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ALGEBRAS OF ACYCLIC TYPE

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Introduction. In this paper we consider the problem of determining when an algebra of formal power series over a commutative ring R is the homomorphic image of a reduced incidence algebra $R(P, \sim)$. The question of when two such algebras are isomorphic is answered in Section 8 of [**1**]. A slight generalization of their notion of full binomial type is introduced here.

Section 1 contains background material together with a summary of the results of [**1**]. In Section 2 we present the desired characterization, and to conclude an application appears in Section 3. In Section 3 the tools of Section 2 are used to derive an equation of R. W. Robinson and R. P. Stanley which counts labelled, acyclic digraphs.

1. Preliminaries. Let $(P, \leqq)$ be a locally finite, partially ordered set. For each pair r and s in P with $r \leqq s$, we let $[r, s]$ denote the set

$$[r, s] = \{t: r \leqq t \leqq s\}.$$

This set, when given the ordering inherited from P, is called the *interval* between r and s. We let $S(P)$ denote the set of all intervals in P.

Let $\mathscr{R}$ be a commutative ring containing the rationals $\mathbf{Q}$. Consider the $\mathscr{R}$-algebra of all functions f mapping $S(P)$ to $\mathscr{R}$, with addition and multiplication of functions defined as

$$(f + g)[r, s] = f[r, s] + g[r, s]$$

$$(f^*g)[r, s] = \sum_{t \in [r,s]} f[r, t]g[t, s]$$

and with multiplication of a function f by a scalar $a \in \mathscr{R}$ defined as

$$(af)[r, s] = af[r, s].$$

This (associative) algebra is called the *incidence algebra of* P and is denoted $\mathscr{I}(P)$. Three functions in $\mathscr{I}(P)$ will be of particular interest to us. The multiplicative identity is the Kronecker delta function δ defined by

$$\delta[r, s] = \begin{cases} 1 \text{ if } r = s \\ 0 \text{ otherwise.} \end{cases}$$

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The zeta function, ζ, takes on the value 1 everywhere. It is easy to show that ζ is invertible; its inverse is the Möbius function of P, denoted μ. Those interested in a more thorough discussion of incidence algebras should see [**3**], or [**1**, Section 3].

It is worth noting that $\mathscr{I}(P)$ is commutative only if P is trivial. For suppose there exist r and s in P with $r < s$. Define $f, g \in \mathscr{I}(P)$ by

$$f[u, v] = \begin{cases} 1 \text{ if } u = r \text{ and } v = s \\ 0 \text{ otherwise} \end{cases}$$

$$g[u, v] = \begin{cases} 1 \text{ if } u = s \text{ and } v = s \\ 0 \text{ otherwise.} \end{cases}$$

It is easy to check that $(f^*g)[r, s] = 1$ whereas $(g^*f)[r, s] = 0$ and so $\mathscr{I}(P)$ does not commute. For this reason, a nontrivial partially ordered set can never have an incidence algebra which is isomorphic to an algebra of formal power series in one variable.

Nonetheless, certain generating function identities can be interpreted within incidence algebras, and so it is appropriate to ask when an incidence algebra $\mathscr{I}(P)$ may have subalgebras which are isomorphic to algebras of formal power series. In answer to this question, Doubilet, Rota and Stanley [**1**] introduced the notion of a reduced incidence algebra.

Definition 1. Let $(P, \leqq)$ be a locally finite partially ordered set, let $\sim$ be an equivalence relation on the set of intervals $S(P)$ and let $R(P, \sim)$ denote the set of all functions in $\mathscr{I}(P)$ which are constant on equivalent intervals; i.e.,

$$R(P, \sim) = \{f \in \mathscr{I}(P) : f[r, s] = f[r', s'] \text{ whenever } [r, s] \sim [r', s']\}.$$

The equivalence relation $\sim$ is called *order compatible* if $R(P, \sim)$ is a subalgebra of $\mathscr{I}(P)$ and in this case, $R(P, \sim)$ is called the *reduced incidence algebra* associated with $\sim$.

Note that $R(P, \sim)$ is a submodule of $\mathscr{I}(P)$ for any equivalence relation $\sim$, and so the condition that it be a subalgebra is precisely the condition that it be closed under multiplication.

The equivalence classes of intervals will be called *types* and denoted with small Greek letters. If f is a function in $R(P, \sim)$ and α is a type, $f(\alpha)$ will denote the value f assumes on the intervals of type α.

