that all associated eigenvalues $\mu (= i\lambda)$ be pure imaginary and semisimple. Then all solutions generated by the eigenvalues are spanned by functions of the form $x\exp(\mu t)$ and are bounded for all real t. It will be shown that when (1.3) is relaxed to

$$|B| \geq kI + k^{-1}C \tag{1.6}$$

for some $k > 0$, then all eigenvalues of μ are pure imaginary but not necessarily semisimple. Accordingly $L(\lambda)$ is said to be *almost gyroscopically stabilized,* or an *AGS system* when $C > 0$, B is invertible and indefinite, and (1.6) is satisfied.

The analogue in the theory of damped oscillators is the *weakly hyperbolic* condition, i.e. with $C > 0$,

$$(Bx, x)^2 \geq 4(x, x)(Cx, x), \tag{1.7}$$

for all $x \in H$. (Note this also implies $B > 0$ or $B < 0$.) Or, when $B > 0$,

$$B \geq kI + k^{-1}C \tag{1.8}$$

for some $k > 0$.

Note that the condition

$$B^2 > (kI + k^{-1}C)^2 \tag{1.9}$$

for some $k > 0$ implies (1.3), but not conversely. Although (1.9) is stronger than (1.3) it may be a computationally convenient sufficient condition for a GS system.

2. **Preliminaries.** For a quadratic operator polynomial $L(\lambda) = I\lambda^2 + B\lambda + C$ where $I, B, C \in \mathcal{L}(H)$, the *spectrum* of $L(\lambda)$ is the set of $\lambda_0 \in \mathbb{C}$ for which $L(\lambda_0)^{-1} \notin \mathcal{L}(H)$ and is denoted by $\sigma\big(L(\lambda)\big)$, or $\sigma(L)$. When B and C are selfadjoint $\sigma(L)$ is symmetric with respect to the real axis of the complex plane. If λ_0 has the property that $\operatorname{Ker} L(\lambda_0) \neq \{0\}$ then λ_0 is called an *eigenvalue* of L, and any nonzero $x \in H$ for which $L(\lambda_0)x = 0$ is an *eigenvector* of L corresponding to eigenvalue λ_0. Clearly, eigenvalues belong to $\sigma(L)$.

The *multiplicity* of an eigenvalue λ_0 is the number of vectors in a canonical system of Jordan chains for $L(\lambda)$ at λ_0 and is denoted by $m(\lambda_0)$. (See p. 57 of [12], or [6], for example, where Jordan chains are described as chains of eigenvectors and associated vectors). Note that, if a real eigenvalue of λ_0 is an isolated *Fredholm point* of the selfadjoint polynomial $L(\lambda)$ (i.e. $\dim \operatorname{Ker} L(\lambda_0) < \infty$ and $\operatorname{Im} L(\lambda_0)$ is closed) then we have $m(\lambda_0) < \infty$ (see [11], for example).

An eigenvalue λ_0 of $L(\lambda)$ is said to be *semisimple* if it has no generalized eigenvectors (associated vectors), i.e. for any corresponding eigenvector x the equation $L(\lambda_0)y = -L'(\lambda_0)x$ where $L'(\lambda) = 2I\lambda + B$, has no solution. If $\dim \operatorname{Ker} L(\lambda_0) < \infty$ then λ_0 is semisimple if and only if $m(\lambda_0) = \dim \operatorname{Ker} L(\lambda_0)$.

A real eigenvalue λ_0 of $L(\lambda)\big(= L^*(\lambda)\big)$ is said to be of *positive,* or *negative type* if $\big(L'(\lambda_0)x, x\big) > 0$, or $\big(L'(\lambda_0)x, x\big) < 0$, respectively, for all nonzero $x \in \operatorname{Ker} L(\lambda_0)$. A real eigenvalue is said to be of *mixed type* if it is not of positive or negative type. (In reference [13], the three types are described as "plus", "minus", and "neutral", respectively.)

LEMMA 2.1. *Let $L(\lambda) = I\lambda^2 + B\lambda + C$ where $B^* = B$ and $C^* = C$.*

(a) If a real eigenvalue of L is not semisimple then it is of mixed type.

