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THE RESIDUAL FINITENESS OF THE CLASSICAL KNOT GROUPS

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1. Introduction. The purpose of this paper is to extend the class of knot groups whose commutator subgroups are known to be residually a finite p-group (i.e., residually of order a power of the prime p). Such a knot group is known to be residually finite (see, e.g., [**10**]), and although this class is quite restricted we will show that it includes all the groups of knots in the classical knot table [**15**].

Of course fibred, or Neuwirth, knots belong to the class since they are characterized by having commutator subgroups which are free of finite rank, and free groups are well known to be residually a finite p-group [**7**].

For non-fibred knots, Neuwirth [**14**] showed that the commutator subgroup can be built up as a direct limit under a countable sequence of generalized free products. To handle this construction inductively, it is convenient to employ Baumslag's concept of a parafree group. (Denoting the terms of the lower central series of G by $\gamma_1 G = G, \gamma_2 G, \ldots$, we define G to be *parafree* (in the variety of all groups) if G is residually nilpotent and G has the same sequence of quotients $G/\gamma_2 G, G/\gamma_3 G, \ldots$ by the terms of its lower central series as some free group F. If $G/\gamma_2 G$ is free abelian of rank r, then we also say that G is parafree of *rank* r. See [**3**].) In [**10**] some sufficient conditions were found for the commutator subgroup of the group of a non-fibred knot to be a direct limit of parafree groups and therefore to be residually a finite p-group. In [**11**] the commutator subgroups of the (one-relator) groups of two-bridge knots were shown to satisfy these conditions. Also it is not hard to show by direct calculations that these conditions are satisfied by the groups of the non-fibred knots of the classical knot table (see, e.g., [**15**]) except for the six knots 8_{15}, 9_{25}, 9_{35}, 9_{38}, 9_{41}, and 9_{49}.

In establishing these results it is actually shown that the generalized free products involved reduce to the adjunction of a countable sequence of roots to a free group. Here we will develop further conditions sufficient to imply that the commutator subgroup of a knot group is the ascending union of parafree groups and therefore residually a finite p-group. In particular, as was announced in [**12**], these conditions are satisfied by groups of the above six remaining classical knots (in which the parafree subgroups involved cannot be built up from the (free) group of the complement of a minimal spanning surface by the adjunction of roots).

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To begin with we establish in § 2 the residual nilpotence of certain generalized free products of free and/or parafree groups. Since so few generalized free products are residually nilpotent, it is not surprising that these results are quite technical. They are derived ultimately from Baumslag's technique of using compatible filters [**2**] and were motivated by Allenby and Gregorac's results on nilpotently separable subgroups [**1**].

In § 3 we follow Neuwirth's analysis of the structure of the commutator subgroup, and show that the results of § 2 will sometimes imply the parafreeness of the generalized free products involved.

To state our conditions, suppose the knot k spans an algebraically unknotted surface S of minimal genus, and let H and K be the two inclusion-induced images of $\pi_1(S) \cong F_{2g}$ in the group $X \cong F_{2g}$ of the manifold "S^3-split-along-S". (See [**14**; **10**; or **17**].) Then our main result is the following theorem:

THEOREM 3.2. *Let G be the group of the knot k, and H, K, and X all as above. Suppose H and K are free factors of the (free) groups $H \cdot \gamma_2 X$ and $K \cdot \gamma_2 X$ respectively. If $|X : H \cdot \gamma_2 X| = p^n$, then $\gamma_2 G$ is an ascending union of parafree groups and residually a finite q-group for any prime $q \neq p$.*

By Corollary 2.3 of [**10**] we have

COROLLARY 3.5. G (as above) *is residually a finite solvable group.*

By a theorem of G. Baumslag [**4**], it follows

COROLLARY 3.6. *Any two-generator subgroup of $\gamma_2 G$ is free.*

Finally in § 4 we give some examples and remarks.