Definition 2. Let $\mathrm{R}(P, \sim)$ be a reduced incidence algebra. For types α, β, γ define the *incidence coefficients* $\begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}$ as follows; choose an interval $[r, s]$ of type γ and let $\begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}$ be the number of elements t in $[r, s]$ such that $[r, t]$ is of type α and $[t, s]$ is of type β.

We need to check that $\begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}$ is well-defined; i.e., independent of the choice of interval $[r, s]$. Define functions f_α and f_β by

$$f_\alpha[r, s] = \begin{cases} 1 \text{ if } [r, s] \text{ is of type } \alpha \\ 0 \text{ otherwise} \end{cases}$$

$$f_\beta[r, s] = \begin{cases} 1 \text{ if } [r, s] \text{ is of type } \beta \\ 0 \text{ otherwise.} \end{cases}$$

Clearly these functions are in the reduced incidence algebra $R(P, \sim)$, so their product is as well. Hence $f_\alpha * f_\beta$ takes on the same value for each interval $[r, s]$ of type γ. But this value is simply the number $\begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}$.

The incidence coefficients describe the multiplication of $R(P, \sim)$ in the following way. Let f and g be in $R(P, \sim)$ and let γ be a type. Then

$$(f * g)(\gamma) = \sum_{\alpha, \beta} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} f(\alpha) g(\beta).$$

An example of an order compatible equivalence relation is to let P be the set of all finite subsets of $\{0, 1, 2, \ldots\}$ ordered by inclusion and to let $\sim$ be defined by $(S, T) \sim (S', T')$ if and only if the cardinality of the set $T - S$ equals the cardinality of the set $T' - S'$. In this case, the types are in one to one correspondence with the natural numbers so we denote the types by integers. The incidence coefficients are the binomial coefficients

$$\begin{bmatrix} n \\ k, n - k \end{bmatrix} = \binom{n}{k}$$

and the map

$$f \mapsto \sum_{l=0}^{\infty} \frac{f(l) x^l}{l!}$$

is an isomorphism from $R(P, \sim)$ onto the algebra of exponential generating functions in one variable x.

It is worth noting that if the equivalence relation $\sim$ is order compatible and if the intervals $[r, s]$ and $[r', s']$ are equivalent then the length of the longest chain in $[r, s]$ equals the length of the longest chain in $[r', s']$. This can be seen as follows. Note that ζ is in $R(P, \sim)$ hence ζ^2 is in $R(P, \sim)$. It is easy to check that $\zeta^2[r, s]$ is the cardinality of the interval $[r, s]$ and so the function c defined by

$$c[r, s] = \begin{cases} 1 \text{ if } s \text{ covers } r \text{ (i.e., if } \zeta^2[r, s] = 2) \\ 0 \text{ otherwise} \end{cases}$$

is in $R(P, \sim)$. Now observe that the length of the longest chain in $[r, s]$ is the maximal positive integer l such that $c^l[r, s] \neq 0$. As c^l must be constant on equivalent intervals, the result follows.

In view of this, it makes sense to define the *length of a type* α, denoted $l(\alpha)$, to be the length of the longest chain in those intervals of type α. The reduced incidence algebra of the previous example is characteristic of those in which the map

$$f \mapsto \sum_{\alpha} \frac{f(\alpha)x^{l(\alpha)}}{B(l(\alpha))}$$

is an $\mathscr{R}$-algebra isomorphism for an appropriately chosen sequence $\{B(n)\}$. Doubilet, Rota and Stanley [**1**] have characterized exactly when this happens; we next present their characterization.

Definition 3. A reduced incidence algebra $R(P, \sim)$ is of *full binomial type* if the map $l(\alpha)$ is one to one.

This condition is very strong for it says that two intervals are equivalent if and only if they have the same length. We note that in this case the function $l(\alpha)$ maps the set of types one to one and onto a set of the form $\{0, 1, 2, \ldots, N\}$. We consider only the case where $N = \infty$.

THEOREM 1. ([**1**]). *Let $R(P, \sim)$ be a reduced incidence algebra. Then:*
(i) The map ψ defined by

$$\psi(f) = \sum_{\alpha} \frac{f(\alpha)x^{l(\alpha)}}{B(l(\alpha))}$$

is an $\mathscr{R}$-algebra isomorphism for an appropriately chosen sequence of ring elements $B(n)$ if and only if $R(P, \sim)$ is of full binomial type.
(ii) Suppose Ψ is an R-algebra homomorphism. By the change of variables $x' = x/B(1)$ we will normalise $B(1)$ to equal 1. Then $B(n)$ is the number of maximal chains in the intervals of P of length n.