(b) An eigenvalue $\lambda_0 \in \mathbb{R}$ is of mixed type if and only if there is an eigenvector x corresponding to λ_0 such that $\big(L'(\lambda_0)x, x\big) = 0$.

PROOF. (a) If λ_0 is not semisimple then there is a corresponding generalized eigenvector. Thus, there is an eigenvector x corresponding to λ_0 and a $y \in H$ for which

$$L(\lambda_0)y + L'(\lambda_0)x = 0.$$

Then, since $\lambda_0 \in \mathbb{R}$,

$$\begin{aligned}\big(L'(\lambda_0)x, x\big) &= \big(y, L(\lambda_0)x\big) + \big(L'(\lambda_0)x, x\big)\\ &= \big(L(\lambda_0)y + L'(\lambda_0)x, x\big) = 0.\end{aligned}$$

Hence λ_0 is of mixed type.

(b) If λ_0 is of mixed type then there are corresponding eigenvectors y and z for which

$$\big(L'(\lambda_0)y, y\big) \geq 0, \quad \big(L'(\lambda_0)z, z\big) \leq 0.$$

It is easily seen that there is a linear combination x ($\neq 0$) of y and z for which $(L'(\lambda_0)x, x) = 0$. ■

Now let $B \in \mathcal{L}(H)$ with $B^* = B$ and $B^{-1} \in \mathcal{L}(H)$. It is well-known that there are mutually orthogonal, B-invariant subspaces H_1 and H_2 of H such that $H = H_1 \oplus H_2$ and (when H_1, H_2 are not trivial) $B|_{H_1} \gg 0$ and $B|_{H_2} \ll 0$. We say that B is *indefinite* when $H_1 \neq \{0\}$ and $H_2 \neq \{0\}$.

Notice that in the context of GS systems, when $H_2 = \{0\}$ then conditions (1.3) and (1.5) coincide and the pencil is of the well-understood hyperbolic type. Similarly for $H_1 = \{0\}$.

3. **The main theorems.**

THEOREM 3.1. *The spectrum of an AGS system is real.*

PROOF. Let $L(\lambda) = I\lambda^2 + B\lambda + C$ be an AGS system.

Step 1. We first prove that any eigenvalue of $L(\lambda)$ is real. Using the decomposition $H = H_1 \oplus H_2$ of Section 2 let the corresponding representations of B and C be

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & -B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix}$$

where $B_1 \gg 0$, $B_2 \gg 0$. Then we may write

$$L(\lambda) = \begin{bmatrix} I\lambda^2 + B_1\lambda + C_{11} & C_{12} \\ C_{12}^* & I\lambda^2 - B_2\lambda + C_{22} \end{bmatrix} \tag{3.1}$$

and

$$|B| = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

the unique positive definite square root of B^2. Then (1.6) implies

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \geq \begin{bmatrix} kI + k^{-1}C_{11} & C_{12} \\ C_{12}^* & kI + k^{-1}C_{22} \end{bmatrix} \tag{3.2}$$

Let λ be an eigenvalue of $L(\lambda)$ with eigenvector x. Write $x = y_1 + y_2$, $y_1 \in H_1$, $y_2 \in H_2$ and we have

$$(I\lambda^2 + B_1\lambda + C_{11})y_1 + C_{12}y_2 = 0, \tag{3.3}$$

$$C_{12}^*y_1 + (I\lambda^2 - B_2\lambda + C_{22})y_2 = 0. \tag{3.4}$$

Let

$$L_1(\lambda) = I\lambda^2 + B_1\lambda + C_{11}, \quad L_2(\lambda) = I\lambda^2 - B_2\lambda + C_{22}, \tag{3.5}$$

and it follows from (3.2) (see also (1.8)) that $L_1(\lambda)$ and $L_2(\lambda)$ are weakly hyperbolic.