2. The residual nilpotence of some generalized free products. If C is a class of groups, we say that the group G is *residually-C* provided for each element $1 \neq x$ in G there exists a group H in C and a homomorphism Φ from G onto H such that $1 \neq \Phi(x)$. In generalization of this concept we say that a subgroup K of G is *C-separable* or *separable by C-groups* if for each element $x \notin K$ there exists a group H in C and a homomorphism Φ on G onto H such that $\Phi(x) \notin \Phi(K)$. Mal'cev, Baumslag, and others have studied subgroups of solvable or nilpotent groups which are separable by finite groups and finite p-groups (see [**1**]), but we shall need to show that subgroups of free groups and other not necessarily nilpotent groups are *p-separable*, or separable by finite p-groups. For a general reference on combinatorial group theory, see [**9**].

We begin, given a prime p, by writing $G^{p^j} = \mathrm{gp}\langle g^{p^j} | g \text{ in } G\rangle$ and $G(i, j)_p$ or just $G(i, j) = \gamma_i G \cdot G^{p^j}$ for any group G. In this notation a finitely generated group G is residually a finite p-group if and only if $\bigcap_{i,j}^{\infty} G(i, j) = 1$ and $H \leqq G$ is p-separable if and only if $\bigcap_{i,j}^{\infty} \{G(i, j) \cdot H\} = H$.

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We are now prepared to state our calculational tools, the first of which is trivial (since a finite p-extension of a finite p-group is a finite p-group).

LEMMA 2.1. *Let $H \leqq K \leqq G$ with $\bigcap \{K(i, j) \cdot H\} = H$ and $|G : K| = p^t$. Then $\bigcap \{G(i, j) \cdot H\} = H$.*

PROPOSITION 2.2. *Let $G = A * B$ be residually a finite p-group and $H \leqq A$ with $\bigcap \{A(i, j) \cdot H\} = H$. Then $\bigcap \{G(i, j) \cdot H\} = H$.*

Proof. Consideration of the homomorphism θ from G onto A which maps A identically and B to 1, shows that $G(i, j) \cap A = A(i, j)$.

Applying θ to $x \in \bigcap \{G(i, j) \cdot H\}$ we can write

$$x = h_1 a_1 \cdot t_1 = h_2 a_2 \cdot t_2 = \ldots$$

where $h_k a_k \in H \cdot A(k, k)$ and $t_k \in B^G \cap G(k, k)$. But $G = A * B$, so $h_k a_k \cdot t_k = h_{k+1} a_{k+1} \cdot t_{k+1}$ implies $a_k^{-1} \cdot h_k^{-1} \cdot h_{k+1} \cdot a_{k+1} = t_k t_{k+1}^{-1} \in A \cap B^G = 1$. Therefore $t_1 = t_2 = \ldots \in \bigcap G(i, j) = 1$, so $x = h_1 a_1 = h_2 a_2 = \ldots \in \bigcap \{A(i, j) \cdot H\} = H$.

PROPOSITION 2.3. *Let R be parafree of rank n, $F, H, K \cong F_n$, and $G = F *_{H=K} R$. Suppose H is a free factor of $\gamma_2 F \cdot H$, $|F : \gamma_2 F \cdot H| = p^m$ and K is p-separable in R. Then G is residually a finite p-group.*

COROLLARY 2.4. *If, in addition, $G/\gamma_2 G \cong F_n/\gamma_2 F_n$, then G is parafree of rank n and hence residually a finite q-group for all primes q.*

Proof of Proposition 2.3. Let M be the kernel of the canonical mapping $G \to F/H \cdot \gamma_2 F$. By Reidemeister-Schreier rewriting (or the Karrass-Solitar subgroup theorem [**8**]), M is isomorphic to the free product of p^m (isomorphic) conjugates of $N = H \cdot \gamma_2 F *_{H=K} R$, amalgamated along the respective conjugates of $H \cdot \gamma_2 F$. For convenience, by abuse of notation, we will label these conjugates $N, N_1, N_2, \ldots, N_\lambda, \ldots, N_{p^m - 1}$.

(We remark that M can also be represented as a "graph of groups" with one copy of $H \cdot \gamma_2 F$ amalgamated to p^m different conjugates of R along the respective conjugates of H.)

We consider an element $W \neq 1$ in G. If $W \notin M$, then we can represent W non-trivially in the finite p-group G/M. Otherwise we write $W \in M$ canonically as $W = W_0 W_1 W_2 \ldots W_r$ with only W_0 amalgamated and consecutive non-amalgamated syllables from strictly different factors.