In the next section, we consider the case where the map $l(\alpha)$ is allowed to be many-one. We know that the map

$$f \mapsto \sum_{\alpha} \frac{f(\alpha)x^{l(\alpha)}}{B(l(\alpha))}$$

will fail to be an isomorphism. We instead look for conditions that will force it to be a homomorphism.

2. Algebras of acyclic type. In this section we introduce the notion of an algebra of acyclic type and prove that a reduced incidence algebra is acyclic if and only if the natural map ψ is a homomorphism. It is readily seen that every algebra of full binomial type is acyclic; the example in Section 3 will show that the converse does not hold.

Definition 4. Let $R(P, \sim)$ be a reduced incidence algebra. We say $R(P, \sim)$ is of *acyclic type* if for each α and β, the sum

$$\sum_{\gamma} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}$$

is a function only of $l(\alpha)$ and $l(\beta)$. To put this another way, $R(P, \sim)$ is of acyclic type if and only if whenever $l(\alpha) = l(\alpha')$ and $l(\beta) = l(\beta')$ then

$$\sum_{\gamma} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} = \sum_{\gamma} \begin{bmatrix} \gamma \\ \alpha', \beta' \end{bmatrix}.$$

THEOREM 2. *Let $R(P, \sim)$ be a reduced incidence algebra. Then*
(*i*) *the map ψ defined by*

$$\psi(f) = \sum_{\alpha} \frac{f(\alpha)x^{l(\alpha)}}{B(l(\alpha))}$$

is an $\mathscr{R}$-algebra homomorphism for an appropriately chosen sequence of ring elements $B(n)$ if and only if $R(P, \sim)$ is of acyclic type.

(*ii*) *In this case, if $B(1)$ is normalized to equal 1, then $B(n)$ is the sum, taken over all types α of length n, of the number of chains of length n in α.*

Proof. We first prove (*i*).

($\Rightarrow$) Suppose ψ is an $\mathscr{R}$-algebra homomorphism. Let α and β be types with $l(\alpha) = k$ and $l(\beta) = m$. We will show that

$$\sum_{\gamma} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} = \frac{B(k+m)}{B(k)B(m)}.$$

Define f_α and f_β in $R(P, \sim)$ by

$$f_\alpha(\eta) = \begin{cases} 1 \text{ if } \alpha = \eta \\ 0 \text{ otherwise} \end{cases}$$

$$f_\beta(\eta) = \begin{cases} 1 \text{ if } \beta = \eta \\ 0 \text{ otherwise.} \end{cases}$$

Note that

$$\psi(f_\alpha) = \sum_{\eta} \frac{f_\alpha(\eta)x^{l(\eta)}}{B(l(\eta))} = \frac{x^k}{B(k)}.$$

Similarly

$$\psi(f_\beta) = x^m/B(m).$$

Now

$$\begin{aligned}\psi(f_\alpha * f_\beta) &= \sum_\gamma \frac{(f_\alpha * f_\beta)(\gamma) x^{l(\gamma)}}{B(l(\gamma))} \\ &= \sum_\gamma \left(\sum_{\eta_1,\eta_2} \begin{bmatrix} \gamma \\ \eta_1, \eta_2 \end{bmatrix} f_\alpha(\eta_1) f_\beta(\eta_2) \right) \frac{x^{l(\gamma)}}{B(l(\gamma))} \\ &= \sum_\gamma \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} \frac{x^{l(\gamma)}}{B(l(\gamma))}.\end{aligned}$$

Observe that if $\begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} \neq 0$ then $l(\gamma) = l(\alpha) + l(\beta) = k + m$ and so the above sum can be rewritten as

$$\psi(f_\alpha * f_\beta) = \frac{x^{k+m}}{B(k+m)} \left(\sum_\gamma \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} \right).$$

Lastly, we use the fact that ψ is a homomorphism to get

$$\frac{x^{k+m}}{B(k+m)} \left(\sum_\gamma \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} \right) = \psi(f_\alpha * f_\beta) = \psi(f_\alpha)\psi(f_\beta) = \frac{x^k x^m}{B(k)B(m)},$$

and so

$$\sum_\gamma \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} = \frac{B(k+m)}{B(k)B(m)}.$$

The right hand side depends only on $k = l(\alpha)$ and $m = l(\beta)$ so $R(P, \sim)$ is acyclic.