Taking inner products of (3.3) and (3.4) with y_1 and y_2, respectively, and defining $\beta = (Cy_2, y_1)$,

$$\alpha_j = \|y_j\|^2, \quad c_j = (Cy_j, y_j), \quad b_j = (B_j y_j, y_j)$$

for $j = 1, 2$, we obtain

$$\begin{aligned} (\alpha_1\lambda^2 + b_1\lambda + c_1) + \beta &= 0 \\ \bar{\beta} + (\alpha_2\lambda^2 - b_2\lambda + c_2) &= 0 \end{aligned}. \tag{3.6}$$

Since $y \neq 0$ we cannot have both $y_1 = 0$ and $y_2 = 0$. If one of y_1 or y_2 is zero, equations (3.6) reduce to a single quadratic equation with real zeros so that $\lambda \in \mathbb{R}$. Hence, we may assume that $y_1 \neq 0$ and $y_2 \neq 0$.

If we define

$$\rho(\lambda, \beta) = (\alpha_1\lambda^2 + b_1\lambda + c_1)(\alpha_2\lambda^2 - b_2\lambda + c_2) - |\beta|^2, \tag{3.7}$$

the eigenvalue λ is a root of the real polynomial equation $\rho(\lambda, \beta) = 0$.

Since $L_1(\lambda)$ and $L_2(\lambda)$ are weakly hyperbolic $\rho(\lambda, 0)$ will have only real zeros with the possibility of either one or two double zeros (see Figure 1).

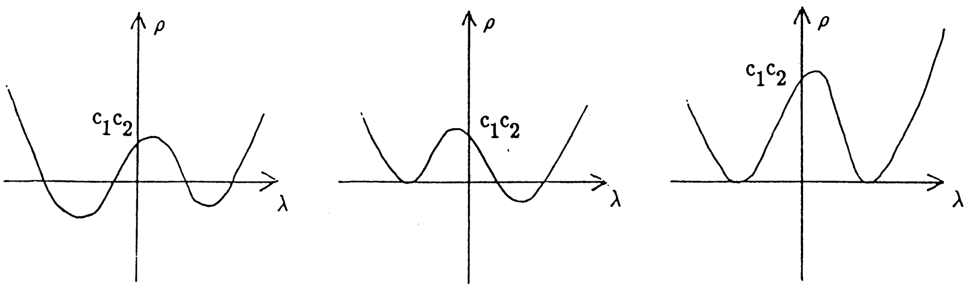

Figure 1. Some possible forms for $\rho(\lambda, 0)$.

By Schwarz' inequality we have

$$|\beta|^2 = |(Cy_2, y_1)|^2 = |(C^{1/2}y_2, C^{1/2}y_1)|^2 \leq (Cy_1, y_1)(Cy_2, y_2) = c_1c_2,$$

i.e. $\rho(0, \beta) = c_1c_2 - |\beta|^2 \geq 0$. It is clear that (by translating the graphs of Figure 1) $\rho(\lambda, \beta)$ has only real zeros and so $\lambda \in \mathbb{R}$, as required.

For future reference we note that, for a GS system the strict inequality (1.3) holds, $L_1(\lambda)$ and $L_2(\lambda)$ are hyperbolic and $\rho(\lambda, \beta)$ does not have multiple zeros $\lambda \neq 0$.

Step 2. Now we take advantage of the argument of Step 1 to show that $\sigma(L) \subset \mathbb{R}$. Let $\lambda \in \partial\sigma(L)$. Since $\partial\sigma(L)$ is contained in the approximate point spectrum (Theorem 57.7 of [2]), there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for each n and $\|L(\lambda)x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Form the decompositions $x_n = y_{1,n} + y_{2,n}$, $y_{1,n} \in H_1$, $y_{2,n} \in H_2$, for each n.