Now H is a free factor of $\gamma_2 F \cdot H$, so we can write $\gamma_2 F \cdot H = T * H$ and hence $N = T * R$. Therefore N is residually a finite p-group. Furthermore, the image of F in this presentation is $T * K$, so F is p-separable in N by Proposition 2.2. It follows that we can separate each syllable W_K from the amalgamated subgroup in a quotient group of order a power of p. More specifically, let $\theta : M \to N$ be induced by the natural "identity" isomorphisms of the various factors N_λ to N. Then, by choosing i and j sufficiently large, we can assume that $W_0 \notin N(i, j)_p$ and that $\theta(W_k)$ is not in the image $T * K$ of the amalgamated subgroup F, for $k = 1, \ldots, r$.

Clearly if W has length 0 or 1, $\theta(W)$ and hence W will have a non-trivial image in the finite p-group $N/N(i, j)$. Otherwise we form the generalized free product, P, of the p^m conjugate quotient groups $N_\lambda/N_\lambda(i, j)$ with the images of $(T * K) \cdot N(i, j)/N(i, j)$ in each conjugate amalgamated. But (as in the proof of Proposition 2.1 in [**4**]) P, itself, is residually a finite p-group, for if $\bar{\theta} : P \to N/N(i, j)$ is induced by θ, then ker $\bar{\theta}$ is a free group (by a well-known theorem of H. Neumann [**13**]) so P is an extension of a free group by a finite p-group.

It follows that W can be represented non-trivially in a finite p-group so that M and G are residually a finite p-group. This completes the proof of Proposition 2.3.

If $G/\gamma_2 G \cong F_n/\gamma_2 F_n$, then it follows from Proposition 1 in [**3**] that G has the same lower central sequence as F_n. Thus G is parafree, residually torsion-free nilpotent, and, by a theorem of Gruenberg [**6**], residually a finite q-group for all primes q. This proves Corollary 2.4.

3. Applications to knot groups. Let k be a knot, and $G = \pi_1(S^3 - k)$ be its knot group. Then $G/\gamma_2 G$ is free cyclic and, by Neuwirth's analysis of the structure of knot groups (see [**14**]), $\gamma_2 G$ is a generalized free product of the form

$$\text{(3.1)} \quad \gamma_2 G = \ldots * X_{-1} *_{F_{2g}} X_0 *_{F_{2g}} X_1 * \ldots .$$

Here each X_i is isomorphic to X, the fundamental group of the three-manifold "S^3-split-along-S" for some fixed spanning surface S of minimal genus, and the amalgamation from X_i to X_{i+1} represents the identification of the appropriate copies $K_i \subseteq X_i$ and $H_{i+1} \subseteq X_{i+1}$ of the inclusion induced images K and H of the group of the spanning surface in X. As indicated, $\pi_1(S) \cong F_{2g}$, and we will restrict our attention to knots k for which S is algebraically unknotted, that is for which X is free (of rank $2g$).

We restate our main result.

THEOREM 3.2. *Let G, H, K, and X be as above. Suppose H and K are free factors of the (free) groups $H \cdot \gamma_2 X$ and $K \cdot \gamma_2 X$ respectively. If $|X : H \cdot \gamma_2 X| = p^m$, then $\gamma_2 G$ is an ascending union of parafree groups and residually a finite q-group for any prime $q \neq p$.*

Proof. Denote the subgroup $X_1 *_{F_{2g}} \cdot X_2 * \ldots * X_n$ of $\gamma_2 G$ (given by 3.1) by $X_1{}^n$. Similarly, using the homogeneity of the presentation, write X^n for the subgroup of 3.1 generated by any n consecutive factors. We will show X^n is parafree by induction.