($\Leftarrow$) Suppose $R(P, \sim)$ is acyclic. We let $S(m, k)$ denote the sum

$$S(m, k) = \sum_\gamma \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}$$

for α and β with $l(\alpha) = k$ and $l(\beta) = m$. Also we let T_k denote the set of types of length k.

Recall that the function c defined by

$$c[r, s] = \begin{cases} 1 \text{ if } s \text{ covers } r \\ 0 \text{ otherwise} \end{cases}$$

is in $R(P, \sim)$. Observe that $c^n[r, s]$ is the number of maximal chains in the interval $[r, s]$ of length n.

Define $B(n)$ to be the sum, taken over all types α of length n, of the number of chains of length n in α. So

$$B(n) = \sum_{\gamma \in T_n} c^n(\gamma).$$

Let k be less than or equal to n. Then we can calculate $B(n)$ in a different way by

$$B(n) = \sum_{\gamma\in T_n} c^n(\gamma) = \sum_{\gamma\in T_n} (c^k * c^{n-k})(\gamma) = \sum_{\gamma\in T_n} \left(\sum_{\alpha,\beta} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} c^k(\alpha)c^{n-k}(\beta)\right).$$

At this point, observe that if $\begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} c^k(\alpha)c^{n-k}(\beta)$ is nonzero, then $\alpha \in T_k$ and $\beta \in T_{n-k}$. This follows from the three observations that

(i) $\begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} \neq 0$ implies that $\alpha \in T_l$ and $\beta \in T_m$ where $l + m = n$;
(ii) $c^k(\alpha) \neq 0$ implies that $\alpha \in T_l$ where $l \geqq k$;
(iii) $c^{n-k}(\beta) \neq 0$ implies that $\beta \in T_m$ where $m \geqq n - k$.

So the above sum becomes

$$\begin{aligned} B(n) &= \sum_{\gamma\in T_n} \left(\sum_{\alpha\in T_k} \sum_{\beta\in T_{n-k}} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} c^k(\alpha)c^{n-k}(\beta)\right) \\ &= \sum_{\alpha\in T_k} \sum_{\beta\in T_{n-k}} c^k(\alpha)c^{n-k}(\beta)\left(\sum_{\gamma\in T_n} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}\right) \\ &= \sum_{\alpha\in T_k} c^k(\alpha) \sum_{\beta\in T_{n-k}} c^{n-k}(\beta)S(k, n-k) \\ &= S(k, n-k)\left(\sum_{\alpha\in T_k} c^k(\alpha)\right)\left(\sum_{\beta\in T_{n-k}} c^{n-k}(\beta)\right) \\ &= S(k, n-k)B(k)B(n-k). \end{aligned}$$

So for each k and m, $S(k, m) = B(k + m)/B(k)B(m)$.

Using this fact, we can now show that ψ is a homomorphism. It is clearly linear; consider next $\psi(f * g)$.

$$\begin{aligned} \psi(f*g) &= \sum_{\gamma} (f*g)(\gamma) \frac{x^{l(\gamma)}}{B(l(\gamma))} \\ &= \sum_{n=0}^{\infty} \left(\sum_{\gamma\in T_n} (f*g)(\gamma)\right) \frac{x^n}{B(n)} \\ &= \sum_{n=0}^{\infty} \sum_{\gamma\in T_n} \left(\sum_{k=0}^{n} \sum_{\alpha\in T_k} \sum_{\beta\in T_{n-k}} f(\alpha)g(\beta) \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}\right) \frac{x^k x^{n-k}}{B(n)} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{\alpha\in T_k} \sum_{\beta\in T_{n-k}} \left(\frac{f(\alpha)x^k}{B(k)}\right)\left(\frac{g(\beta)x^{n-k}}{B(n-k)}\right) \frac{B(k)B(n-k)}{B(n)} \\ &\times \left(\sum_{\gamma\in T_n} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{\alpha\in T_k} \sum_{\beta\in T_{n-k}} \left(\frac{f(\alpha)k^k}{B(k)} \frac{g(\beta)x^{n-k}}{B(n-k)}\right) \\ &\times \frac{1}{S(k, n-k)} S(k, n-k) = \left(\sum_{k=0}^{\infty} \sum_{\alpha\in T_k} \frac{f(\alpha)x^k}{B(k)}\right) \\ &\times \left(\sum_{m=0}^{\infty} \sum_{\beta\in T_m} \frac{g(\beta)x^m}{B(m)}\right) = \psi(f)\psi(g). \end{aligned}$$

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So ψ is a homomorphism which completes the proof of (i).