Using the line of argument of Step 1 it is found that, as $n \longrightarrow \infty$

$$(3.8) \qquad \begin{aligned} &(\alpha_{1,n}\lambda^2 + b_{1,n}\lambda + c_{1,n}) + \beta_n \longrightarrow 0 \\ &\bar{\beta}_n + (\alpha_{2,n}\lambda^2 - b_{2,n}\lambda + c_{2,n}) \longrightarrow 0 \end{aligned}$$

where $\beta_n = (Cy_{2,n}, y_{1,n})$ and for $j = 1, 2$

$$\alpha_{j,n} = \|y_{j,n}\|^2, \quad c_{j,n} = (Cy_{j,n}, y_{j,n}), \quad b_{j,n} = (B_j y_{j,n}, y_{j,n}).$$

Since the sequences $\{\alpha_{j,n}\}_{n=1}^{\infty}$, $\{c_{j,n}\}_{n=1}^{\infty}$, $\{b_{j,n}\}_{n=1}^{\infty}$, $(j = 1, 2)$ and $\{\beta_n\}_{n=1}^{\infty}$ are bounded there is a subsequence of indices $\{n_k\}_{k=1}^{\infty}$ for which we may define $\alpha_j = \lim_{k\to\infty} \alpha_{j,n_k}$ $(j = 1, 2)$, $\beta = \lim_{k\to\infty} \beta_{n_k}$, etc. With these definitions equations (3.6) are obtained from (3.8). As before, it follows that $\lambda \in \mathbb{R}$. Thus $\partial\sigma(L) \subset \mathbb{R}$ and hence $\sigma(L) \subset \mathbb{R}$. ∎

THEOREM 3.2. *The spectrum of a GS system is real and all eigenvalues are either of positive type or negative type.*

PROOF. Suppose now that $L(\lambda)$ is a GS system. By the previous theorem $\sigma(L) \subset \mathbb{R}$. Let λ be an eigenvalue of $L(\lambda)$ of mixed type. Then, by Lemma 2.1, there is a corresponding eigenvector x for which $\big(L'(\lambda)x, x\big) = 0$. Let $x = y_1 + y_2$, $y_1 \in H_1$, $y_2 \in H_2$ and using the representations of (3.1) along with (3.5) it is found that

$$\big(L_1'(\lambda)y_1, y_1\big) + \big(L_2'(\lambda)y_2, y_2\big) = 0.$$

and hence

$$(3.9) \qquad 2\alpha_1\lambda + b_1 + 2\alpha_2\lambda - b_2 = 0$$

(with the notations of equations (3.6)). From (3.7) we have

$$\rho'(\lambda, \beta) = (2\alpha_1\lambda + b_1)(\alpha_2\lambda^2 - b_2\lambda + c_2) + (\alpha_1\lambda^2 + b_1\lambda + c_1)(2\alpha_2\lambda - b_2).$$

From (3.6) it is seen that, because $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}$ and, using equations (3.6) and (3.9)

$$\rho'(\lambda, \beta) = -\beta(2\alpha_1\lambda + b_1 + 2\alpha_2\lambda - b_2) = 0.$$

Thus λ is a multiple zero of $\rho(\lambda, \beta)$. But, as noted at the end of Step 1 of the proof of Theorem 3.1, all zeros $\lambda \neq 0$ of $\rho(\lambda, \beta)$ are distinct and we have our contradiction. ∎

REMARK 3.1. Recalling the definitions of hyperbolic and weakly hyperbolic pencils (and definitions (3.5)), and on checking the proofs of Theorems 3.1 and 3.2, it is easily seen that the conclusions follow from the hypotheses that $L_1(\lambda)$ and $L_2(\lambda)$ are weakly hyperbolic (for Theorem 3.1) or hyperbolic (for Theorem 3.2) together with $C > 0$. Thus: let

$$L(\lambda) = \begin{bmatrix} L_1(\lambda) & C_{12} \\ C_{12}^* & L_2(\lambda) \end{bmatrix}$$

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where $L_1(\lambda) = I\lambda^2 + B_1\lambda + C_{11}$, $L_2(\lambda) = I\lambda^2 - B_2\lambda + C_{22}$ are weakly hyperbolic and $B_1 > 0$, $B_2 > 0$. If also

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix} > 0$$

then $\sigma(L) \subset \mathbb{R}$.