We claim first that $X^n/\gamma_2 X^n \cong F_{2g}/\gamma_2 F_{2g}$. (This was surely known, for example, to Seifert, but we sketch a proof for completeness.) Let V and V^t denote the Seifert linking matrix [**16**] for S and its transpose. Then for $J =$

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$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, it follows that $V - V^t$ is the $2g \times 2g$ block matrix

$$(3.3) \quad \begin{bmatrix} J & 0 & & & \{0\} \\ 0 & J & & & \\ & & \cdot & & \\ & & & \cdot & \\ & \{0\} & & & \cdot \\ & & & & J \end{bmatrix}$$

having J on the diagonal and zero elsewhere. Under an appropriate choice of bases in K, H, and X we will have V and V^t defining the inclusion-induced mappings $H/\gamma_2 H \to X/\gamma_2 X$ and $K/\gamma_2 K \to X/\gamma_2 X$ (See [**17**]). It follows that the abelian group $X^n/\gamma_2 X^n$ is presented by the n by $n - 1$ block matrix L_n, given by (3.4).

$$(3.4) \quad \begin{bmatrix} V & 0 & & & & & & \\ -V^t & -V & 0 & & & & & \\ 0 & V^t & V & & & & & \\ & 0 & -V^t & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & \pm V & 0 \\ & & & & & & \mp V^t & \mp V \\ & & & & & & 0 & \pm V^t \end{bmatrix}$$

The n by $n - 1$ block matrix L_n

Now all of these relation matrices present a free abelian group of rank $2g$ (corresponding to the empty matrix), since considering L_n as an n by $n - 1$ matrix of blocks we can reduce L_n to L_{n-1} by the following sequence of equivalences corresponding to Tietze transformations of appropriate factors or amalgamated subgroups in $X^n/\gamma_2 X^n$.

(1) Replace row 1 by the sum of all the rows.

(2) Replace column m by the sum of column 1 and column m for $m = 2, \ldots, n$.

(3) Delete row 1 using column 1.

(4) Replace row m by the sum of row m and row $m + 1$ for $m = 1, \ldots, n - 2$.

(5) Multiply by -1.

Since any one factor X is embedded in X^n and its generators are independent modulo $\gamma_2 X^n$, we have $X/\gamma_k X$ embeds in $X^n/\gamma_k X^n$, and therefore X^n has the same lower central sequence as a free group of rank $2g$.

We will now show X^n is residually a finite p-group by induction on n. Since $X^1 \cong F_{2g}$ and $X_1^{n+1} \cong (X_1^n * X_{n+1}^1; K_n = H_{n+1})$, it suffices by Proposition 2.3 to show that K_n is p-separable in X_1^n.

We have already remarked that the abelian quotients $X/\gamma_2 X \cdot H$ and $X/\gamma_2 X \cdot K$ are presented by the Seifert linking matrix and its transpose. It follows that $\gamma_2 X \cdot K$ is also of index p^m in X. We write $X_1^n = (X_1^1 * X_2^{n-1}; K_1 = H_2)$ and present (as in the proof of Proposition 2.3) the kernel of the canonical homomorphism from X_1^n to $X_1^1/\gamma_2 X_1^1 \cdot K_1$ as a graph (generalized free product) of p^m copies of $N = (\gamma_2 X_1^1 \cdot K_1 * X_2^{n-1}; K_1 = H_2)$ amalgamated along their respective (free) conjugate subgroups $\gamma_2 X_1^1 \cdot K_1$.

Now K_n is p-separable in X_2^{n-1} by induction, (and Lemma 2.1), and therefore in N by Proposition 2.2. It follows that if $U = U_1 U_2 \cdot \ldots \cdot U_n \notin K_n$ has length zero or one, in the graph, then U can be separated from the image of K_n in some finite p-group $N/N(i, j)$. But the graph is residually {a generalized free product of p^n isomorphic finite p groups}, where the image of K_n is *finite* and consists of elements of length one. Since such generalized free products are in turn residually a finite p-group, the image of any U of length two or greater can be separated from the finitely many members of the image of K_n. Thus K_n is p-separable in the graph, therefore, clearly, p-separable in all of X_1^n. Therefore all X^n are parafree, completing the proof of Theorem 3.2.

As indicated in the introduction, we have the following immediate consequences.