The proof of (ii) follows from the proof just given for (i).

The above result assumed that the poset P contained intervals of arbitrarily large finite length. As in [**1**], this assumption can be dropped by considering the power series $\psi(f)$ modulo x^{N+1} where N is the length of the longest interval.

3. An example. In this section we apply the methods of Section 2 to the problem of counting labelled acyclic digraphs.

Definition 5. An *n-labelled digraph r of order k* is a pair $(V(r), E(r))$ such that

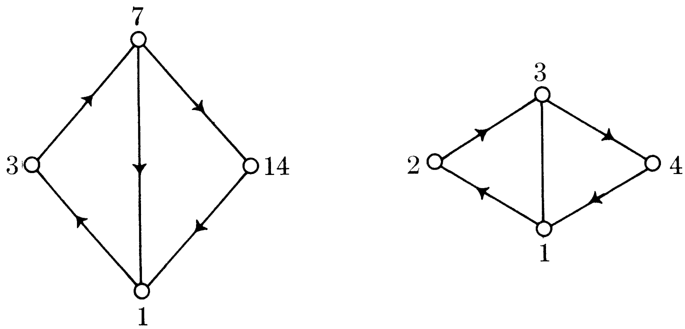
(*i*) $V(r) \subseteq \{1, 2, \ldots, n\}$ and $|V(r)| = k$;

(*ii*) $E(r)$ is a set of ordered pairs of elements of $V(r)$.

Clearly k must be less than or equal to n. If $k = n$ then r is called *canonically labelled.*

To each n-labelled digraph r or order k is associated a canonically labelled digraph of the same order which is obtained by relabelling the vertices of r with the numbers $1, 2, \ldots, k$ in such a way as to respect the natural ordering of the vertices. This new digraph will be called the *cononical representative of r* and will be denoted $\hat{r}$.

For example, a 15-labelled digraph r or order 4 is shown below along with $\hat{r}$.



r $\hat{r}$

Definition 6. Let $r = (V(r), E(r))$ be an n-labelled digraph of order k. A *directed cycle* in r is a sequence of vertices $v_0, v_1, \ldots, v_l$ with $(v_i, v_{i+1}) \in E(r)$ for $i = 0, 1, \ldots, l - 1$. r is *acyclic* if it contains no directed cycles. Let P_n be the set of all n-labelled acyclic digraphs and let $P = \bigcup_{n=0}^{\infty} P_n$.

Define the partial ordering $\leqq$ on P by $r \leqq s$ if there exists some t in P such that

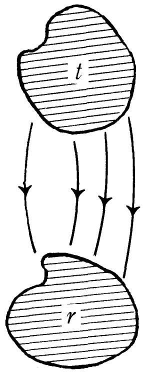
(i) $V(s) = V(r) \cup V(t)$ and $V(r) \cap V(t) = \emptyset$;

(ii) $E(r) \cup E(r) \subseteq E(s)$;

(iii) Those edges of s not in $E(r) \cup E(t)$ are directed from vertices in t to vertices in r.

This definition states that s is constructed by placing t above r and then inserting directed edges from t to r.

Note that if $r \leqq s$ then the element t is uniquely defined to be the subdigraph of s induced by the set of vertices $V(s) - V(r)$.

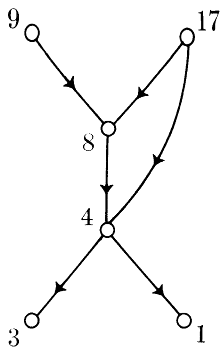


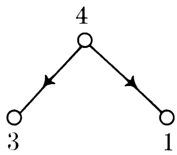
This digraph t will be denoted $I(r, s)$. Note that as a partially ordered set, the interval from r to s is isomorphic to the interval from the empty digraph to $I(r, s)$.