4. **The distribution of eigenvalues.** In this section some results are established concerning the distribution of eigenvalues of AGS and GS systems under the further hypothesis that $C \gg 0$. It is convenient here to introduce the idea of the *total multiplicity* of certain sets of complex numbers with respect to $L(\lambda)$, and to establish two lemmas which isolate important ideas in the proof of the subsequent theorems. For these purposes $L(\lambda)$ (or $L(\lambda, \alpha)$) will denote a selfadjoint monic operator polynomial of any degree. The polynomial of (1.1) with $B^* = B$ and $C^* = C$ is included, of course.

Let $D \subset \mathbb{C}$ be a domain with $\partial D \cap \sigma\big(L(\lambda)\big) = \emptyset$. If D contains not more than a finite number of points of $\sigma(L)$, say $\lambda_1, \lambda_2, \ldots, \lambda_r$, and each of these is a Fredholm point of $\sigma\big(L(\lambda)\big)$, then $m\big(D, L(\lambda)\big)$ denotes the sum of the multiplicities of $L(\lambda)$ at $\lambda_1, \lambda_2, \ldots, \lambda_r$. If the preceding conditions are not satisfied, then we define $m\big(D, L(\lambda)\big) = \infty$. The number $m\big(D, L(\lambda)\big)$ is called the *total multiplicity* of D with respect to $L(\lambda)$.

The above definition is modified in a natural way if D is replaced by an interval $(a, b) \subset \mathbb{R}$ or if $L(\lambda)$ is replaced by an operator $B \in \mathcal{L}(H)$. Thus, $m\big((a, b), B\big)$ denotes the total multiplicity of the interval (a, b) with respect to $B \in \mathcal{L}(H)$.

LEMMA 4.1. *For $\alpha \in [0, 1]$, let $L(\lambda, \alpha)$ be a family of selfadjoint monic operator polynomials on H which depend continuously on α and assume that, for each $\alpha \in [0, 1]$, $\sigma\big(L(\lambda, \alpha)\big) \subset \mathbb{R}$. Let (a, b) be an interval for which $a, b \notin \sigma\big(L(\lambda)\big)$ for any $\alpha \in [0, 1]$. Then*

$$m\big((a, b), L(\lambda, 1)\big) = m\big((a, b), L(\lambda, 0)\big).$$

PROOF. Let D be the open disc in the complex plane with centre $\frac{1}{2}(b + a)$ and radius $\frac{1}{2}(b - a)$. The hypotheses imply that all points of ∂D are regular points of $L(\lambda, \alpha)$ for each $\alpha \in [0, 1]$. The result now follows from the invariance of the total multiplicity of D with respect to α (see [6], for example). ∎

LEMMA 4.2. *Given the hypotheses of Lemma 4.1 assume in addition that, for each $\alpha \in [0, 1]$,*

$$m\big((a, b), L(\lambda, \alpha)\big) < \infty$$

and $L(\lambda, \alpha)$ has no eigenvalues in (a, b) of mixed type. Then:

(a) *The number of eigenvalues in (a, b) of positive (of negative) type (i.e. the sum of their multiplicities) is the same for $L(\lambda, 1)$ and $L(\lambda, 0)$.*

(b) *If, for $L(\lambda, 0)$, and the interval (a, b), all eigenvalues of positive type exceed all eigenvalues of negative type, then the same is true for $L(\lambda, 1)$.*

PROOF. Note first of all that the eigenvalues of $L(\lambda, \alpha)$ are all real and depend continuously on α. Then it follows from Theorem III.1.1 of [5] that eigenvalues of positive type (or of negative type) retain that type under small selfadjoint perturbations of

$L(\lambda, \alpha)$, unless there is a confluence of eigenvalues of different types. But this is excluded by our hypotheses for $0 \leq \alpha \leq 1$. Statements (a) and (b) follow immediately from these properties. ∎

THEOREM 4.3. *Let $L(\lambda) = I\lambda^2 + B\lambda + C$ be an AGS system with $C \gg 0$. Then*

$$m\big((0,\infty), B\big) = p \text{ implies } m\big((-\infty,0), L(\lambda)\big) = 2p \tag{4.1}$$

and

$$m\big((-\infty,0), B\big) = q \text{ implies } m\big((0,\infty), L(\lambda)\big) = 2q \tag{4.2}$$