COROLLARY 3.5. *The knot group G* (as in Theorem 3.2) *is residually a finite solvable group.*

COROLLARY 3.6. *Any two-generator subgroup of $\gamma_2 G$ is free.*

4. Examples. We indicate briefly that the group of the knot 9_{35} (which was not previously known to be residually finite) satisfies the hypothesis of Theorem 3.2. The groups $X = \langle a, b | - \rangle$, $H = \langle (ab)^2 a, (ab)^2 ba | - \rangle$ and $K = \langle aba^2, (ab)^2 b | - \rangle$ are found from the pretzel projection as in Figure 4.1. Evidently $V = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$ and $\det V = 7$, so $|X : H \cdot \gamma_2 X| = 7$. To present $H \cdot \gamma_2 X$ choose coset representatives $a, a^2, \ldots, a^7 = 1$ and representative function $a \to a$, $b \to a^2$. Writing A and B_i, $i = 0, 1, \ldots, 6$, for the generators corresponding to a^7 and $a^i b a^{-i}$, we compute

$$\tau(H \cdot \gamma_2 X) \cong F_8 = \langle A, B_0, \ldots, B_6 | - \rangle$$

and $\tau(H) = \langle B_1 B_4 A,\ B_1 B_4 B_6 | - \rangle$. The computation for K is analagous.

Unfortunately all pretzel knots do not have residually nilpotent commutator subgroups. Seifert [**16**] gave an example with trivial polynomial and therefore perfect commutator subgroup. In this case the commutator subgroup has no nilpotent homomorphic images. Also the main result of [**5**] is the existence of pretzel knots for which $\det V = 1$ but the inclusion maps of H and K into X are not onto. In this case H is not nilpotently separable in X so the X^n are not

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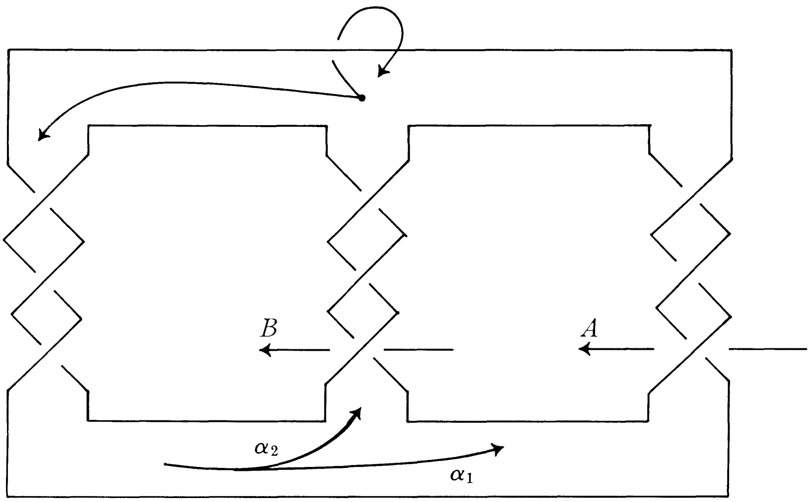

$$\begin{array}{ccccc} ABA^2 & \longleftarrow\!\!\dashv & \alpha_1 & \longmapsto & (AB)^2\mathrm{A} \\ (AB)^2B & \longleftarrow\!\!\dashv & \alpha_2 & \longmapsto & (AB)^2BA^{-1} \\ K & & & & H \end{array}$$

residually nilpotent. Such a knot spans an algebraically unknotted, weakly incompressible surface (the inclusion induced maps $H/\gamma_2 H \to X/\gamma_2 X$ and $K/\gamma_2 K \to X/\gamma_2 X$ are one-to-one), but does not have a residually nilpotent commutator subgroup. Nevertheless we still conjecture that the commutator subgroup of an alternating knot is residually a finite p-group for any prime p not dividing det V.

To further this study we would have to strengthen Proposition 2.3. We feel that many similar generalized free products are residually nilpotent. In particular, for applications to knot groups, we expect to be able to weaken the requirement that the index $|F : \gamma_2 F \cdot H|$ is a prime power to merely that it is finite. However we give just one example of the sorts of problems encountered in generalized free products of this type to show that such theorems will not hold without additional hypotheses.

Let $G = \langle a, b, c, d | a^3 = c^2, b^2 = d^3\rangle$. Then $G = \langle a, b | - \rangle *_{F_2} \langle c, d | - \rangle$ and the amalgamated subgroup is a free factor of a subgroup of index 6 with (abelian) quotient $Z_2 \oplus Z_3$ on each side. However, $G \cong \langle a, c | a^3 = c^2\rangle * \langle b, d | b^2 = d^3\rangle$ is a free product of two knot groups, hence not residually nilpotent.

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