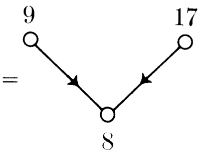
Recall that an order ideal in a partially ordered set $(Q, \leqq)$ is a subset J of Q having the property that if $x \leqq y$ and $y \in J$ then $x \in J$. Let $\mathscr{J}(Q)$ denote the set of all order ideals of Q. It is easily verified that if J and K are order ideals in Q then so are $J \cup K$ and $J \cap K$. Hence, if Q is a finite partially ordered set, then $J(Q)$, ordered by inclusion, is a sublattice of B_Q where B_Q denotes the lattice of subsets of Q. In particular, $\mathscr{J}(Q)$ is distributive.

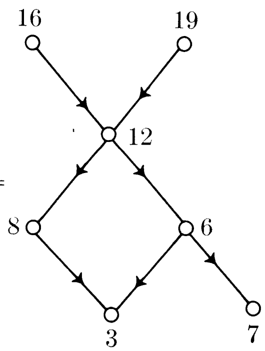
Observe that for $t \in P$, the interval from 0 to t is isomorphic as a partially ordered set to $\mathscr{J}(\bar{t})$, the lattice of order ideals of $\bar{t}$, where $\bar{t}$ is the transitive closure of the digraph t. Note that $\bar{t}$ is a partially ordered set since t is acyclic. Hence, each interval $[r, s]$ in P is isomorphic to a finite distributive lattice of height $|I(r, s)|$. Here $|I(r, s)|$ denotes the number of points in $I(r, s)$.

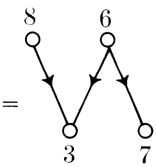
Define an equivalence relation $\sim$ on the intervals of P by $[r, s] \sim [r', s']$ if and only if the canonical representatives of $I[r, s]$ and $I[r', s']$ are equal. For example, if

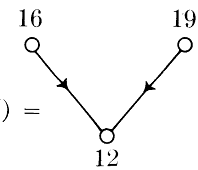
$s =$ 

$r =$ 

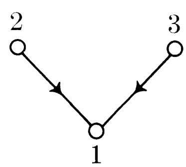
$I(r, s) =$ 

$s' =$ 

$r' =$ 

$I(r', s') =$ 

the canonical representative for both $I(r, s)$ and $I(r', s')$ is

$\widehat{I(r, s)} =$  $= \widehat{I(r', s')}.$

So $[r, s] \sim [r', s']$.

Observe that within each equivalence class of intervals, there is a unique interval of the form $(0, \alpha)$ where α is a canonically labelled acyclic digraph. So we identify the types with the set $\mathscr{A}$ of canonically labelled acyclic digraphs. The problem we wish to solve is that of enumerating this set $\mathscr{A}$.

The next four propositions explore some of the properties of this partially ordered set and this equivalence relation $\sim$. We use the notion of an *inpoint* in a digraph, this being a point which has no edges leading away from it. 0 will denote the empty digraph.

PROPOSITION 1. *The equivalence relation $\sim$ is order compatible.*

Proof. Let $r \leqq s$. We have noted that the interval from r to s is isomorphic as a partially ordered set to the interval from 0 to $\widehat{I(r, s)}$. Moreover, this isomorphism has the property that if $r \leqq u \leqq v \leqq s$ then the image of the interval $[u, v]$ under this isomorphism is equivalent to $[u, v]$. This implies that if $f, g \in R(P, \sim)$ then

$$(f * g)[r, s] = (f * g)[0, I(r, s)].$$

As $[r, s] \sim [r', s']$ if and only if $\widehat{I(r, s)} = \widehat{I(r', s')}$ we see that $R(P, \sim)$ is closed under multiplication.

Recall that for $\alpha \in A$, the interval $(0, \alpha)$ is isomorphic to the lattice of order ideals of $\bar{\alpha}$, the transitive closure of α. So, the length of $(0, \alpha)$ is equal to $|\alpha|$, the number of points in α.

PROPSOITION 2. *$R(P, \sim)$ is acyclic.*

Proof. Fix $\alpha, \beta \in \mathscr{A}$ and consider the pairs $[r, \gamma]$ where $r \leqq \gamma$, where $[0, r] \sim [0, \alpha]$ and where $[r, \gamma] \sim [0, \beta]$. Note that the number of these pairs is precisely the sum $\sum_{\gamma} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix}$. Such pairs are constructed by first placing β above α, then choosing labels for α and for β from amongst the set $\{1, 2, \ldots, |\alpha| + |\beta|\}$, and then choosing edges to be inserted from β to α. The labels can be chosen in $\begin{pmatrix} |\alpha| + |\beta| \\ |\alpha| \end{pmatrix}$ ways and the edges can be inserted in $2^{|\alpha||\beta|}$ ways. So

$$\sum_{\gamma} \begin{bmatrix} \gamma \\ \alpha, \beta \end{bmatrix} = \begin{pmatrix} |\alpha| + |\beta| \\ |\alpha| \end{pmatrix} 2^{|\alpha||\beta|}$$

which is clearly a function only of $|\alpha| = l(\alpha)$ and $|\beta| = l(\beta)$.