PROOF. Write $L(\lambda)$ as in (3.1) and recall the definitions (3.5). Consider the family of monic, quadratic polynomials

$$L(\lambda, \alpha) = \begin{bmatrix} L_1(\lambda) & \alpha C_{12} \\ \alpha C_{12}^* & L_2(\lambda) \end{bmatrix} \tag{4.3}$$

where $\alpha \in [0,1]$. Thus, $L(\lambda) = L(\lambda, 1)$. It is easily verified that $C \gg 0$ implies

$$L(0, \alpha) = \begin{bmatrix} C_{11} & \alpha C_{12} \\ \alpha C_{12}^* & C_{22} \end{bmatrix} \gg 0.$$

Hence $0 \notin \sigma(L(\lambda, \alpha))$ for any $\alpha \in [0,1]$.

Since $L_1(\lambda)$ and $L_2(\lambda)$ are weakly hyperbolic we may use Remark 3.1 to deduce that, for any $\alpha \in [0,1]$, $\sigma\big(L(\lambda, \alpha)\big) \subset \mathbb{R}$. Furthermore, properties (4.1) and (4.2) follow for $L(\lambda, 0)$ from the weakly hyperbolic character of $L_1(\lambda)$ and $L_2(\lambda)$ (see [12], for example). Lemma 4.1 can now be applied to any interval $[-r, 0]$ or $[0, r]$ with sufficiently large $r > 0$ to establish (4.1) and (4.2) for $L(\lambda, 1) = L(\lambda)$. (See p. 140 of [12] for an estimate of r.) ∎

THEOREM 4.4. *Let $L(\lambda) = I\lambda^2 + B\lambda + C$ be a GS system with $C \gg 0$ and let $m\big((0,\infty), B\big) = p < \infty$. Then of the $2p$ negative eigenvalues of $L(\lambda)$ there are p of positive type, p of negative type, and all eigenvalues of positive type exceed all eigenvalues of negative type.*

PROOF. Consider the family $L(\lambda, \alpha)$ of equation (4.3) where $\alpha \in [0,1]$. Using Theorem 3.1 and Remark 3.2 it is seen that, for $\alpha \in [0.1]$, $L(\lambda, \alpha)$ has no eigenvalues of mixed type. Since $p < \infty$, Lemma 4.2 applies if we choose an interval $[-r, 0]$ for $[a, b]$ with r arbitrarily large. Since the assertion of the theorem holds for $L(\lambda, 0)$, the lemma implies the same result for $L(\lambda, 1)$, as required. ∎

REMARK 4.1. If $m\big((-\infty,0), B\big) = q < \infty$, then a similar assertion about the positive eigenvalues of $L(\lambda)$ holds.

REMARK 4.2. As with Remark 3.1, the conclusions of Theorems 4.3 and 4.4 hold given that $L_1(\lambda)$, $L_2(\lambda)$ are weakly hyperbolic, or hyperbolic, respectively, and $C \gg 0$.

REMARK 4.3. The separation properties claimed here require conditions of the form (1.6), or those of the preceding remark. The 2×2 polynomial

$$L(\lambda) = \begin{bmatrix} \lambda^2 + 2\lambda + 4 & \sqrt{12} \\ \sqrt{12} & \lambda^2 - 2\lambda + 4 \end{bmatrix}$$

has $C \gg 0$, B indefinite, but $L_1(\lambda)$ and $L_2(\lambda)$ are not weakly hyperbolic. Since all eigenvalues of $L(\lambda)$ are pure imaginary the root separation properties do not hold.

REMARK 4.4. In the physical problems giving rise to this investigation C is a real $n \times n$ matrix and $B = iG$ where G is a real skew-symmetric matrix (see [1]). In this case the spectrum of L is symmetric with respect to the imaginary axis.

5. **Mixed eigenvalues of AGS systems.** It has been seen in Theorem 3.2 that GS systems have no eigenvalues of mixed type. Our next result uses a similar result of Langer for weakly hyperbolic systems to show that, for AGS systems, there cannot be more than two eigenvalues of mixed type. This is consistent with the separation properties established in Section 4.

THEOREM 5.1. *Let $L(\lambda)$ be an AGS system. Then $L(\lambda)$ has not more than two eigenvalues of mixed type. If there are two then one is positive and the other negative. The lengths of Jordan chains corresponding to eigenvalues of mixed type cannot exceed two.*

PROOF. As above, write $L(\lambda)$ in the form

$$L(\lambda) = \begin{bmatrix} L_1(\lambda) & C_{12} \\ C_{12}^* & L_2(\lambda) \end{bmatrix}$$

where $L_1(\lambda)$ and $L_2(\lambda)$ are weakly hyperbolic. Also, if $\lambda \in \sigma(L_1)$ then $\lambda < 0$ and if $\lambda \in \sigma(L_2)$ then $\lambda > 0$.

Let λ_0 be an eigenvalue of $L(\lambda)$ of mixed type with eigenvector $y = y_1 + y_2$, $y_1 \in H_1$, $y_2 \in H_2$. If $y_2 = 0$ (or $y_1 = 0$) it is obvious that λ_0 is an eigenvalue of mixed type of $L_1(\lambda)$ (or of $L_2(\lambda)$). The results then follow from those of Langer on weakly hyperbolic pencils (see [10] or Theorem 31.10 of [12]).

Now it will be shown that, if λ_0 is an eigenvalue of mixed type, then the case $y_1 \neq 0$ and $y_2 \neq 0$ cannot arise. It follows from the proof of Theorem 3.2 that such a λ_0 is a multiple zero of the polynomial $\rho(\lambda, \beta)$ of equation (3.7). Since $\rho(\lambda, 0)$ has only real zeros and the graph of $\rho(\lambda, \beta)$ is obtained from that of $\rho(\lambda, 0)$ by a shift downward through $|\beta|^2$, it is clear that multiple zero $\lambda_0 \neq 0$ can only occur in the case $\beta = 0$. However, when $\beta = 0$ the first equation of (3.6) has only negative roots and the second has only positive roots. Hence the system (3.6) has *no* solution. ∎

6. **Bases of eigenvectors.** When H is finite dimensional the existence of a basis for H consisting of eigenvectors (more precisely, of Jordan chains) for $L(\lambda)$ of (1.1) is equivalent to the existence of a nontrivial factorization of $L(\lambda)$. In the monograph [4], for example, it is shown that factorization of $L(\lambda)$ is always possible and the construction of

a right-divisor $I\lambda - K$ depends on the notion of eigenvalue (or eigenvector) types used above. It follows from that general theory that for GS (and AGS) systems there exist bases of eigenvectors corresponding to eigenvalues of positive (positive and mixed) type. Bases can be constructed starting with negative type eigenvalues in a similar way. Here, we give a more direct argument applicable in the GS case, and using some ideas from the paper [7].

LEMMA 6.1. *Let* $\dim H = n < \infty$ *and* $L(\lambda) = I\lambda^2 + B\lambda + C$ *where* $B, C \in \mathcal{L}(H)$ *and are selfadjoint. Assume that* $\sigma(L)$ *is real with no eigenvalues of mixed type. Then there are two bases for* H *consisting of eigenvectors and having the following properties: One basis consists of the union of bases for eigenspaces of the eigenvalues of positive type. The other is the union of bases for eigenspaces of the eigenvalues of negative type.*

PROOF. Let f be orthogonal to all eigenvectors corresponding to eigenvalues of positive type and define $F(\lambda) = (L^{-1}(\lambda)f, f)$. Since $F(\lambda) = O(|\lambda|^{-2})$ as $|\lambda| \to \infty$ we have

$$\int_{|\lambda|=R} F(\lambda)\,d\lambda = 0 \tag{6.1}$$

for sufficiently large R. If λ_0 is a pole of $F(\lambda)$ then, because $L(\lambda)$ has no eigenvalues of mixed type, λ_0 is a simple pole and the corresponding residue is

$$\sum_{j=1}^{r} \frac{|(f, e_j)|^2}{(L'(\lambda_0)e_j, e_j)} \tag{6.2}$$

where $e_1, e_2, \ldots, e_r$ is an orthonormal basis for $\operatorname{Ker} L(\lambda_0)$ (see Lemma 2.1 of [13]). By hypothesis, all the numerators in (6.2) vanish at eigenvalues of positive type, so $F(\lambda)$ may have poles only at eigenvalues of negative type, and if so, then the residue is negative.