PROPOSITION 3. $B(n) = n!2^{\binom{n}{2}}$.

Proof. By Theorem 2 (ii), $B(n)$ is the sum, taken over all types α of length n, of the number of maximal chains of length n in α. Viewing the intervals as order ideals, it is clear that a maximal chain of length n consists of an ordering of the vertices $1, 2, \ldots, n$, together with a choice of edges from the later vertices to the earlier vertices. This can be done in $n!2^{\binom{n}{2}}$ ways.

These initial three Propositions dealt with the reduced algebra $R(P, \sim)$. Our final Proposition deals with the partial ordering P, in particular with its Mobius function.

PROPOSITION 4. *Let $r, s \in P$ with $r \leqq s$. Then*

$$\mu[r, s] = \begin{cases} (-1)^n \textit{ if } I(r, s) \textit{ has } n \textit{ points and no edges} \\ 0 \textit{ otherwise.} \end{cases}$$

Proof. As $\sim$ is order compatible, μ is constant on equivalent intervals and so

$$\mu[r, s] = \mu[0, \widehat{I(r, s)}].$$

Thus it suffices to consider $\mu[0, \alpha]$ where $\alpha \in \mathscr{A}$.

The atoms of $[0, \alpha]$ are the single point digraphs consisting of those inpoints of α. The join of a set of these inpoints can have no edges, since an edge between inpoints is clearly impossible. Since every interval of P is a distributive lattice, the proposition follows immediately (see [**3**], Ex. 1, pp. 349-350).

We may now proceed with counting acyclic digraphs, that is, with enumerating the set of types $\mathscr{A}$. Let Δ be the linear operator on generating functions which takes $f(x) = \sum_{n=0}^{\infty} f_n x^n$ to

$$\Delta(f(x)) = \sum_{n=0}^{\infty} \frac{f_n x^n}{2^{\binom{n}{2}}}.$$

THEOREM 3. ([**2**] and [**4**].) *Let $A(x)$ be the exponential generating function for acyclic digraphs; i.e., let*

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

where a_n is the number of canonically labelled acyclic digraphs on n vertices. Then

$$(\Delta A(x))(\Delta e^{-k}) = 1.$$

Proof. We consider the map ψ of Theorem 2. We have

$$\text{(i) } \psi(\delta) = \sum_{\alpha} \frac{\delta(\alpha) x^{|\alpha|}}{B(|\alpha|)} = 1 \quad \text{as } \delta(\alpha) = 0$$

unless α is the empty digraph;

$$\text{(ii) } \psi(\zeta) = \sum_{\alpha} \frac{\zeta(\alpha) x^{|\alpha|}}{B(|\alpha|)} = \sum_{\alpha} \frac{x^{|\alpha|}}{|\alpha|! 2^{\binom{\alpha}{2}}}$$

$$= \sum_{n=0}^{\infty} \frac{a_n x^n}{n! 2^{\binom{n}{2}}} = \Delta A(x);$$

$$\text{(iii) } \psi(\mu) = \sum_{\alpha} \frac{\mu(\alpha) x^{|\alpha|}}{|\alpha|! 2^{\binom{|\alpha|}{2}}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n! 2^{\binom{n}{2}}} = \Delta e^{-x},$$

here using Proposition 4.

In the reduced incidence algebra $R(P, \sim)$ we have

$$\zeta^* \mu = \delta.$$

Applying ψ and using the fact that ψ is a homomorphism gives

$$1 = \psi(\delta) = \psi(\zeta^* \mu) = \psi(\zeta)\psi(\mu) = (\Delta A(x))(\Delta e^{-k}).$$

Conclusion. A similar result to the one in Section 2 can be proven for algebras of triangular type; again the isomorphism question was settled in [**1**]. An interesting question is to classify all reduced incidence albegras for which the natural map ψ is a homomorphism onto an algebra of power series in two variables. This would require that the intervals of P have two additive invariants and this would probably require that conditions be imposed on the partially ordered set P as well as on the equivalence relation $\sim$.

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