But then, using the residue theorem and (6.1), it follows that $F(\lambda)$ can have *no* poles. Then Liouville's theorem implies that $F(\lambda) \equiv 0$. But we have $L^{-1}(\lambda) > 0$ for λ large enough, and so $f = 0$. Thus the eigenvectors corresponding to eigenvalues of positive type are complete. Similarly, the eigenvectors corresponding to eigenvalues of negative type are complete.

Let X_+ and X_- be the unions of bases for all eigenspaces of eigenvalues of positive and negative types, respectively. Then $X_+ \cup X_-$ consists of $2n$ vectors and since both X_+ and X_- are complete, they each consist of n vectors. ∎

THEOREM 6.2. *If* $\dim H = n < \infty$ *and* $L(\lambda)$ *is a GS system then the assertions of Lemma 6.1 hold.*

PROOF. By Theorem 3.2, a GS system has no eigenvalues of mixed type and so Lemma 6.1 applies. ∎

REMARK 6.1. The general theory referred to above shows that, even for AGS systems, there is a basis consisting of eigenvectors which are positive with respect to $L'(\lambda_j)$ for some eigenvalues λ_j of positive type, with the possible addition of some eigenvectors x for which $\big(L'(\lambda_k)x, x\big) \geq 0$, and λ_k is an eigenvalue of mixed type (see Theorem 5.1). A similar statement applies begining with eigenvectors of eigenvalues with negative type. These statements may also be proved by the method used in the proof of Lemma 6.1.

7. **Concluding remarks.** Let us summarize the results in the finite dimensional case, $\dim H = n < \infty$ for a GS system $L(\lambda)$. Let $In\, B = (p, n-p, 0)$, $0 \le p \le n$ ("*In*" denotes the *inertia* of B; the number of eigenvalues in the open right and left half-planes and on the imaginary axis, respectively. A similar convention is used for the eigenvalues of a polynomial $L(\lambda)$.) Then

$$In\, L = \big(2(n-p), 2p, 0\big)$$

and the eigenvalues lie in four disjoint intervals of the real line as indicated in Figure 2.

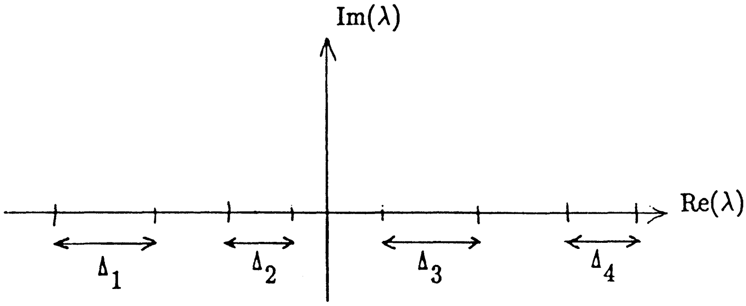

Figure 2. Distribution of eigenvalues for finite dimensional GS systems.

In Δ_1 there are p eigenvalues of negative type
In Δ_2 there are p eigenvalues of positive type
In Δ_3 there are $n-p$ eigenvalues of negative type
In Δ_4 there are $n-p$ eigenvalues of positive type

There is a basis of eigenvectors corresponding to eigenvalues in $\Delta_1 \cup \Delta_3$ and another basis of eigenvectors corresponding to $\Delta_2 \cup \Delta_4$.

In this case interesting properties concerning stability and perturbation of GS systems follow immediately from the results of Chapter III.1 of [5]. For example, if B and C depend analytically on a real parameter τ in a neighbourhood U of $\tau = 0$ and are hermitian on U, then there is a neighbourhood U_0 of $\tau = 0$ on which all eigenvalues are analytic functions of τ, and all eigenspaces are spanned by analytic eigenvector functions.

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